Operator spaces and Quantum Information Theory

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- Operator systems and Operator spaces
- Bell inequalities
- Entropy

Operator Systems and Spaces

Definition: 1) An Operator System is subspace $X \subset B(H)$, the bounded operators on a Hilbert space, such that

$$1 \in X \; , \; X^{\dagger} = X \; .$$

2) A Operator Space is subspace $X \subset B(H)$.

Structure: For an operator system we consider the collection

$$M_n(X) \subset M_n(B(H)) \; = \; B(H \oplus \cdots H)$$

of X valued $n \times n$ matrices entries and study the positive part

$$M_n(X)_+ \; = \; \{x \; = \; (x_{ij}) : x_{ij} \in X \; , \; x \; \geq \; 0\}$$

For an operator space we also consider $M_n(X)$, but now investigate the sequence of norms $\|x\|_n \; = \; \|(x_{ij})\|_n \; = \; \|(x_{ij})\|_{M_n(B(H))}$.

Connection

$\|x\| \le 1$ iff $\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix} \ge 0$. A selfadjoint element is positive iff

$$\|x\| \le 1 \quad \text{and} \quad \|1-x\| \le 1 \, .$$

In operator algebras one frequently uses an order morphism of the form

$$\Phi : B(H) \to B(H)_* \, , \; \Phi(x) \; = \; D^{1/2} x D^{1/2}$$

where D is is the density of a normal state $\varphi_D(x) = tr(Dx)$. The range of $\Phi(B(H)_+)$ is given by

$$\{\psi : B(H) \to \mathbb{C} \; : \; \exists_{C>0} \; : \; 0 \; \le \; \psi \; \le \; C\varphi_D\} \, .$$

For beautiful applications see the paper of Effros/Lance on nuclear C^*-algebras.

Morphism

Morphisms: A morphism between operator systems is linear unital map $u : X \to Y$ such that

$$x = (x_{ij}) \ \geq \ 0 \Rightarrow (u(x_{ij})) \ \geq \ 0 \,,$$

i.e. a unital completely positive map.

A morphism between operator spaces is a linear map $u : X \to Y$ such that $\|u\|_{cb} = \sup_n \|id_{M_n} \otimes u : M_n(X) \to M_n(Y)\|$ remains bounded.

Pro's and Con's:

- Operator system and completely positive maps are very well-known in operator algebras, and positivity is important.
- Operator spaces are closed under taking dual spaces and can be studied with the help of Banach space techniques.

Features of Operator Spaces

Let $X \subset B(H)$ be an operator space. Due to Ruan's theorem there exists en embedding $\iota : X^* \to B(H)$ such that

$$M_n(X^*) \;=\; CB(X, M_n) \quad \text{isometrically.}$$

Examples: 1) $X = C = B(\mathbb{C}, \ell_2)$. Then $C^* = R = B(\ell_2, \mathbb{C})$.
2) (Paulsen) $X = \mathbb{C}^n = \ell_\infty^n$. Then
$X^* = \ell_1^n = \mathrm{span}\{g_i : 1 \le i \le n\} \subset C^*(\mathbb{F}_n)$, the full C^*-algebra of the free group.
3) (J.-Palazuelos) The dual space NSG^* of the space of non-signally probabilities is a subspace of the full free product $*_{i=1}^m \ell_\infty^n$. Here the positive elements of norm 1 in NSG are given by probabilities $\{(a_{jk}) \;:\; a_{jk} \;\ge\; 0 \;,\; \forall_j \sum_k a_{jk} = 1\}$.

Additional features: Connection to harmonic analysis, Grothendieck inequality/Grothendieck program is developed, many noncommutative functions spaces, in particular vector-valued $L_p(L_q(X))$ are available, the *Haagerup tensor product*; and

Tensor products

1) For two operator spaces $\subset B(H)$ and $Y \subset B(K)$ we can define the minimal tensor product

$$X \otimes_{\min} Y \subset B(H \otimes K)$$

as the closure of finite rank tensors.

