Perspectives on Expansions: Stability/ NIP

John T. Baldwin

February 9, 2009

SETTING
M is a structure for a language L, A is a subset of M.

$L^* = L(P)$ is the expansion of L by one unary predicate
(M, A) is the L^*-structure where P is interpreted by A.

When does (M, A) have the same stability class as M?

Goal and Acknowledgements

1. Describe 30 years of work in this area.

2. Pose many new questions -especially about unstable theories.

This account explictly relies on work by Adler, Baizhanov, Baldwin, Benedikt, Bouscaren, Casanovas, Poizat, Polskawa, Shelah, Ziegler.

Results and definitions are rephrased anachronistically for coherence of this presentation.

1 Background

PERSPECTIVES
I. Constructing Expansions:
What hath Hrushovski wrought?

II. Analysis of Arbitrary expansions

Vocabulary
locally: one formula at a time
uniformly: across all $L(P)$-structures
small: generalizes *belles paires*

General Program
Reduce the 'stability' of the pair (M, A) to the 'stability' of the theory "induced" on A.

FOUR FACTORS

1. stability class, simple, nip,
2. What kind of creature is A?
3. How does A 'sit in' M?
4. What structure does M 'induce' on A?

Creature?

A may be:

1. submodel
2. sequence/set of indiscernibles
3. arbitrary subset

By a routine translation we can transfer result about arbitrary subsets of M to arbitrary relations on M.

[BB04]

FCP (over A)

Definition 1 (Keisler; Casanovas-Ziegler)**.** We say M has the *finite cover property over* A if there is a formula $\phi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ such that for each $k < \omega$ there are a tuple $\mathbf{m} \in M$ and a family $(\boldsymbol{a}_i)_{i \in I}$ of tuples from A such that the set

$$\{\phi(\mathbf{x}, \boldsymbol{a}_i, \mathbf{m}) : i \in I\}$$

is k-consistent but not consistent.

The ordinary fcp arises when $A = M$.

Note that 'fcp over A' is preserved by $L(P)$-elementary equivalence.

nfcp is *strictly* stronger than eliminating there exists infinitely many [She78, CZ01]

Strength of FCP

NFCP implies stable.

NFCP over A does not.

SITS: small

Definition 2. M is ω-saturated over A, (A is *small* in M), if for every finite sequence $\boldsymbol{a} \in M - A$, every L-type $p \in S(\boldsymbol{a}A)$ is realized in M.

(M, A) is pseudosmall if $(M, A) \equiv (N, B)$ and (N, B) is small.

[BB00, BB04]

Beautiful pairs

Recall:

Definition 3 (Poizat)**.** (M, A) is a *belle pair* if

1. $A \prec M$
2. M is an $\aleph_1$-saturated L-structure
3. A is small in M.

[Poi83]

Canonical Example

The theory T of an equivalence relation with one class of size n for each finite n has fcp.

Note that if M is an $\aleph_1$-L-saturated model of T and P is defined so that:

1. all finite classes are contained in $P(M)$
2. every infinite equivalence class contains infinitely many elements in P and infinitely many elements not in P

then is M is not small. It omits:

$$\{\neg E(x, a) : a \in P(M)\}$$

Canonical Example: different expansion

Note that if M is an $\aleph_1$-L-saturated model of T and P is defined so that:

1. infinitely many finite classes are contained in $P(M)$;
2. infinitely many finite classes are half in and half out of $P(M)$;
3. infinitely many infinite classes that do not intersect $P(M)$.
4. infinitely many infinite equivalence class contains infinitely many elements in P and infinitely many elements not in P

then is (M, A) is small but not ω-saturated in $L(P)$.

It omits:

$$\{\exists^{\geq n} y E(x, y) \wedge E(x, y) \rightarrow P(y) : a \in P(M)\}$$

[BB04]

From Example to Theorem

Theorem 4 (BaldwinBaizhanov)**.** *If M has fcp over A, then (M, A) is not both small and ω_1-saturated in $L(P)$.*

Question 5. *ω-saturated in $L(P)$?*

[BB04]

INDUCE?

