The Additive Collapse

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1. Starting Theories

T countable complete

M, N models

$\mathbb{C}$ monster model of T

$\langle X \rangle$ substructure generated by X

$\langle X \rangle^{\ell}$ linear hull

P(I) The models M of T are $\mathbb{F}_q$-vectorspaces with additional structure, where $\mathbb{F}_q$ is a finite field.
Furthermore we have a unary predicate $R(x)$ for a subspace of M. For all $M \models T$ we have $\langle R(M) \rangle = M$.

Mainly we consider finite subspaces A, B, C of $R(M)$. U, V, W are used for arbitrary subspaces of $R(M)$.

P(II) We have a pregeometry "$a \in \mathrm{cl}_d(A)$" on $R(M)$ and a notion "A is a strong subspace in $R(M)$" (short $A \leq M$). Both notions are invariant under automorphisms of $\mathbb{C}$. $\langle 0 \rangle^\ell \leq M$. For every B there exists a finite algebraic extension that is strong in M. Algebraic extensions of strong subspaces are strong. If M, N are models of T $A \subseteq R(M)$, $B \subseteq R(N)$, $\mathrm{tp}^M(A) = \mathrm{tp}^N(B)$ and a and b are geometrically independent of A and B respectively, then $\mathrm{tp}^M(a, A) = \mathrm{tp}^N(b, B)$. If furthermore $A \leq M$, then $\langle Aa \rangle^\ell \leq M$. $d(\mathbb{C})$ is infinite, where d is the geometrical dimension.

Let $B \subseteq R(M)$ be strong in M.

A is a minimal strong transcendental extension, if $A = \langle B, a\rangle^{\ell}$ and $a \notin \mathrm{cl}_d(B)$.

A is a minimal strong algebraic extension, if $A = \langle B, a\rangle^{\ell}$ and a is algebraic over B.

We extend the notions in P(II) to infinite subspaces U of $R(M)$ by the following definitions:

Definition $a \in \mathrm{cl}_d(U)$, if $a \in \mathrm{cl}_d(A)$ for some finite subspace A of U.

Definition $U \leq M$, if for every finite $B \subseteq U$ there is a finite $A \subseteq U$ with $B \subseteq A$ and $A \leq M$.

P(III) There is a set $\mathcal{X}$ of formulas $\varphi(\bar{x}, \bar{y})$ in L^{eq} such that $\varphi(\bar{x}, \bar{b})$ is either empty or strongly minimal. Furthermore $\varphi(\bar{x}, \bar{b}) \sim \varphi(\bar{x}, \bar{b}')$ implies $\bar{b} = \bar{b}'$. Length$(\bar{x}) = n_\varphi \geq 2$, $\varphi(\bar{x}, \bar{y})$ implies $x_i \in R$ and the linear independence of $x_1, \ldots, x_{n_\varphi}$. If $\bar{b}$ is in $\mathrm{dcl}^{\mathrm{eq}}(U)$ and $M \models \varphi(\bar{a}, \bar{b})$, then $\bar{a} \in \mathrm{cl}_d(U)$. If furthermore $U \leq M$, then either $\bar{a} \subseteq U$ or $\bar{a}$ is a generic solution over U. In the generic case $\langle U\bar{a}\rangle^{\ell} \leq M$. $\mathcal{X}$ is closed under affine transformations.

Let $B \subseteq R(M)$ be strong in M.

A is a minimal strong prealgebraic extension of B, if $A = \langle B, \bar{a} \rangle^{\ell}$ and $\bar{a}$ is a solution of some $\varphi(\bar{x}, \bar{b})$ in $\mathcal{X}$ generic over B with $b \in \mathrm{dcl}^{\mathrm{eq}}(B)$.

P(IV) If $A \leq M$, $B \leq M$, and $\langle A \rangle \cong \langle B \rangle$, then $\mathrm{tp}(A) = \mathrm{tp}(B)$.
If $B \leq M$, $A \leq M$ and $B \subseteq A \subseteq \mathrm{cl}_d(B)$, then there is a chain $B = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = A$ where $A_i \leq M$ and A_{i+1} is a minimal strong algebraic or prealgebraic extension of A_i.

$B = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = A$ above is a geometrical construction of A over B.

