Absolutely connected groups

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Stability Theoretic Methods in Unstable Theories
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It is work in progress.

General references:

- "G-compactness and groups" L. Newelski, J.G., Archive for Mathematical Logic, 47 (2008), no. 5, p. 479-501
- A preliminary version of Ph.D. thesis at `www.math.uni.wroc.pl/~gismat`

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E.g. when G has NIP, ${G^*}^{\infty}$, ${G^*}^{00}$ and ${G^*}^{0}$ exist.

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Problem

Find a group G with

$$ {G^*}^{00}_A \neq {G^*}^\infty_A $$

for some small A.

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Lemma

$$G^{*\infty}_A = \left\langle \bigcap \{P \subseteq G^* : P \textit{ is } A\textit{-def. and thick } \} \right\rangle$$

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$$G - G = K.$$

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Moreover

$$(K^{*\times})_A^\infty - (K^{*\times})_A^\infty = K^*,$$

where K^* is a monster model of an arbitrary first order expansion of K and $A \subset K^*$ is small.

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If $(K^*,+)^{\infty}$ exists (e.g. K has NIP), then

$$(K^*,+)^{\infty} = K^*,$$

because then $(K^*,+)^{\infty}$ is an ideal in K^* (for $(K^*,+)^{00}$ it was noticed by A. Pillay).

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Now use description of the conjugacy classes: in 1. results of E. A. Bertram '73 and G. Moran '76; in 2. — V. A. Tolstykh '06; in 3. — A. Lev '96. □

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Proposition

Either $\forall N$, $\mathcal{C}_\infty \neq \mathcal{C}_N$ (so there is a group G with $G^{*\infty}_{\emptyset} \neq G^{*00}_{\emptyset}$) or there is an absolutely connected group an with infinite commutator width.

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Thank you for your attention