The basic formulas induced on A can be:

L^*: the traces on A of parameter free L-formulas (*induced structure*);

$L^{\#}$: the traces on A of parameter free $L(P)$-formulas ($L^{\#}$-*induced structure*, $A^{\#}$);

[BB04]

EXAMPLE: The notions of Induce are different

(Benedikt): Form a structure M with a two sorted universe:

1. The complex numbers.
2. A fibering over the complex numbers.

One sort contains the complex field, a binary relation E links the two with each field element indexing one member of a partition of the second sort into infinite sets.

[BB04]

EXAMPLE continued

Let N extend M by putting one new point in the fiber over a if and only if a is a real number.

Now M and N are isomorphic and are ω-stable nfcp. But the structure (N, M) is unstable.

EXAMPLE continued

The $*$-induced structure on M is stable since in fact no new sets are definable.

In the $\#$-induced structure

$$(\exists x)E(x, y) \wedge x \notin P$$

defines the reals so the $\#$-induced structure is unstable.

Moreover (N, M) is not *small*.

$\# = *$ **?**

Theorem 6 (Polkowska)**.** *If M is stable, (M, A) is small and M has nfcp over A then the $*$-induced and $\#$-induced theories on A are the same.*

[Pol05, CZ01]

We just saw that 'small' is necessary.

Question 7. *Is stable? What about the converse?*

SITS: benign

slogan: L-strong types over A determine $L(P)$ types over A.

Definition 8. 1. The set A is *weakly benign* in M if for every $\alpha, \beta \in M$ if:

$$\mathrm{stp}(\alpha/A) = \mathrm{stp}(\beta/A)$$

implies

$$\mathrm{tp}_*(\alpha/A) = \mathrm{tp}_*(\beta/A).$$

[BB04]

SITS: uniformly weakly benign

2. (M, A) is *uniformly weakly benign* if every (N, B) which is $L(P)$-elementarily equivalent to (M, A) is weakly benign.

Thus, this is a property of the theory T^*.

SITS: Locally homogeneous

The pair (M, A) is *locally homogeneous* if for every finite $\Delta \subseteq L$ and any α and β that realize the same L-type over A:

If a Δ-type

$$q(\mathbf{x}, \alpha, A)$$

is realized in M, so is

$$q(\mathbf{x}, \beta, A).$$

SITS: Uniformly locally homogeneous

The pair (M, A) is *locally homogeneous* if for every finite $\Delta \subseteq L$ there is a finite Δ' such that for any α and β that realize the same Δ'-type over A:

If a Δ-type

$$q(\mathbf{x}, \alpha, A)$$

is realized in M, so is

$$q(\mathbf{x}, \beta, A)$$

SITS: Dividing Lines

The following are preserved by $L(P)$-elementary equivalence:

1. (M, A) is uniformly weakly benign or *equivalently* uniformly locally homogeneous.

2. (M, A) has nfcp over A.

Question 9. *Is there a classification for theories with a predicate?*

[BB04, CZ01]

2 Stable Theories

SUFFICIENT CONDITIONS I

Theorem 10 (Poizat)**.** *If T is stable without fcp then the theory of 'belles paires' is complete, stable and nfpc.*

[Poi83]

SUFFICIENT CONDITIONS II

Theorem 11 (Baldwin-Benedikt)**.** *If M is stable and I is a set of indiscernibles so that (M, I) is small, then (M, I) is stable.*

[BB00]

On the hunting of mammoths

FCP (over A)

Definition 12. We say M has the *finite cover property over A* if there is a formula $\phi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ such that for each $k < \omega$ there are a tuple $\mathbf{m} \in M$ and a family $(\boldsymbol{a}_i)_{i \in I}$ of tuples from A such that the set

$$\{\phi(\mathbf{x}, \boldsymbol{a}_i, \mathbf{m}) : i \in I\}$$

is k-consistent but not consistent.

nfcp implies stability but nfcp over A implies only 'stability over A.
Note that 'fcp over A' is preserved by $L(P)$-elementary equivalence.
[CZ01]

WEAKER SUFFICIENT CONDITIONS
Explaining Baldwin-Benedikt and Poizat:

Theorem 13 (Casanovas-Ziegler)**.** *If M is stable, (M, A) has the nfcp (over A) and is small, and the $*$-induced theory on A is stable then (M, A) is stable.*

[CZ01]

STILL WEAKER SUFFICIENT CONDITIONS
Extending Casanovas-Ziegler,

Theorem 14 (Baizhanov-Baldwin)**.** *If (M, A) is uniformly weakly benign and M is stable then (M, A) has the same stability class as the $\#$-induced theory on A.*

[BB04]

What entails weakly benign?

