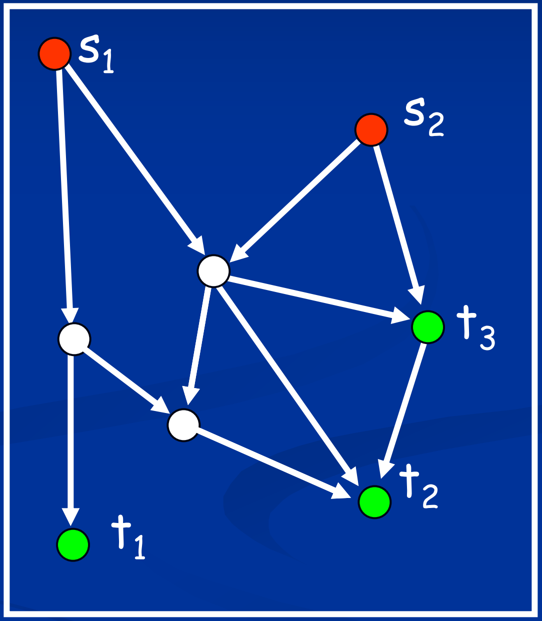
Algorithmic Complexity of Network Coding

Michael Langberg
Open University of Israel

General Network Coding

- Directed (acyclic) graph G=(V,E).
  - Assume unit capacity lossless edges.
- Source vertices S.
- Terminal vertices T.
- Requirement:
  - Transfer information from S to T.



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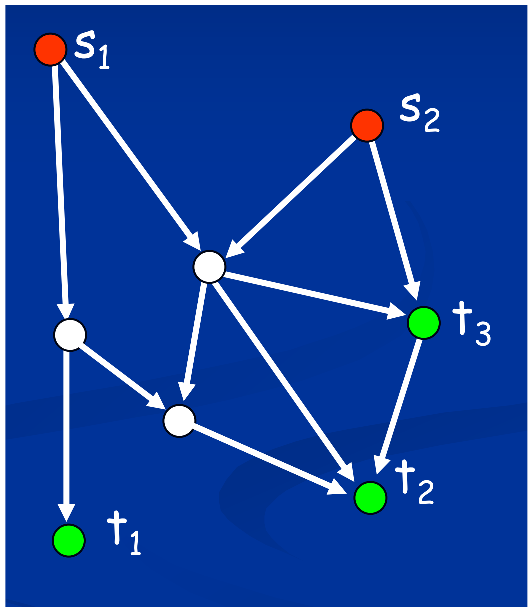

G=(V,E)

- Objectives:
  - Decide if given requirements are feasible using NC.
  - Design information flow using Network Coding.
  - Coding functions may be (Scalar/Vector) Linear or General.

- Multicast:
  - Single source, all terminals want all information at source.
  - Well understood.
  - Efficient solutions.
- General NC:
  - Not well understood.
- This talk:
  - Present evidence that the general problem is "hard".
  - Focus on "negative" results.

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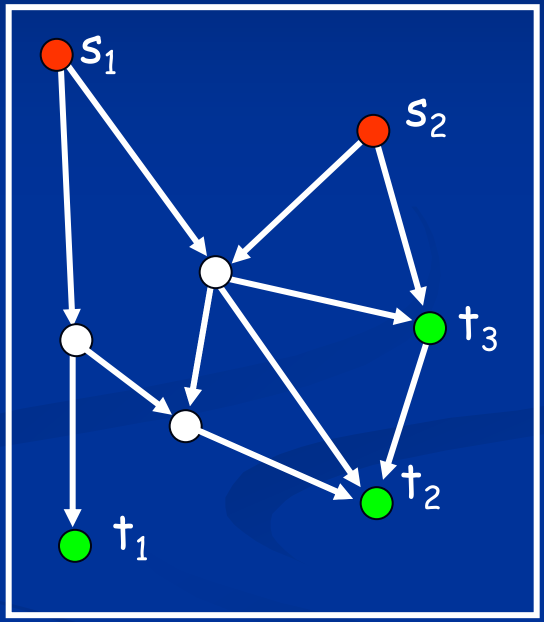

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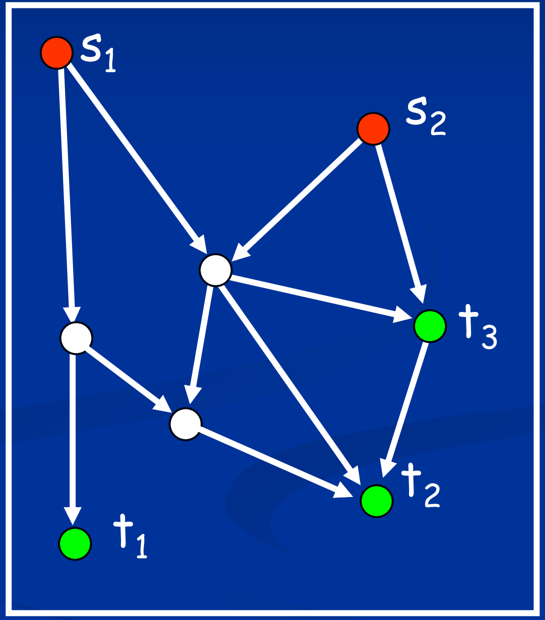

