Entropy functions, information inequalities, and polymatroids

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Applications of Matroid Theory and Combinatorial Optimization

Banff, August 6, 2009

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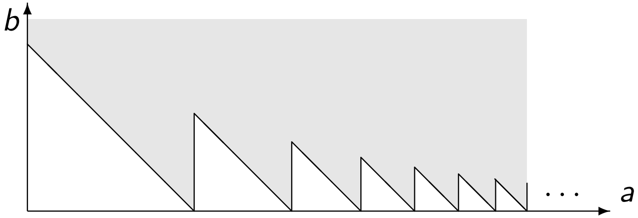

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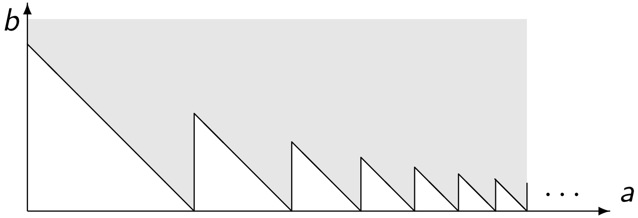

$a + b \geqslant \ln\lceil e^{a} \rceil$ (FM, Jan 2006, *Trans. IT IE*^{3})

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... H_N^{ent} and $cl(H_N^{\mathrm{ent}})$ differ only on the boundary of the cone

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$$H_N \supseteq cl(H_N^{\text{ent}}) \supseteq H_N^{\text{ent}} \supseteq ri(cl(H_N^{\text{ent}}))$$

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$t\, r \in H_N^{\mathrm{ent}}$ implies $t \in \{\ln d \colon d = 1, 2, \ldots\}$, and
$(\ln d)\, r = h_\xi$ implies that ξ_I takes $d^{r(I)}$ values
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… secret sharing matroids, almost affine codes, …

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A p-representation of $U_{2,4}$ of the degree $d = 10$ exists.

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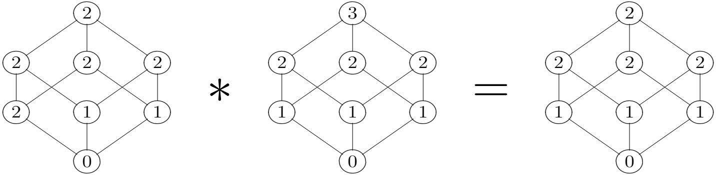

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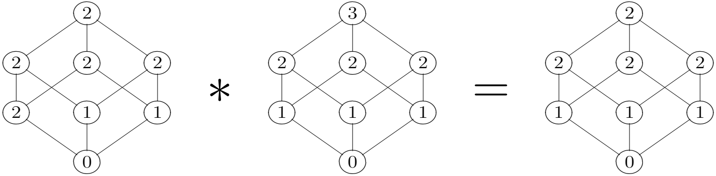

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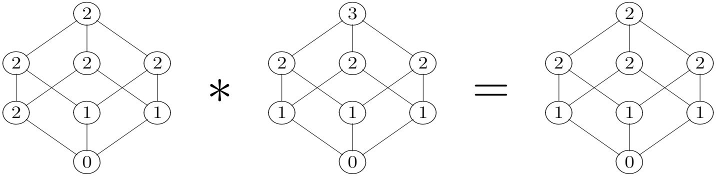

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[$cl(H_N^{\mathrm{ent}})$ is closed to the convolutions with modular polymatroids.]

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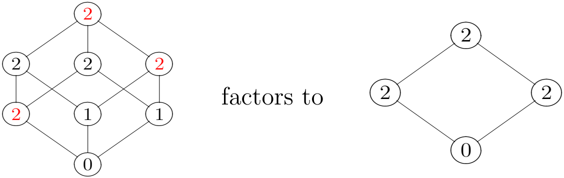

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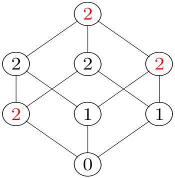

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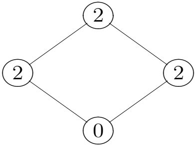

Every integer polymatroid is a factor of a matroid.