P(I) – P(IV) implies

- T is ω-stable.
- $R(x)$ is connected.
- tp(A) can be described by chains as in P(IV) using also minimal strong transcendental extensions.

Let $\mathop{\smile\!\!\!\!\!|}$ be the non-forking independence in T. Besides genericity of solutions $\bar{a}$ of $\varphi_\alpha(\bar{x}, \bar{b})$ we introduce $\mathop{\smile\!\!\!\!\!|}^w$-genericity for these solutions. If $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(B)$, then in the known examples $\mathop{\smile\!\!\!\!\!|}^w$-genericity of $\bar{a}$ over B means that $\bar{a}$ is linearly independent over B and $\delta(\bar{a}/B) = 0$.

P(V) Let $\varphi(\bar{x},\bar{y}) \in \mathcal{X}$, V a subspace of $R(M)$, and $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(V)$. Then the $\downfree$-generic type of $\varphi(\bar{x},\bar{b})$ over V is $\downfree^w$-generic over V and the $\downfree^w$-generics of $\varphi(\bar{x},\bar{b})$ over V have the same isomorphism type over V. They are $\downfree^w$-generic over every $U \subseteq V$ with $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(U)$. Furthermore if $\varphi(\bar{x},\bar{y}) \in \mathcal{X}$, $U \leq M$, $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(B)$, and $\bar{e}_0, \bar{e}_1, \ldots$ are solutions of $\varphi(\bar{x},\bar{b})$ linearly independent over B with $\bar{e}_i \not\subseteq \langle U, B, \bar{e}_0, \ldots, \bar{e}_{i-1}\rangle^{\ell}$, then there are at most l.dim(B/U) many i such that $\bar{e}_i$ is not $\downfree^w$-generic over $\langle U, B, \bar{e}_0, \ldots, \bar{e}_{i-1}\rangle^{\ell}$.

Let V be a strong subspace of M, $b \in \mathrm{dcl}^{\mathrm{eq}}(V)$ and $\varphi(\bar{x}, \bar{y}) \in \mathcal{X}$. In this case P(III) and P(V) imply that the solutions of $\varphi(\bar{x}, \bar{b})$ $\smile^{w}$-generic over V are exactly the $\smile$-generic solutions over V.

$\varphi(\bar{x}, \bar{b})$ defines a group set, if the generic type of $\varphi(\bar{x}, \bar{b})$ is the generic type of a definable subgroup.

$\varphi(\bar{x}, \bar{b})$ defines a torsor set, if the generic type of $\varphi(\bar{x}, \bar{b})$ is the generic type of a coset of a definable subgroup.

P(VI) Assume $C \supseteq B \subseteq A$ are strong subspaces of $R(M)$ linearly independent over B and both minimal strong extensions of B given by generic solutions of formulas in $\mathcal{X}$. If $b \in \mathrm{dcl}^{\mathrm{eq}}(E)$, $E \subseteq A + C$, and there is a solution $\bar{a}$ of some $\varphi(\bar{x}, \bar{b})$ in $\mathcal{X}$ $\downarrow^{w}$-generic over $C + E$ and over $A + E$, then $\varphi(\bar{x}, \bar{b})$ defines a torsor set. If it defines a group set, then $\bar{b}$ is in $\mathrm{dcl}^{\mathrm{eq}}(B)$.

P(VII) Either $M = R(M)$ and therefore connected,

or M is connected and there is a quantifier free formula $\theta(\bar{x}, y)$ in $\mathcal{X}$ such that for every $B \subseteq R(M)$ and every tuple $\bar{a}$ of geometrically independent generics over B in $R(M)$ $\quad M \models \theta(\bar{a}, b)$ implies that the canonical parameter b is a generic of M over B and $b \in \mathrm{dcl}(\bar{a})$,

or for every substructure $H \subseteq M \models T$ with $\mathrm{acl}(R(H)) \cap R(M) = R(H)$ and $\langle R(H) \rangle = H$ we have some quantifier free definable function $\eta(\bar{x}) = y$ such that

$$H = \{b : M \models \eta(\bar{a}) = b \text{ for some } \bar{a} \text{ in } R(H)\}.$$

2. Codes and Difference Sequences

Work in T^{eq}.