2) Note that if in addition X and Y are operator systems, then $X \otimes_{\min} Y$ is an operator system.

3) There is a largest projective tensor norm $X \hat{\otimes} Y$ such that $(X \hat{\otimes} Y)^* = CB(X, Y^*)$ holds completely isometrically.

More tensor norms

In C^*-algebra theory the maximal tensor norm on $A \otimes B$ is given by

$$\|\sum_k a_k \otimes b_k\|_{\max} = \sup_{[\pi(a),\sigma(b)]=0} \|\sum_k \pi(a_k)\sigma(b_k)\|_{B(H)}$$

The supremum is taken over all *-representation.

Problem: When does a map $u : A \to B$ remains bounded from $u \otimes id : A \otimes_{\min} C \to B \otimes_{\max} C$ for all C?

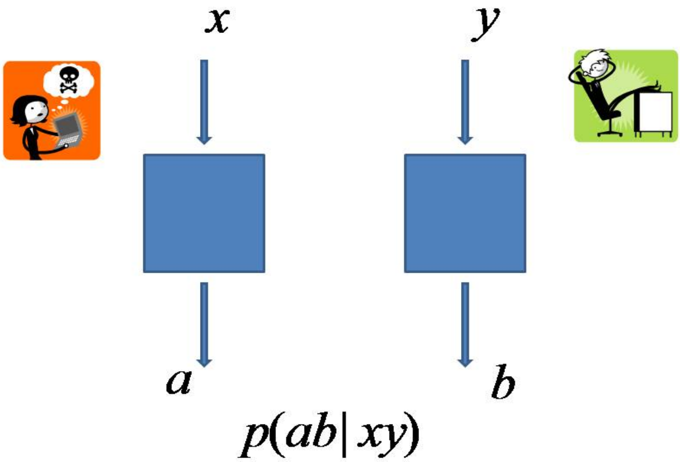
- Variations of this norm have been studied for operator spaces and lead to important results due to work by Pisier/Ozawa/LeMerdy/...
- Operator system analogues have recently been studied by Paulsen-with connections to entanglement breaking channels!

Probabilities

✍ In Bell's Gedankenexperiment one considers probabilities $p(a,b|x,y)$ which are obtained by averaging over independently performed experiments with input x for Alice, and y for Bob and output a for Alice and output b for Bob:

$$p_{loc}(a,b|x,y) \;=\; \int_{\Omega} p_a^x(\lambda) q_b^y(\lambda) d\mu(\lambda)$$

such that $\sum_a p_a^x(\lambda) = 1 = \sum_b q_b^y(\lambda)$ holds for all x, y and $\lambda \in \Omega$.


x
y
a
b
p(ab| xy)

Quantum version

The quantum version of this experiment replaces the commuting variables $p_a^x(\lambda)$ and $q_b^y(\lambda)$ by commuting operators

$$p_{qua}(a,b|x,y) \;=\; (h|(T_a^x \otimes S_b^y)h)\;,\; h \in H \otimes H$$

such that for all experiments x, y

$$T_a^x \;\geq\; 0, S_b^y \;\geq\; 0 \quad , \quad \sum_a T_a^x \;=\; 1 \;=\; \sum_b S_b^y \;.$$

For tripartite systems one may consider $(h|(T_a^x \otimes S_b^y \otimes R_c^z)h)$.
Theorem: (Bell) There are quantum probabilities which are not local.