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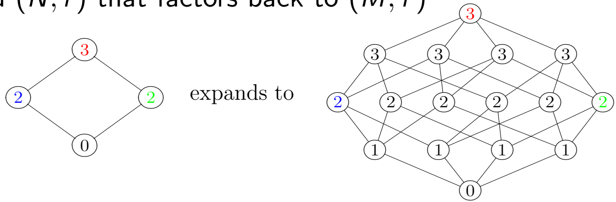

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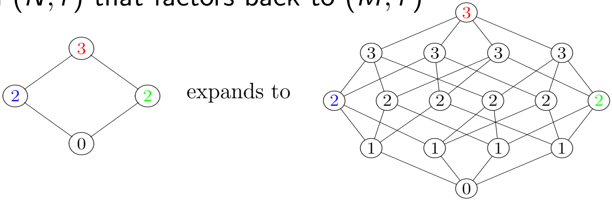

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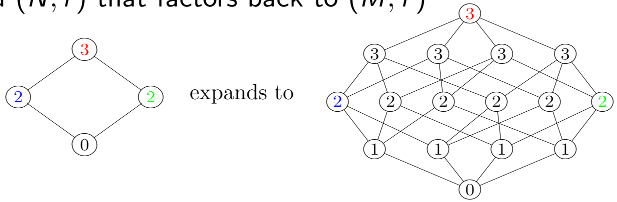

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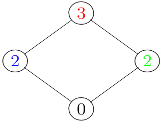

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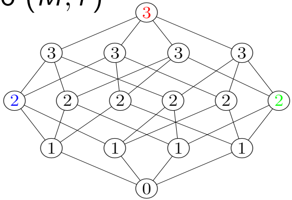

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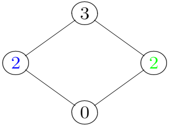

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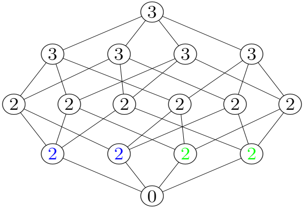

Expansion of an integer polymatroid (M, f)

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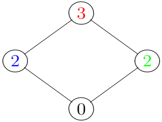

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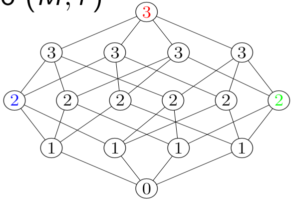
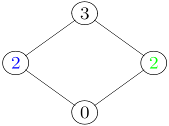
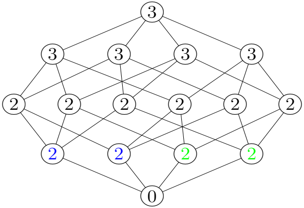


Free expansion can be constructed in two steps

1. make $f(m)$ parallel copies of each $m \in M$



parallelizes to



2. convolve with the free matroid $I \mapsto |I|$

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is a factor of an aent matroid. (FM, Jan 2007, *Trans. IT IE*^{3})

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The cone $cl(H_N^{\text{ent}})$ can be described in terms of aent matroids:
scalings of factors of aent matroids are dense $cl(H_N^{\text{ent}})$.

A point $c = (c_I)_{I \subseteq N}$ of $\mathbb{R}^{\mathcal{P}(N)}$ generates
a (linear unconditional) information inequality if
$\sum_{I \subseteq N} c_I \cdot g(I) \leqslant 0$ for the entropic points $g = (g(I))_{I \subseteq N}$.

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If $|N| \leqslant 3$ then $H_N = cl(H_N^{\text{ent}})$ whence all info inequalities are of Shannon type.