Replace $\mathcal{X}$ by a set of good codes C such that P(I) – P(VII) remain true and some additional properties are fulfilled.

If $U \leq V$ both strong in $\mathbb{C}$, and V is linearly generated over U by a generic solution of a formula $\varphi_\alpha(\bar{x}, \bar{b})$ in C, then $\varphi_\alpha(\bar{x}, \bar{y})$ and $\bar{b}$ are uniquely determined.

Let $\bar{e}_0, \ldots, \bar{e}_\lambda$, $\bar{f}$ be a initial segment of a Morley sequence of some $\varphi_\alpha(\bar{x}, \bar{b})$ in C.

We create a formula ψ_α such that

$$\mathbb{C} \models \psi_\alpha(\bar{e}_0 - \bar{f}, \ldots, \bar{e}_\lambda - \bar{f}) \text{ and}$$

ψ_α describes some important properties of the sequence above.
A realization of ψ_α is called a difference sequence.

There is some m_α such that m_α-many common solutions of $\varphi_\alpha(\bar{x}, \bar{c})$ and $\varphi_\alpha(\bar{x}, \bar{e})$ imply that $\varphi_\alpha(\bar{x}, \bar{c})$ and $\varphi_\alpha(\bar{x}, \bar{e})$ almost coincide.

$\mathbb{C} \models \psi_\alpha(\bar{e}_0, \ldots, \bar{e}_\lambda)$ implies:

There exists a unique $\bar{b}'$ such that $\mathbb{C} \models \varphi_\alpha(\bar{e}_i, \bar{b}')$ for all i and $\bar{b}' \in \mathrm{dcl}^{\mathrm{eq}}(\bar{e}_{i_1}, \ldots, \bar{e}_{i_{m_\alpha}})$ for all $i_1 < \ldots < i_{m_\alpha}$.

Furthermore:

$$\mathbb{C} \models \psi_\alpha(\bar{e}_0 - \bar{e}_i, \ldots, \bar{e}_{i-1} - \bar{e}_i, -\bar{e}_i, \bar{e}_{i+1} - \bar{e}_i, \ldots, \bar{e}_\lambda)$$

and ψ_α holds for every permutation of the e_i.

3. Amalgamation

We consider functions $\mu(\alpha) > \mu^*(\alpha)$ from the set of good codes into the natural numbers that allow the results of this chapter.

Definition Let $\mathbb{K}^\mu$ be the class of all strong subspaces U of $R(\mathbb{C})$, such that for every good code α there is no difference sequence for α of length $\mu(\alpha)+1$ in U. $\mathbb{K}^\mu_{\text{fin}}$ are the finite spaces in $\mathbb{K}^\mu$.

Lemma Let D be in $\mathbb{K}^\mu$ and $D \leq D'$ be a minimal strong extension. If D' has linear dimension one over D, then D' is in $\mathbb{K}^\mu$. Otherwise, in the prealgebraic case, D' is in $\mathbb{K}^\mu$ if and only if none of the following two conditions holds:

a) There is a code $\alpha \in C$ and a difference sequence $\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)}$ for α in D' such that

   i) $\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)-1}$ are contained in D.

   ii) $D' = \langle D\bar{e}_{\mu(\alpha)}\rangle^\ell$.

   iii) In this case α is the unique good code that describes D' over D.

b) There exists a code $\alpha \in C$ and a difference sequence for α in D' of length $\mu(\alpha)+1$ with canonical parameter $\bar{b}$ with a subsequence $\bar{e}_0, \ldots, \bar{e}_{\mu^*(\alpha)-1}$ such that $\bar{e}_i$ is $\perp^w$-generic over $D + \langle \bar{e}_0, \ldots, \bar{e}_{i-1}\rangle^\ell$.

Theorem Assume T satisfies P(I) – P(VI). The set $\mathbb{K}^{\mu}_{\mathrm{fin}}$ has the amalgamation property with respect to partial elementary maps.

Definition Let D be a subspace of $R(M)$. D is called rich if it is in $\mathbb{K}^\mu$ and if for every finite $B \subseteq A$ in K^μ with $B \leq M$ and $B \subseteq D$, there exists an A' with $B \subseteq A' \subseteq D$ and $\text{tp}(A'/B) = \text{tp}(A/B)$.