Linear constraints

- Following Tsirelson, we want to show that there are significantly more quantum probabilities than classical or local probabilities.
- We use linear testing (constraints)

$$\|[M_{ab,xy}]\|_{loc} \; = \; \sup_{p \text{ local}} |\sum_{abxy} M_{ab}^{xy} p(a,b|xy)|$$

and

$$\|[M_{ab,xy}]\|_{qua} \; = \; \sup_{p \text{ quantum}} |\sum_{abxy} M_{ab}^{xy} p(a,b|xy)| \; .$$

- The violation for a matrix M is given by the ration

$$\mathrm{viol}(M) \; = \; \frac{\|M\|_{qua}}{\|M\|_{loc}} \; .$$

Connection to OS

- ✍ $\ell_1^n(\ell_\infty^m)$ is an operator space.
- ✍ The unit ball of $CB(\ell_1^n(\ell_\infty^m), B(H))$ is given by sequences $(\Phi_i)_{i=1}^n$ such that
$$\Phi_i(e_\alpha) \;=\; a_\alpha^* b_\alpha$$
and $\sum_\alpha a_\alpha^* a_\alpha \leq 1$, $\sum_\alpha b_\alpha^* b_\alpha \leq 1$.
- ✍ We can also define an operator system $NSG^*(n,m)$ such that $CPU(NSG^*, B(H))$ is exactly given by all sequences (Φ_i) such that
$$\sum_\alpha \Phi_i(e_\alpha) \;=\; 1 \quad \text{for all i} \,.$$
- ✍ NSG^* and $\ell_1^n(\ell_\infty^m)$ are closely related (see work with Carlos P.)

min versus ε

For two Banach spaces X and Y the minimal tensor norm is given by

$$\|\sum_k x_k \otimes y_k\|_\varepsilon \;=\; \sup_{\|x^*\|\le 1, \|y^*\|\le 1} |\sum_k x^*(x_k) y^*(y_k)| \;.$$

For a matrix M_{ab}^{xy} we see that

$$\|[M_{ab,xy}]\|_{loc} \;=\; \|\sum_{ab,xy} M_{ab,xy} e_{x,a} \otimes e_{y,b}\|_{NSG^* \otimes_\varepsilon NSG^*)}$$

and

$$\|[M_{ab,xy}]\|_{qua} \;=\; \|\sum_{ab,xy} M_{ab,xy} e_{x,a} \otimes e_{y,b}\|_{NSG^* \otimes_{\min} NSG^*)} \;.$$

Conclusion: Quantum versus local allows a one to one translation in terms of ε versus min tensor norm.

Comments

- Tsirelson showed that for correlations (no *a*'s and *b*'s) Grothendieck's inequality implies (is even equivalent to) $\ell_1 \otimes_{\min} \ell_1 = \ell_1 \otimes_\varepsilon \ell_1$ isomorphically. Hence the violation for correlations is bounded.
- With Garcia-Perez, Villanueovo, Palazuelos, and Wolff, we showed that $\ell_1 \otimes_{\min} \ell_1 \otimes_{\min} \ell_1 = \ell_1 \otimes_\varepsilon \ell_1 \otimes_\varepsilon \ell_1$ fails dramatically, and hence unbounded violation can occur for tripartite systems.
- Asymptotics for more than three parties are unknown.
- It is open whether $\ell_1 \otimes_{\min} \ell_1(\ell_\infty) = \ell_1 \otimes_\varepsilon \ell_1(\ell_\infty)$ holds.

Classical Entropy

Definition: Let $a = (a_j)$ be a probability measure on $\{1, .., n\}$. The entropy is given by

$$\mathrm{Ent}(a) \; = \; -\sum_{k=1}^{n} a_k \ln(a_k) \, .$$

Note: If $\sum_k a_k = 1$, we have

$$\mathrm{Ent}(a) = -\frac{d}{dp} \|a\|_p |_{p=1}$$

where

$$\|a\|_p \; = \; \left(\sum_k a_k^p \right)^{\frac{1}{p}} \; .$$

Entropy of a channel

For a channel $T:\ell_1\to\ell_1$ the minimal entropy is given by

$$\mathrm{Ent}(T) \;=\; \min_{\|a\|_1=1, a\,\geq\,0} \mathrm{Ent}(T(a))\;.$$

Note: For a positivity preserving, probability preserving map we have

$$\mathrm{Ent}(T) = -\frac{d}{dp}\|T:\ell_1\to\ell_p\|\;.$$

Mixed norms

For matrices (a_{ij}) we define

$$\|x\|_{\ell_p(\ell_q)} = \left(\sum_i \left(\sum_j |a_{ij}|^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} .$$

Note: $\ell_p(\ell_q) \subset \ell_q(\ell_p)$ contractively if $q \geq p$.