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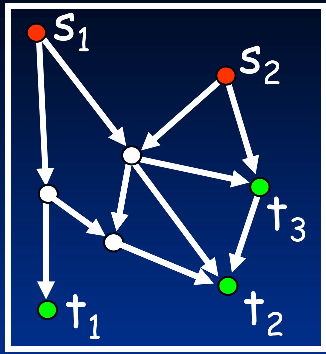
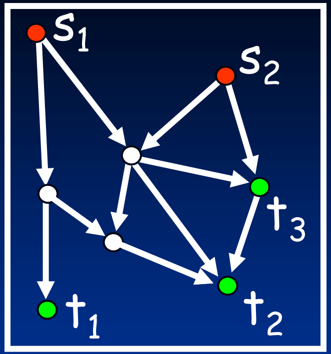
Scalar Linear Capacity



e.g. [Dougherty Freiling Zeger].

- General acyclic NC instance.
- Requirement matrix: $R=[r_{ij}]$
  - $r_{ij} \in \{0,1\}$, $r_{ij}=1 \Leftrightarrow$ terminal t_j wants info. of source s_i.
- Alphabet Σ of size q (here Σ will be a field).
- Scalar Linear: Each source has single character from Σ.
- Objective: design linear network code which is feasible.

Scalar Linear Capacity



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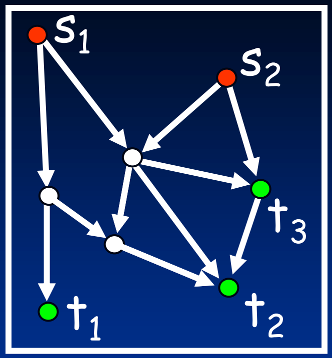
- Scalar Linear: Each source has single character from Σ.
- Objective: design linear network code which is feasible.
- A Network Code enables communication at rate $1/m$:
  - Each link can carry m characters from Σ during transmission.

Capacity: C = maximum rate feasible linear NC.

Problem



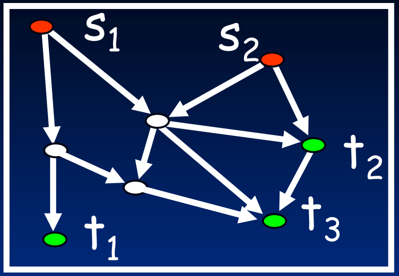
Input:

- Instance to the NC problem ($G,S,T,R=[r_{ij}]$).
- Desired communication rate r.

Output:

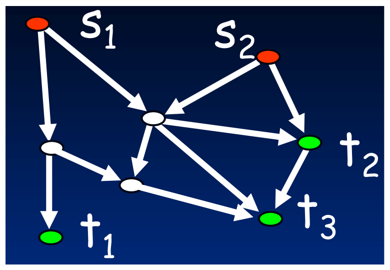
- Decision: Is the instance feasible (scalar linear).
- Construction: If so find corresponding NC.

Scalar Linear Coding



Q: Given an instance G with requirements $R=[r_{ij}]$, can one determine if instance has scalar linear capacity of 1.

Scalar Linear Coding



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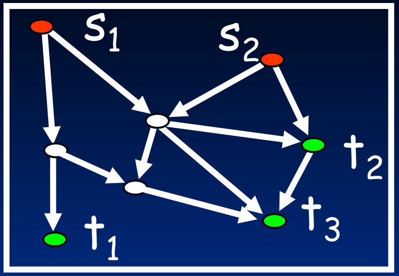
Scalar Linear Coding



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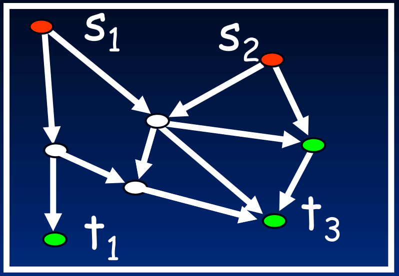
Proof Technique by reduction:

Show that solving the problem at hand efficiently will enable the efficient solution of a "hard" problem.

- Instance to hard problem $\Rightarrow$ Network Coding instance
- Solution to hard problem $\Leftarrow$ Solution to NC problem

Proof Technique by reduction:

Show that solving the problem at hand will solve a problem considered to be hard.



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A: "No" not efficiently [Lehman Lehman].

- NP-hard to determine scalar linear feasibility ($C=1$).
  - We are not even asking to find a network code!!
- Reduction from the 3-SAT problem.
- 3-SAT: given 3-CNF formula, determine if satisfiable.

$$(x_1 \cup x_2 \cup x_3) \cap (x_4 \cup \bar{x}_2 \cup x_7) \cap \ldots \cap (\bar{x}_1 \cup \bar{x}_5 \cup x_6) \cap (\bar{x}_4 \cup x_9 \cup \bar{x}_5)$$

- 3-SAT is a classical NP-Complete problem.