Bouscaren showed (in our language):

Theorem 15. *If N is superstable and $M \prec N$, then (N, M) is uniformly weakly benign.*

[Bou89]

Baizhanov, Baldwin, Shelah showed:

Theorem 16. *If M is superstable (M, A) is uniformly weakly benign for any A.*

[BBS05]

SITS: benign vrs weakly benign

slogan: L-types over A determine $L(P)$ types over A.

Conclusion

Theorem 17. *If M is superstable and the #-induced theory on A is superstable then (M, A) is superstable.*

Contrasting Result

Theorem 18 (Bouscaren)**.** *Let T be a superstable theory. TFAE:*

1. *T has NDOP*
2. *All theories of pairs of T are stable.*
3. *All theories of pairs of T are superstable.*

[Bou89]

Why weakly benign?

Example 19 (BaizhanovBaldwinShelah). Let $E(x, y, z)$ index a pair of cross cutting equivalence relations:

$$E_a := E(x, y, a), E_b := E(x, y, b).$$

Let I be a set of elements which are pairwise equivalent under each equivalence relation. I intersects every E_a class and all but one E_b class.

Then, $\mathrm{tp}_L(a/I) = \mathrm{tp}_L(b/I)$
but
$\mathrm{tp}_{L(P)}(a/I) \neq \mathrm{tp}_{L(P)}(b/I)$.
[BBS05]

Major Question

Question 20. *If M is stable must (M, A) be uniformly weakly benign for any A?*

Indiscernibles Again

Question 21. *Is there a stable structure M and an infinite set of indiscernibles I such that I is not indiscernible in (M, I)?*

Question 22. *Is there a superstable structure M and an infinite set of indiscernibles I such that $I^{\#}$ is not superstable?*

[BB04]

Codimension

Definition. If I is an infinite set of indiscernibles in M such that for some infinite $J \subset M$, $I \cup J$ is a set of indiscernibles, we say I has *infinite codimension* (in M); otherwise I has finite codimension.

Local Saturation

Definition 23. (M, A) is *locally saturated* if for any $\mathbf{b} \in M$, for any L-formula $\phi(\mathbf{x}, \mathbf{y}, \mathbf{u})$, any $\phi(\mathbf{x}, \mathbf{y}, \mathbf{b})$-type over A is realized in M.

[BB04]

Characterizing local saturation

Theorem 24 (BaizahnovBaldwin)**.** *Suppose (M, I) is $L(P) - \omega$-saturated. The following are equivalent.*

1. *(M, I) is locally saturated*
2. *I has infinite codimension*
3. *I is small*

[BB]

Partial Success

Now applying the Baldwin-Benedikt main result:

Theorem 25. *If M is stable, I is an infinite set of indiscernibles in M with infinite codimension then (M, I) is stable.*

Finite Codimension Result

Theorem 26. *Let $I \subset M$ be an indiscernible set and M be stable. Suppose (M, I) is ω-saturated in $L(P)$. The following are equivalent:*

1. *I has finite codimension*
2. *For some ϕ, a canonically defined equivalence relation E_ϕ has less than N_ϕ classes that do not intersect I (and infinitely many classes on I).*

Finite Codimension Questions

Conjecture 27.

1. *Show for an appropriate notion of* nontrivial *that if I has finite codimension, forking is trivial on I.*
2. *Show (possibly using the triviality) that if (M, I) has finite codimension I is indiscernible in $L(P)$.*
3. *Show that if I has finite codimension (M, I) is weakly benign.*

Note that 2) and 3) yield (M, I) is stable by [BB04].

"Strong" Indiscernibility

Theorem 28 (Baldwin-Benedikt)**.** *Let T be a stable theory, M a model of T, and I a set of indiscernibles in M with (M, I) saturated and (M, I) pseudo-small. Then every permutation of I extends to an automorphism of M.*

This works for nip as well.

Question 29.