Lemma: T and S linear maps and $1 \leq p$. Then

$$\| T \otimes S : \ell_1^{nm} \to \ell_p^{nm} \| = \| T \| \| S \| .$$

Proof

- $\|id \otimes T : \ell_1^m(\ell_1^n) \to \ell_1^m(\ell_p^n)\| \leq \|T\|$.
- Since $\ell_1^m(\ell_p^n) \subset \ell_p^n(\ell_1^m)$ contractively, we find

$$\|\,\text{flip}\; T\; \text{flip} : \ell_1^n(\ell_1^m) \to \ell_p^n(\ell_1^m)\| \leq \|T\| \, .$$

- We compose with

$$\|id \otimes S : \ell_p^n(\ell_1^m) \to \ell_p^n(\ell_p^m)\| \;\leq\; \|S\|$$

and find

$$\|T \otimes S : \ell_1^{nm} \to \ell_p^{nm}\| \leq \|T\|\|S\| \, .$$

Classical Additivity

Theorem: $\mathrm{Ent}(T \otimes S) = \mathrm{Ent}(T)\,\mathrm{Ent}(S)$

Proof: For any channel R we define $f_R(p) = \|T : \ell_1 \to \ell_p\|$. Then

$$f_{T\otimes S}(p) \;=\; f_p(T)f_p(S)$$

and hence

$$f'_{T\otimes S}(p) \;=\; f'_p(T)f_p(S) + f_p(T)f'_p(S)\,.$$

For $p = 1$ we have $f_1(S) = 1 = f_1(T)$ and hence

$$-\,\mathrm{Ent}(T \otimes S) \;=\; \mathrm{Ent}(T) + \mathrm{Ent}(S)\,.$$

Noncommutative Entropy

Definition: $\mathrm{Ent}(\rho) = -tr(\rho \ln(\rho))$ and

$$\mathrm{Ent}(\Phi) \; = \; \min_{tr(\rho)=1} \mathrm{Ent}(\Phi(\rho)) \, .$$

Theorem: (Hastings 2009) The minimal entropy is not additive.

Abstract Entropy

Observation: Assume that we have norms $\| \, \|_p$ on complex $n \times n$ matrices and mixed norms $\| \, \|_{p,q}$ with corresponding spaces $L_p(M_n)$, $L_p(M_n; L_q(M_m))$ such that

- $L_p(M_n \otimes M_m) = L_p(M_n; L_p(M_n))$;
- $L_p(M_n; L_q(M_m)) \subset L_q(M_m; L_p(M_n))$ contractively.

Then the expression

$$F_\Phi(p) \;=\; \|id \otimes \Phi : L_1(M_n; L_p(M_m)) \to L_1(M_n; L_p(M_m))$$

is (sub-) multiplicative and

$$\mathrm{Ent}^F(\Phi) \;=\; -\frac{d}{dp} F_\Phi(p)|_{p=1}$$

is (sub-) additive for linear maps satisfying

$$\|id \otimes \Phi : L_1(M_{nm}) \to L_1(M_{nm})\| \;=\; 1 \, .$$

Comments

- Nobody explored other values than $p = 1$, not even for *cb*-entropy from above.
- The cb-entropy should be related to the operator space structure of the spaces considered by Szarek.
- We are working on new channels using finite dimensional quantum groups.
- There seem to be more connections between operator space theory and quantum capacity (with or without assisted entanglement).