For the entropy functions $g = h_\xi$ over $N = \{i, j, k, l\}$

$$\begin{aligned} 3\big[g(ik) + g(il) + g(kl)\big] + g(jk) + g(jl) \\ \geqslant g(i) + 2\big[g(k) + g(l)\big] + g(ij) + 4g(ikl) + g(jkl) \end{aligned}$$

(Zhang & Yeung 1998, *Trans. IT IE*³)

i, j, k, l considered for singletons, the signs for union omitted

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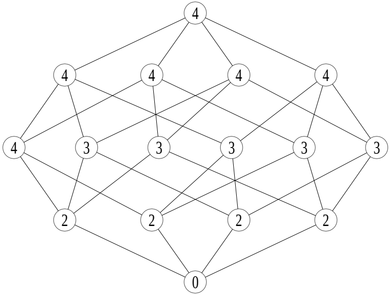

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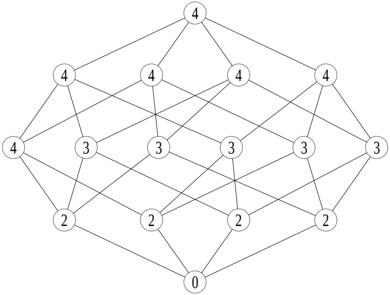
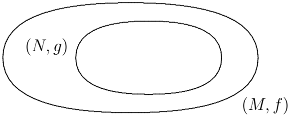


hence it is not of Shannon type and $cl(H_N^{\mathrm{ent}}) \subsetneq H_N$.


(N, g)
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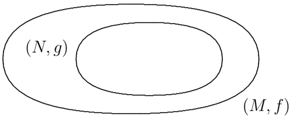
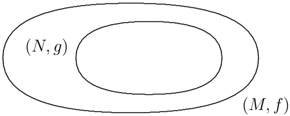


If $N \subseteq M$ and $g(I) = f(I)$ for $I \subseteq N$ then
g is the *restriction* of f and f is an *extension* of g.



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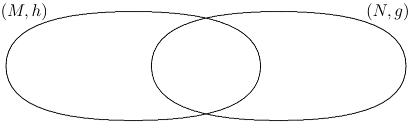

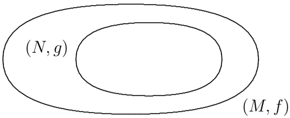

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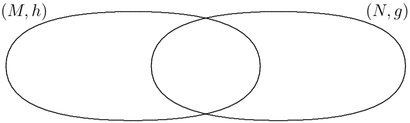
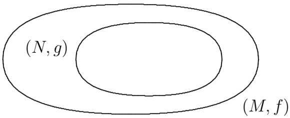


Polymatroids g and h are adhesive if
a polymatroid $(M \cup N, f)$ extends both and

$$f(M) + f(N) = f(M \cup N) + f(M \cap N)$$



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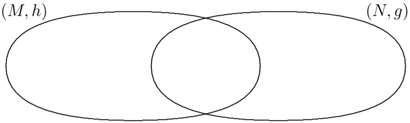

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(f ... an adhesive extension of g and h).

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if (N, g) and its *copy* (M, h) *along* I adhere.

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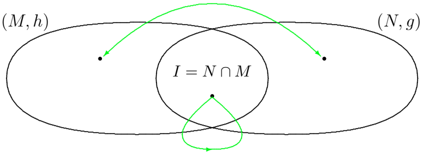

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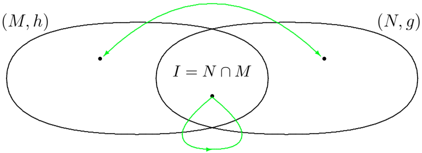

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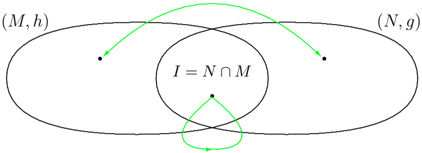

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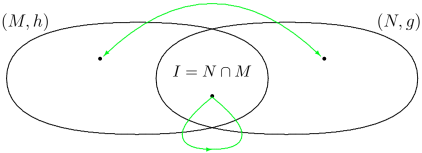

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(FM 2007, *Discrete Math*)

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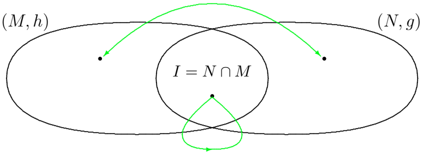

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The entropy functions are selfadhesive.

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$H_N^{\mathrm{ia}} = \bigcap_{s \geqslant 0} H_N^{\mathrm{ar}}(s)$ … the polymatroids with inner adhesivity.