By P(II) $A' \leq \mathbb{C}$. We call a substructure V of $\mathbb{C}$ rich, if $\langle R(V) \rangle = V$ and $R(V)$ is rich.

Corollary There is a unique (up to automorphisms) countable rich subspace of $R(\mathbb{C})$.

L^μ is the extension of L by a unary predicate P^μ.

Definition We call an L^μ-structure $M = (M \upharpoonright L, P^\mu(M))$ rich, if $M \upharpoonright L \models T$, $P^\mu(M) \cap R(M) = R^\mu(M)$ is rich. $P^\mu(M) = \langle R^\mu(M) \rangle$ is defined by a L-formula χ, and $d(R(M)/R^\mu(M)) \geq \aleph_0$.

d is the geometrical dimension.

Lemma Let M be a rich L^μ-structure. Code-formulas have only finitely many solutions in $R^\mu(M)$.

Theorem Let M and N be rich L^μ-structures, $\bar{a} \in R^\mu(M)$ and $\bar{b} \in R^\mu(N)$. If $\mathrm{tp}^M \upharpoonright L(\bar{a}) = \mathrm{tp}^N \upharpoonright L(\bar{b})$, then $(M, \bar{a})$ and $(N, \bar{b})$ are $L^\mu_{\infty,\omega}$-equivalent.

Definition Let T^μ be the L^μ-theory of all rich L^μ-structures.

Corollary T^μ is complete.

4. Axiomatization of T^μ

T^μ 1) $M \upharpoonright L$ is a model of T.

T^μ 2) $\mathrm{acl}^L(R^\mu(M)) \cap R(M) = R^\mu(M)$, $P^\mu(M) = \langle R^\mu(M) \rangle$ described by χ. $d(R^\mu(M))$ and $d(R(M)/R^\mu(M))$ are infinite for ω-saturated models.

T^μ 3) $R^\mu(M)$ is in $\mathbb{K}^\mu$.

T^μ 4) If $\bar{b}$ is in $\mathrm{dcl}^{\mathrm{eq}}(R^\mu(M)))$ and $\bar{a}$ is a solution of $\varphi_\alpha(\bar{x}, \bar{b})$ in M generic over $R^\mu(M)$ for some code formula $\varphi_\alpha(\bar{x}, \bar{b})$, then $R^\mu(M) + \langle \bar{a} \rangle^\ell$ is not in K^μ.

Theorem An L^{μ}-structure M that satisfies T^{μ} 1), T^{μ} 2) and T^{μ} 3) is rich if and only if it is an ω-saturated model of T^{μ}.

Corollary

i) The deductive closure of $T^{\mu}1) - T^{\mu}4)$ is the complete theory T^{μ}.

ii) $R^{\mu}(x)$ is strongly minimal.

iii) $P^{\mu}(x)$ is of finite Morley rank.

iv) T^{μ} is ω-stable.

5. Reduction

Let T be a countable complete theory with P(I) – P(VII)

Definition Let $\Gamma(T^{\mu})$ be the L-theory of all $P^{\mu}(M)$ where $M \models T^{\mu}$.

Theorem $\Gamma(T^{\mu})$ is uncountably categorical. $R(x)$ is a strongly minimal formula in this theory. The pregeometry cl_d of $R(x)$ is given by acl.

Theorem Every subset of $P^{\mu}(M)^n$ L^{μ}-defined in M can be defined in the L-structure $P^{\mu}(M)$.

6. New uncountably categorical groups

M 2-nilpotent graded $\mathbb{F}_q$-Lie algebra

$M = M_1 \oplus M_2$ as $\mathbb{F}_q$-vectorspace

$[\,,\,]$ Lie multiplication

$[M_1, M_1] \subseteq M_2$, $[M_1, M_2] = 0$, $[M_2, M_2] = 0$

L vectorspace language in addition with

$[,]$, R_1 for M_1, R_2 for M_2

c constant

Free algebra $F(M_1)$ is given by
$(F(M_1))_2 = \Lambda^2 M_1$

$$\begin{array}{ccc} M_1 \times M_1 & \xrightarrow{\Lambda} & \Lambda^2 M_1 \\ & \searrow^{[\,,\,]} & \downarrow \gamma \\ & & M_2 \end{array}$$

γ vectorspace homomorphism

Let $N(M)$ be the kernel of γ.

