Quantum error correction and operator algebras

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Quantum error correction

- Traditional view of QEC is in the Schrödinger picture for quantum time evolution (evolution of states):
- A quantum channel is a completely positive trace-preserving map

$$\mathcal{E} : \mathcal{B}_t(\mathcal{H}_1) \to \mathcal{B}_t(\mathcal{H}_2),$$

with operators E_i (viewed as error operators in QEC) such that

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^* \quad \forall \rho.$$

- Given such a channel, we look for a set $\mathbf{S}$ of states ρ (density operators) and a channel $\mathcal{R}$ such that

$$(\mathcal{R} \circ \mathcal{E})(\rho) = \rho \quad \forall \rho \in \mathbf{S}.$$

Quantum error correction

- On the other hand, we can consider the Heisenberg picture which describes time evolution of *observables* via completely positive unital maps (dual maps)

$$\mathcal{E}^\dagger : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1).$$

- The "sharp" observables are given by self-adjoint operators $X = \sum_k \lambda_k P_k$. The relationship between $\mathcal{E}$, $\mathcal{E}^\dagger$ is

$$\mathrm{Tr}(\mathcal{E}(\rho)P_k) = \mathrm{Tr}(\rho\,\mathcal{E}^\dagger(P_k)),$$

  which gives the probability that event k is measured after the evolution of the system with initial state ρ.
- Thus, the Schrödinger picture evolution $\mathcal{E}_2 \circ \mathcal{E}_1$ corresponds to $(\mathcal{E}_2 \circ \mathcal{E}_1)^\dagger = \mathcal{E}_1^\dagger \circ \mathcal{E}_2^\dagger$ in the Heisenberg picture.

Correction of observables

- Thus we say $X = \sum_k \lambda_k P_k$ is a *correctable (sharp) observable* if there is a channel $\mathcal{R}$ such that

$$(\mathcal{R} \circ \mathcal{E})^\dagger (P_k) = P_k \quad \forall k.$$

- This expression means that measuring X before or after the action of $\mathcal{R} \circ \mathcal{E}$ would yield the same outcomes with the same probabilities no matter what the initial state was.

Correction of observables

- More generally, an observable is given by a *positive operator-valued measure* (POVM). In the case of a discrete measure, a POVM is specified by a family of positive operators $0 \leq A_k \leq I$, called *effects*, such that $\sum_k A_k = I$. If A_k is a projection it is called a *sharp effect*.
- Thus, we say an effect A is *correctable for* $\mathcal{E}$ if there is a channel $\mathcal{R}$ such that $(\mathcal{R} \circ \mathcal{E})^\dagger(A) = A$. And a POVM is correctable if all its effects are correctable.

Correction of von Neumann algebras

- **Question:** What are correctable effects for a given channel $\mathcal{E}$?
- **Investigate:** Suppose P is a correctable sharp effect. Then there is an effect $0 \leq B \leq I$ such that $P = \mathcal{E}^\dagger(B)$ ($B = \mathcal{R}^\dagger(P)$ will do). Then we have

$$\begin{aligned} P^\perp \mathcal{E}^\dagger(B) P^\perp &= 0 & \\ \Rightarrow B E_i P^\perp &= 0 & \forall i \\ \Rightarrow B E_i &= B E_i P & \forall i \end{aligned}$$

- Similarly (since $\mathcal{E}^\dagger$ is unital) we have $E_i P = B E_i P$, and hence

$$B E_i = E_i P \quad \forall i.$$

- Thus $E_i^* E_j P = E_i^* B E_j = P E_i^* E_j$, and we see that if P is correctable for $\mathcal{E}$, then

$$[P, E_i^* E_j] = 0 \quad \forall i, j.$$

Correction of von Neumann algebras

Theorem

A sharp effect P is correctable for the channel $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^*$ if and only if

$$[P, E_i^* E_j] = 0 \quad \text{for all } i, j.$$

For sufficiency, an explicit recovery operation $\mathcal{R}$ can be constructed and (an important point for practical purposes) the *same* recovery operation works for any channel $\mathcal{E}'$ with operators E_i' that belong to the span of the E_i. (In many situations, the precise E_i may not be known, but often the operator system they generate is.)

Correction of von Neumann algebras

The commutant of the operators $E_i^* E_j$ is a von Neumann algebra, and hence the effects it contains are the closed convex hull of its projections. Since all projections in this algebra are corrected by $\mathcal{R}$, so are all the effects it contains, and thus we have the following:

Corollary

The set of effects spanning the von Neumann algebra

$$\mathcal{A} = \{A \in \mathcal{B}(\mathcal{H}_1) : [A, E_i^* E_j] = 0 \text{ for all } i, j\}$$

are all corrected by the channel $\mathcal{R}$ constructed in the theorem above. Moreover, this algebra contains all the correctable sharp effects for $\mathcal{E}$.

