# Combinatorics and geometry in an unstable world

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# Some ways of generalising stability

- Simplicity: good theory; some complications; most important examples not covered; can perhaps be generalised further (NSOP).
- NIP: promising theory, though still very obscure; many important new examples.
- Rosy theories: good theory; does not seem to be a robust dividing line.
- Specialised definitions that do not pretend to be dividing lines: o-minimality, D-minimality, o-stability, metastability, measurable structures.

# Contents

- **Combinatorics – trying to classify classifications**
- Geometry – exploiting the lattice connection

## $\frac{1}{9}$ Combinatorics

"Combinatorial" properties are robust and give rise to good dividing lines; they often restrict the behaviour of indiscernible sequences.

Examples: Stability, superstability, simplicity, supersimplicity, NIP.

## $\frac{2}{9}$ Unifying principles

- Definition-mining Shelah's book.
- Counting types.
- Counting models.
- Interpretability.
- Properties of indiscernibles, of forking.

# $\frac{3}{9}$ Definition-mining Shelah's book

Shelah defined quite a few invariants $\kappa_{xyz}(T)$, for various values of $xyz$. Some of these were essentially ignored.
E.g. $\kappa_{inp}$:

- $\kappa_{inp}(T) < \infty$ iff $T$ does not have $TP_2$.
- $\kappa_{inp}(T) = \aleph_0$ for dependent $T$ iff $T$ is strongly dependent.
- $\kappa_{inp}(T) = \aleph_0$ for simple $T$ iff $T$ has finite weight.

**Definition**
$T$ is strong if $\kappa_{inp}(T) = \aleph_0$.

# Counting types

Recall the stability function:

$$g_T(\kappa) = \sup_{M \models T,\, |M| = \kappa} |S(M)| \,.$$

The six possible stability functions of a countable theory are (Keisler 1974):

$$\kappa, \quad \kappa + 2^{\aleph_0}, \quad \kappa^{\aleph_0},$$
$$\operatorname{ded}\kappa, \quad (\operatorname{ded}\kappa)^{\aleph_0}, \quad 2^{\kappa}.$$

This gives us a unifying principle for total transcendentality, superstability, stability, non-multiorder (possibly) and NIP.

# Counting more types

There is a more delicate version of the stability function:

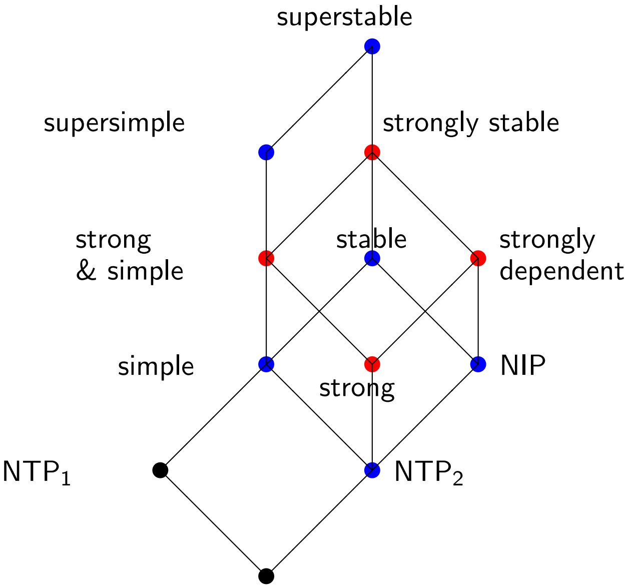
$$\mathrm{NT}_T(\kappa, \lambda) = \sup \left|\{A \mid A \text{ antichain of partial types with } \leq \kappa \text{ formulas over a set of size } \leq \lambda\}\right|.$$

This can also detect supersimplicity and simplicity, but the possible functions have not been described.

**Conjecture**

- There is a finite number of possibilities.
- $\mathrm{NTP}_2$ can also be detected in this way.

# A lattice of dividing lines



# Interpretability

" $T$ is interpretable in $T'$ " gives rise to a preorder on all theories.
A good dividing line defines a cut in this preorder.
For weaker variants of interpretability we get less, sometimes only finitely many, equivalence classes:
A subset $\mathcal{X} \subseteq \mathcal{P}(\omega)$ is <u>represented</u> in $T$ if there are $(\bar{b}_i)_{i<\omega}$ and $\varphi$ such that

$$\mathcal{X} = \big\{\{i < \omega \mid \varphi(\bar{a}, \bar{b}_i)\} \;\big|\; \bar{a} \in M\big\}.$$

Theories with IP are those that represent all subsets of $\mathcal{P}(\omega)$.
Stability can also be detected.
A variant where $(\bar{b}_i)_{i<\omega}$ must be an indiscernible sequence has only 3 equivalence classes: IP, unstable NIP, stable.

# $\frac{8}{9}$ Properties of indiscernibles/forking

Non-forking is bounded by a function $f$ if a type over a set of cardinality $\kappa$ has at most $f(\kappa)$ non-forking global extensions.
If non-forking is bounded, then it is bounded by $f(\kappa) = 2^{2^{\kappa}}$.