Fact If H_1 is a subspace of M_1, then

$$H = \langle H_1 \rangle^M \cong F(H_1)/N(M) \cap \Lambda^2 H_1,$$

since there is a canonical embedding of $F(H_1)$ into $F(M_1)$.

Definition We define $\delta(H) = \mathrm{l.dim}(H_1) - \mathrm{l.dim}(N(H))$ where $N(H) = N(M) \cap \Lambda^2 H_1$.

Definition $B \leq U$ for $B \subseteq U \subseteq M_1$ (B is strong in U), if $\delta(B) \leq \delta(A)$ for all $B \subseteq A \subseteq U$.

Assumption We consider only M with $\langle 0 \rangle \leq M$.

That means $\delta(A) \geq 0$ for all A in M. Hence we can define

Definition $d(A) = \min\{\delta(B) : A \subseteq B \subseteq M\}$. $a \in \mathrm{cl}_d(A_1)$, if $d(A) = d(A \cup \{a\})$.

Lemma

i) $\delta(A+B) \leq \delta(A) + \delta(B) - \delta(A \cap B)$

ii) cl_d defines a pregeometry on subspaces of M_1 with dimension function d.

Let $\mathbb{K}$ be the class of all 2-nilpotent graded $\mathbb{F}_q$-Lie algebras M with $M = \langle M_1 \rangle$ and $c^M \in M_1 \setminus \{0\}$ such that

i) $[a, b] \neq 0$ for linearly independent a, b in M_1.

ii) $\langle 0 \rangle^\ell \leq M_1$ and $\langle c \rangle^\ell \leq M_1$.

Theorem

i) $\mathbb{K}$ has the amalgamation with respect to strong embeddings.

ii) If $B \subseteq U$ and $B \leq A$ for A, B, U in $\mathbb{K}$, then there is an amalgam D of $\langle A \rangle$ and $\langle U \rangle$ over $\langle B \rangle$ such that $U \leq D$.

Theorem There is a countable structure M_{FH} in $\mathbb{K}$ that satisfies the following condition:

(rich) If $B \leq A$ are in $\mathbb{K}$ and there is a strong embedding f of B in M_{FH}, then it is possible to extend f to a strong embedding $\bar{f}$ of A in M_{FH}.

M_{FH} is uniquely determined up to isomorphisms.

Definition A structure M in $\mathbb{K}$ that satisfies the condition (rich) is called a rich $\mathbb{K}$-structure.

Theorem Let M and N be rich $\mathbb{K}$-structures, $\langle\bar{a}\rangle \leq M$, $\langle\bar{b}\rangle \leq N$ and $\langle\bar{a}\rangle \cong \langle\bar{b}\rangle$. Then $(M, \bar{a}) \equiv_{L_{\infty,\omega}} (N, \bar{b})$.

By the above theorem all rich $\mathbb{K}$-structures have the same elementary theory T. To axiomatize T we write the following sets of L-sentences:

T 1) M is a 2-nilpotent graded $\mathbb{F}_q$-Lie algebra with $R_1(c) \wedge c \neq 0$.

T 2) $\forall xy \in R_1('' x$ and y are linearly independent'' $\rightarrow [x, y] \neq 0)$
$\forall xz \exists y (x \in R_1 \wedge x \neq 0 \wedge z \in R_2 \rightarrow [x, y] = z)$.

T 3) $\langle 0 \rangle \leq M$, $\langle c \rangle \leq M$.

T 4) If $B \subseteq M$ and $B \leq A$ are in $\mathbb{K}$, then there is an embedding of A in M.

Theorem

i) A rich $\mathbb{K}$-structure satisfies T 1)–T 4).

ii) Let M be a model of T 1), T 2) and T 3). Then M is a rich $\mathbb{K}$-structure if and only if M is a ω-saturated model of T 1)–T 4).

Theorem T is a theory that satisfies the conditions P(I)–P(VII).

Corollary T provides us uncountably categorical theories $\Gamma(T^{\mu})$ of Morley rank 2 where $R_1(x)$ is a strongly minimal set. By interpretation we get the corresponding theories of nilpotent groups of class 2 and exponent $p > 2$.