1. *Weaken pseudosmall.*
2. *replace (M, I) saturated by M saturated??*
[BB00]

3 Simple Theories

Simple Theories

Polkowska gave conditions, 'bounded PAC', on (M, A) so that:

Theorem 30 (Polkowska)**.** *If T is stable and bounded PAC:*

1. *$th_*(A)$ is simple;*
2. *if, further, T has nfcp and (M, A) is small, then (M, A) is simple.*

[Pol05]

Simple Conjecture

Question 31. *If M is simple (stable), (M, A) is small, and M has nfcp over A and $\mathrm{Th}^*(A)$ is simple must (M, A) be simple?*

Is small essential here?

4 NIP

NIP quantifier reduction

The next result and it's corollary are in L; there is no smallness hypothesis.

Theorem 32 (Baldwin-Benedikt)**.** *If M lacks IP and I is order-indiscernible with order type a complete dense linear order then for every L-formula $\phi(\vec{x}, \vec{y})$ there is a quantifier-free $<$-formula $\psi(\vec{w}, \vec{y})$ such that for every $\vec{m}$ there is a $\vec{c}_{\vec{m}} \in I$ such that*

$$\forall \vec{y} \in P[\psi(\vec{c}_{\vec{m}}, \vec{y}) \equiv \phi(\vec{m}, \vec{y})].$$

In particular, I is order-indiscernible in (M, I).

[BB00]

Another Formulation

Corollary 33 (Baldwin-Benedikt)**.** *If M lacks IP and I is a densely ordered sequence of order-indiscernibles then for every L-formula $\phi(\vec{x}, \vec{y})$ the trace of ϕ on $(I, <)$ is a disjoint union of convex sets. That is, the induced structure on $(I, <)$ is weakly o-minimal.*

Casanovas-Zeigler (stable) and Adler (nip) replace the 'chasing mammoths with stone-axes' proof given by Baldwin-Benedikt by clearer arguments.

[Adl08, CZ01]

$L(P)$**-consequence**

Theorem 34 (BaldwinBenedikt)**.** *If M lacks IP and (M, I) is pseudo-small then I is indiscernible in $L(P)$.*

NIP Questions: Assuming pseudosmall

Suppose M lacks IP and I is order-indiscernible with order type a complete dense linear order.

Does (M, I) have the independence property (even assuming pseudosmall)?

NIP Questions: without small

Question 35. *Suppose M is nip and $(I, <)$ is a set of order indiscernibles.*

1. *Does (M, I) have the independence property ?*
2. *Does infinite codimension imply small?*
3. *Does infinite codimension imply locally saturated? (Very likely)*
4. *Does infinite codimension imply I is order indiscernible in $L(P)$?*
5. *What do we know in the finite codimension case?*

Themes

How are sequences of indiscernibles like models?

Are smallness hypotheses necessary?

Can we extend to unstable theories?

Are stone axes enough?

References

References

[Adl08] H. Adler. Introduction to theories without the independence property. preprint, 2008.

[BB] B. Baizhanov and J.T. Baldwin. Naming an indiscernible set. in preparation: 2009.

[BB00] J.T. Baldwin and M. Benedikt. Stability theory, permutations of indiscernibles and embedded finite models. *Transactions of the American Mathematical Society*, 352:4937–4969, 2000.

[BB04] B. Baizhanov and J.T. Baldwin. Local homogeneity. *Journal of Symbolic Logic*, 69:1243–1260, 2004.

[BBS05] B. Baizhanov, J.T. Baldwin, and S. Shelah. Subsets of superstable structures are weakly benign. *Journal of Symbolic Logic*, 70:142–150, 2005.

[Bou89] E. Bouscaren. Dimensional order property and pairs of models. *Annals of Pure and Applied Logic*, 41:205–231, 1989.

[CZ01] E. Casanovas and M. Ziegler. Stable theories with a new predicate. *Journal of Symbolic Logic*, 66:1127–1140, 2001.

[Poi83] B. Poizat. Paires de structure stables. *The Journal of Symbolic Logic*, 48:239–249, 1983.

[Pol05] D. Polkowska. *Bounded Pseudoalbegraically closed structures*. PhD thesis, University of Illnois at Urbana-Champaign, 2005.

[She78] S. Shelah. *Classification Theory and the Number of Nonisomorphic Models*. North-Holland, 1978.