Standard QEC

- (Shor, Steane, Knill-Laflamme, Bennett-DiVincenzo-Smolin -Wootters, Gottesman, etc) A *code* is given by a subspace $\mathcal{H}_0 \subseteq \mathcal{H}_1$, $\dim \mathcal{H}_0 < \infty$. Then $\mathcal{H}_0$ is *correctable* for $\mathcal{E}$ if $\exists \mathcal{R}$ such that $\mathcal{R}(\mathcal{E}(\rho)) = \rho \ \forall \rho$ supported on $\mathcal{H}_0$.

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- In other words, $\mathcal{R}'(\mathcal{E}(V \rho V^*)) = \rho, \forall \rho \in \mathcal{B}_t(\mathcal{H}_0)$, where $\mathcal{R}'(\rho) = V^* \mathcal{R}(\rho) V$ and $V : \mathcal{H}_0 \hookrightarrow \mathcal{H}_1$.

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- Thus $\mathcal{H}_0$ is correctable for $\mathcal{E}$ iff $\mathcal{B}(\mathcal{H}_0)$ is correctable for $\mathcal{E}$ (in our algebraic sense) iff $\{V^* E_i^* E_j V\}' = \mathcal{B}(\mathcal{H}_0)' = \mathbb{C} I$; i.e., $\exists \lambda_{ij}$ such that

$$V^* E_i^* E_j V = \lambda_{ij} I,$$

which is exactly the Knill-Laflamme condition for QEC.

Subsystem codes

- (K.-Laflamme-Poulin, Klappenecker, Sarvepalli, Aly, Nielsen, etc) A *subsystem code* is defined through a subspace $\mathcal{H}_0 \subseteq \mathcal{H}_1$ with a particular subsystem decomposition $\mathcal{H}_0 = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $V : \mathcal{H}_0 \hookrightarrow \mathcal{H}_1$. Then $\mathcal{H}_A$ is *correctable* for $\mathcal{E}$ if $\exists\ \mathcal{R}$ such that $\forall \rho \in \mathcal{B}_t(\mathcal{H}_A)$, $\forall \tau \in \mathcal{B}_t(\mathcal{H}_B)$, $\exists \tau' \in \mathcal{B}_t(\mathcal{H}_B)$ for which

$$\mathcal{R}(\mathcal{E}(V(\rho \otimes \tau)V^*)) = \rho \otimes \tau'.$$

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- One can show that this is equivalent, in our framework, to the case where the correctable algebra $\mathcal{A}$ is any type I finite-dimensional factor

$$\mathcal{A} = \mathcal{B}(\mathcal{H}_A) \otimes I_B.$$

Type I infinite dimensional example

- Let $\mathcal{H}_0 \subseteq \mathcal{H}_1$ with $\dim \mathcal{H}_0 = \infty = \dim \mathcal{H}_0^\perp$. Let $\{P_i\}_{i=0}^\infty$ be projections onto mutually orthogonal subspaces $\{\mathcal{H}_i\}_{i=0}^\infty$ and partial isometries V_i such that $V_i^* V_i = P_0$ and $V_i V_i^* = P_i$.

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- Suppose we have probabilities $p_i \geq 0$; $\sum_i p_i = 1$. Then we have a channel,

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- Then $V_i^* V_j = \delta_{ij} I_{\mathcal{H}_0}$, and so $\{V_i^* V_j\}' = \mathcal{B}(\mathcal{H}_0)$. Thus, $\mathcal{B}(\mathcal{H}_0)$ is a type I infinite dimensional code, the natural generalization of finite-dimensional codes. In fact this is the prototypical example in *continuous variable* QEC (Braunstein, Lloyd -Slotine).

Type II example: irrational rotation algebra

- Consider two unitaries U, V on infinite dimensional space such that $UV = e^{2\pi i\theta} VU$ with θ irrational. We can take $U = e^{ia\hat{x}}$, $V = e^{ib\hat{p}}$, where $\hat{x}$, $\hat{p}$ are position and momentum operators on $L^2(\mathbb{R})$ satisfying the canonical commutation relations $[\hat{x}, \hat{p}] = i\mathbf{1}$.

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- We can consider a noise model with errors I, U, V as the possible errors. Thus, to find the correctable algebra we compute the commutant of $\{U, V\}$.
- But, in the concrete case above, this commutant is generated by unitaries $U' = e^{i(a/\theta)\hat{x}}$ and $V' = e^{i(b/\theta)\hat{p}}$, and is a factor of type II (Faddeev), and hence we find a naturally arising type II correctable algebra.

References

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