Hyperbolicity in the symplectic category

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In this report, we remember our dearest friend and colleague, Pit-Mann Wong. It was his inspiration which lead to the proposal for this workshop and its organization. In the weeks before the workshop, he was diagnosed with a severe form of liver cancer, and was unable to attend the workshop; unfortunately, he has since passed away, on July 3 of this year. We will remember him and miss him.

1 Overview of the Field

The Kobayashi metric is a key intrinsic quantity associated to complex manifolds, if it is nondegenerate then the manifold is said to be hyperbolic; the study of hyperbolicity is central in much of complex geometry. This workshop aimed to extend notions and theorems regarding hyperbolicity to the (much more general) area of almost-complex and symplectic geometry, thus finding a range of applications to an exciting field of modern mathematics.

Let (M, J) be an almost complex manifold and $\Delta_r, r > 0$, be the disc of radius r, centered at the origin, in the complex plane $\mathbb{C}$. At a point $x \in M$ and a tangent vector $v \in T_x M$, denote by $\mathrm{H}ol(\Delta_r, M)(x, v)$ the space of all J-holomorphic curves from Δ_r into M with the properties that $f(0) = x$ and $f'(0) = v$. The J-Kobayashi pseudo-metric is defined by

$$\kappa_J(x, v) = \inf \frac{1}{r}$$

where the infimum is taken over all $r > 0$ such that $\mathrm{H}ol(\Delta_r, M)(x, v)$ is non-empty. An almost complex manifold (M, J) is said to be *J-Kobayashi hyperbolic* if $\kappa_J(x, v) > 0$ of all $x \in M$ and $v \neq 0$.

A compact almost complex manifold M is said to be J-Brody hyperbolic if there are no non-constant J-holomorphic curves $f : \mathbb{C} \longrightarrow M$. This implies, in particular, there are no rational or elliptic curves in M.

It is easy to see that J-Kobayashi hyperbolic implies J-Brody hyperbolic. The converse is false in general, however it is valid if M is compact;

Lemma 0.1 *For a compact almost complex manifold (M, J), J-Kobayashi hyperbolic is equivalent to J-Brody hyperbolic.*

In the complex case this is a consequence of Brody's reparametrization lemma together with a convergence argument using the fact that M is compact. In the almost complex case the argument is identical since Brody's reparametrization lemma only acts on the domain and the existence of a convergent subsequence follows from Arzela-Ascoli.

2 Recent Developments and Open Problems

In the literature there are several results concerning J-hyperbolicity. Bangert showed in [Ban98] that T^{2n} equipped with a standard symplectic structure ω is not J-Brody-hyperbolic for any ω-tame almost complex structure J. These results were extended by Biolley in her thesis [Bio04], where she proves the same result for a Stein manifold satisfying an algebraic condition in Floer homology. In all of these cases, the manifolds were shown to be not J-Brody hyperbolic for all tamed almost complex structures J.

On the other hand, Duval showed in [Duv04] that the complement of 5 J-holomorphic lines in $(\mathbb{P}^2, \omega_{FS})$, where J is any ω_{FS}-tame almost complex structure, is Kobayashi-hyperbolic.

3 Scientific Progress Made

In the results in the literature concerning J-hyperbolicity described above, a symplectic manifold was shown to be either hyperbolic or not hyperbolic for all tamed almost complex structure. We extend these examples by investigating the hyperbolicity of the complement of a divisor in ruled symplectic surfaces.

We review some necessary background on symplectic ruled surfaces. For details we refer the interested reader to [MS98]. Let $\pi : X \to \Sigma$ be a smooth sphere bundle over a compact genus g Riemann surface Σ. Up to diffeomorphism there are exactly two such bundles for each g, the product $X_0 = S^2 \times \Sigma$ and the non-trivial bundle X_1. The trivial bundle X_0 admits sections σ_{2k} of even self-intersection number $2k$ and the non-trivial bundle admits sections σ_{2k+1} of odd self-intersection number $2k+1$. The second homology group $H_2(X;\mathbb{Z})$ is generated by the class of a section and the class of a fiber f, and we have $[\sigma_n] + f = [\sigma_{n+2}] \in H_2(X;\mathbb{Z})$, $[\sigma_n] \cdot f = 1$, $[\sigma_n] \cdot [\sigma_n] = n$ and $f \cdot f = 0$. It is completely understood which cohomology classes can be represented by symplectic forms and any two cohomologous symplectic forms on X are symplectomorphic.

Examples of such bundles are given by taking a holomorphic line bundle $L \to \Sigma$ and setting $X = \mathbb{P}(L \oplus \mathbb{C}) \to \Sigma$.

Let (X, ω) denote a symplectic sphere bundle over a Riemann surface of genus g and let J be an ω-tame almost complex structure on X. Denote the homology class of a fiber by f and let s denote the section with self-intersection 0 or 1, depending on whether X is the trivial or non-trivial bundle, respectively.

Definition 0.1 *Fix a symplectic ruled surface (X, ω) with tamed almost complex structure J.*

Let m and n be non-negative integers and let L_f be the disjoint union of images of m J-curves in the class f, and define L_σ to be the union of images of n generic smooth J-curve in the class $[\sigma_{k_i}]$ for some integers $k_1, k_2, \dots k_n$, assuming that such curves exist. Here generic means that every J-curve in the class f intersects L_σ in at least $n-1$ distinct points. Set $L = L_f \cup L_\sigma$. Then set

$$X(m, n) = X \setminus L.$$

Theorem 1 *$X(m, n)$ is J-Kobayashi hyperbolic if either*

- *$n \geq 4$, and one of the following holds*
  1. *$g > 2$ or*
  2. *$g = 1$ and $m \geq 1$ or*
  3. *$g = 0$ and $m \geq 3$,*

  or
- *$n = 3$, X is the trivial bundle and all curves in L_σ represent the class of the trivial section $[\sigma_0]$.*

Theorem 2 *$X(m, n)$ is not J-Kobayashi hyperbolic if either*

- *$n < 4$ (unless $n = 3$, X is the trivial bundle and all curves in L_σ represent the class of the trivial section $[\sigma_0]$), or*
- *$g = 0$ and $m \leq 2$, or*
- *$g = 1$ and X is the trivial bundle and $m = 0$.*

We also have one theorem giving a criterion of the non-hyperbolicity of symplectic manifolds admitting a plurisubharmonic exhaustion.

Theorem 3 *Let* (M, J_0) *be a symplectic manifold, possibly with boundary. Suppose there exists a* J_0*-plurisubharmonic exhaustion* ψ *with uniformly bounded gradient with respect to the metric of the compatible triple* $(\omega = d\, d^{\mathbb{C}}\psi, J_0, g_0)$ *and so that the curvature is uniformly bounded.*

Then (M, ω, J) *is hyperbolic for any uniformly tamed* J *that is uniformly bounded w.r.t* g_0.

4 Open Questions

The results concerning the hyperbolicity of the complement of a divisor in a ruled surface are incomplete since the case of the non-trivial bundle over T^2 with no section removed is not addressed. Moreover, the theorem only applies where L is the union of distinct curves as described. It would be nice to extend this to the case where L is a general divisor in the class. But this is much harder and very little is known even in the complex category.

References

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