$cl(H_N^{\mathrm{ent}}) \subseteq H_N^{\mathrm{ia}}$

… because two restrictions of an entropic polymatroid have an adhesive extension that is entropic.

For $N = \{1, 2, 3, 4, 5\}$ and $g \in H_N^{\mathrm{ar}}(s)$, $s \geqslant 1$,

$$s\left[\square_{12,34}\, g + \Delta_{34|5}\, g + \Delta_{45|3}\, g\right] + \Delta_{35|4}\, g + \tfrac{s(s-1)}{2}\left[\Delta_{24|3}\, g + \Delta_{34|2}\, g\right] \geqslant 0.$$

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where $\Delta_{34|5}\, g=g(35)+g(45)-g(5)-g(345)$

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where $\Delta_{34|5}\, g = g(35) + g(45) - g(5) - g(345)$

and $\square_{12,34}\, g = \Delta_{34|1}\, g + \Delta_{34|2}\, g + \Delta_{12|\emptyset}\, g - \Delta_{34|\emptyset}\, g$

For $N = \{1, 2, 3, 4, 5\}$ and $g \in H_N^{\mathrm{ar}}(s)$, $s \geqslant 1$,

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A proof is by induction on s, proving even three sequences of such inequalities simultaneously.

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Hence, all the inequalities hold for the almost entropic points.

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For $s \geqslant 1$ and the entropic function f of $(\xi_1, \xi_2, \xi_3, \xi_4)$

$$s \,\square_{12,34}\, f + \Delta_{24,34}\, f + \frac{s(s+1)}{2}\left[\Delta_{23,34}\, f + \Delta_{23,24}\, f\right] \geqslant 0$$

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In terms of mutual information,

$$s\big[I(\xi_3;\xi_4|\xi_1) + I(\xi_3;\xi_4|\xi_2) + I(\xi_1;\xi_2) - I(\xi_3;\xi_4)\big]$$
$$+I(\xi_2;\xi_3|\xi_4) + \tfrac{s(s+1)}{2}\big[I(\xi_2;\xi_4|\xi_3) + I(\xi_3;\xi_4|\xi_2)\big] \geqslant 0$$

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$s = 1$: ZY inequality.

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$s = 1$: ZY inequality.

$s = 2$: $2\Box_{12,34}\, f + \Delta_{24,34}\, f + 3\Delta_{23,34}\, f + 3\Delta_{23,24}\, f \geqslant 0$

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(Dougherty, Freiling & Zeger, *ISIT 2006*)

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The latter sequence of inequalities is used to arrive at contradiction with a geometrical lemma.

If $C \subseteq \mathbb{R}^d$ is polyhedral and a curve $\varphi\colon [0,1] \to C$ has a tangent $\dot{\varphi}(0_+)$ then C contains the segment with endpoints $\varphi(0)$ and $\varphi(0) + \varepsilon\dot{\varphi}(0_+)$ for some $\varepsilon > 0$.

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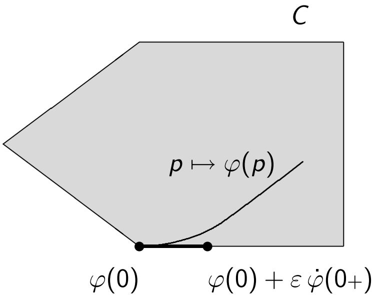

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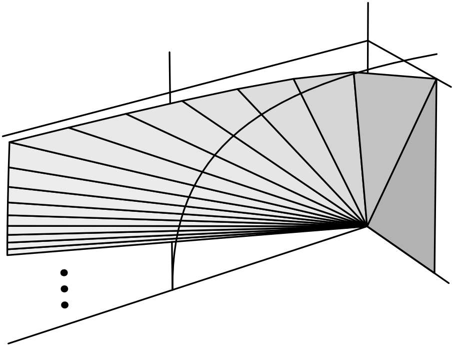
and r_1^{14}, r_1^{23}, r_2^{4} are linear matroids.



A projection to $\mathbb{R}^3$ of the halfspaces given by the new information inequalities and the curve $p \mapsto \varphi(p)$.

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