- $T$ is simple iff forking has local character.
- $T$ is stable iff non-forking is bounded and forking has local character.
- $T$ is dependent iff non-forking is bounded by $f(\kappa) = 2^{\kappa}$.

**Question**
Is $T$ dependent iff non-forking is bounded?
Chernikov and Kaplan: Yes, for $NTP_2$ theories.

# A speculation

**Conjecture**

1. $\exists$ a nice machinery for dependent (NIP) theories, similar to forking, weight and matroids in stable theories.
2. When defined correctly, it specialises to that for simple theories.
3. The right context of generality is $\mathrm{NTP}_2$.

# Contents

# $\frac{1}{9}$ Geometry

"Geometric" properties are in the spirit of lattice theorical properties. They are typically rather fragile and do not give rise to dividing lines. Examples: One-basedness, triviality, CM-triviality, local modularity, rosiness.

Often a "combinatorial" property must be assumed before a specific "geometric" property can even be defined. "Geometric" properties typically do not imply "combinatorial" properties.

When trying to connect "combinatorial" and "geometric" properties, hard issues such as elimination of hyperimaginaries can arise. These connections seem to be harder in the unstable case.

# $\frac{2}{9}$ Another speculation

**Conjecture**
Strong theories give rise to a generalised matroid (a greedoid?) that can help us understand the structure of models.

# $\frac{3}{9}$ Is geometric model theory model theory?

Some parts of model theory have remarkably close analogues in lattice theory. Perhaps they are better thought of as applied lattice theory?

The situation in lattice theory is simpler. This allows us to

- explore a toy problem before attacking the real one, and
- explain fairly advanced ideas to people from outside model theory.

# $\frac{4}{9}$ Strong minimality

### Definition

$T$ is strongly minimal if for all models

- algebraic closure is a matroid, and
- for all $n$, all independent $n$-tuples have the same type.

### Theorem

*If countable $T$ is strongly minimal, then $T$ is uncountably categorical and the cardinality of an uncountable model equals its dimension. The converse is morally true.*

# Independence for more general theories

To extend this dimension theory beyond uncountably categorical theories, we can:

1. drop or weaken the condition on independent *n*-types
2. generalise the notion of matroid, or
3. allow other closure operators instead of acl.

O-minimal theories are pregeometric, i.e. algebraic closure is a matroid. But they are of course not uncountably categorical, and this is an example of 1.

We will interpret a part of the machinery of stability theory as doing 2 in order to get 3.

# Semimodular lattices

### Definition (Wilcox)

$(A, B)$ is a modular pair if for all $C \in [A \wedge B, B]$ we have $(A \vee C) \wedge B = C$.
A lattice is semimodular (M-symmetric) if being a modular pair is a symmetric relation.

• The lattice of closed sets of a matroid is semimodular; the closures of elements are its atoms.
• A semimodular lattice that is generated by its atoms can be interpreted as a matroid.

# $\frac{7}{9}$ Independence in a semimodular lattice

$$A \perp_C B \iff (A \vee C) \wedge (C \vee B) = C, \text{ and } (A, B) \text{ is a modular pair}$$
$$\iff \text{for all } D \in [C, B \vee C]\text{: } (A \vee D) \wedge (B \vee C) = D.$$

## Fact

*In a matroid, $A \perp_C B$ is equivalent to the condition that every subset $A_0 \subset A$ which is independent over $C$ is also independent over $B \cup C$.*

*In an arbitrary semimodular lattice, $\perp$ still deserves the name 'independence'.*

# Weight

**Definition**
In a semimodular lattice, the weight of a lattice element $A$ is the maximal $n$ such that there is an independent sequence $B_0, B_1, \ldots, B_{n-1}$ with $A \not\leq B_0, \ldots, A \not\leq B_{n-1}$.

**Fact**
*The weight 1 elements of a semimodular lattice form a matroid under the following closure operator:*

$$\operatorname{cl} A = \{b \mid \exists A_0 \subseteq A\colon A_0 \text{ independent and } A_0 \not\leq b\}.$$

If there are ‘enough’ weight 1 elements, and if there is some control over the independent sets of weight 1 elements, then the matroid of weight 1 elements helps to understand the models of a theory. This is the case for superstable theories.

# Exploiting the connection to get fresh ideas

Can we define meaningful (non-symmetric) independence and weight in lattices that are not semimodular? Can we still get some kind of dimension theory for weight 1 types?

Instead of a matroid we may get something with a notion of independent sequences, depending on the ordering. E.g. a greedoid:

- The empty tuple is independent.
- If $\bar{a}\bar{b}$ is independent, then so is $\bar{a}$.
- If $\bar{a}$ and $\bar{b}$ are independent and $|\bar{a}| < |\bar{b}|$, then for some $x \in \bar{b}$, $\bar{a}x$ is independent.

Examples: Independent tuples in a matroid. Shelling sequences in a convex geometry.
Problem: It's not clear how to define infinite greedoids!