Proof Technique by reduction:

- Instance to hard problem $\Rightarrow$ Network Coding instance
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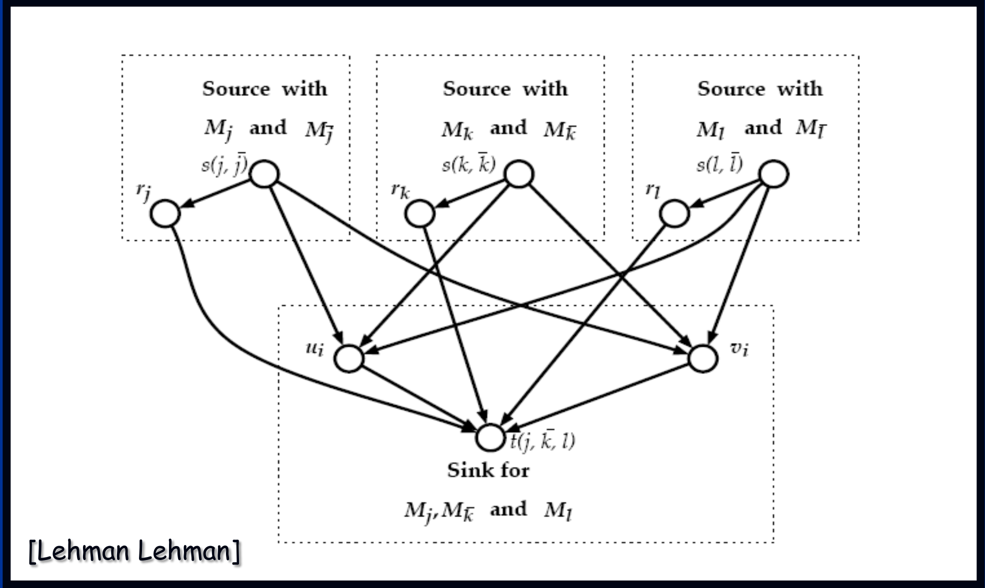
- Given 3-SAT instance φ [Lehman Lehman] construct network coding instance (G,R) such that:
  - Associate 2 sources with each variable corr. to TRUE and FALSE.
  - Single terminal with each clause.
  - With each clause associate a subgraph and terminal requirements.
  - For $(x_j \cup \bar{x}_k \cup x_l)$

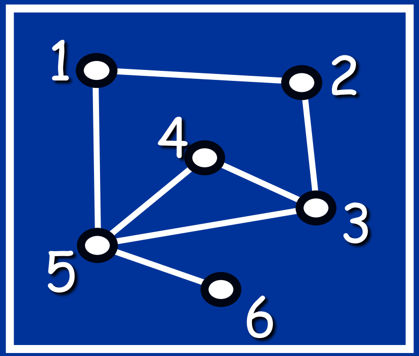


- Reduction works: φ is satisfiable iff (G,R) is feasible.

Index Coding

- Special instances of the general NC problem.
- Instances defined by an undirected base graph *G*.
- Combinatorial properties of *G* now pass on to the NC instance.
- Reduction, enables to establish hardness results via standard and well know combinatorial problems.

G



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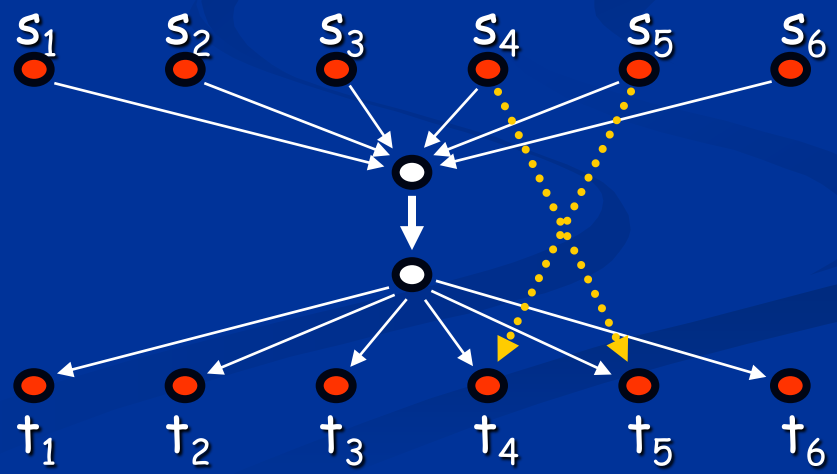

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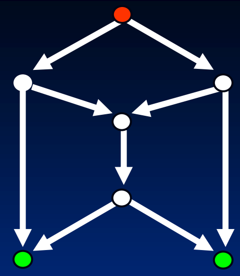

- Very simple construction – generalizes "butterfly".
- Index Coding appears in study of "source coding with side information" [Bar-Yossef et al][Lubetzky Stav][Alon et al][Yazdi et al][L S][Rouayheb S Georghiades] …

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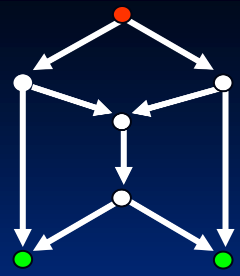

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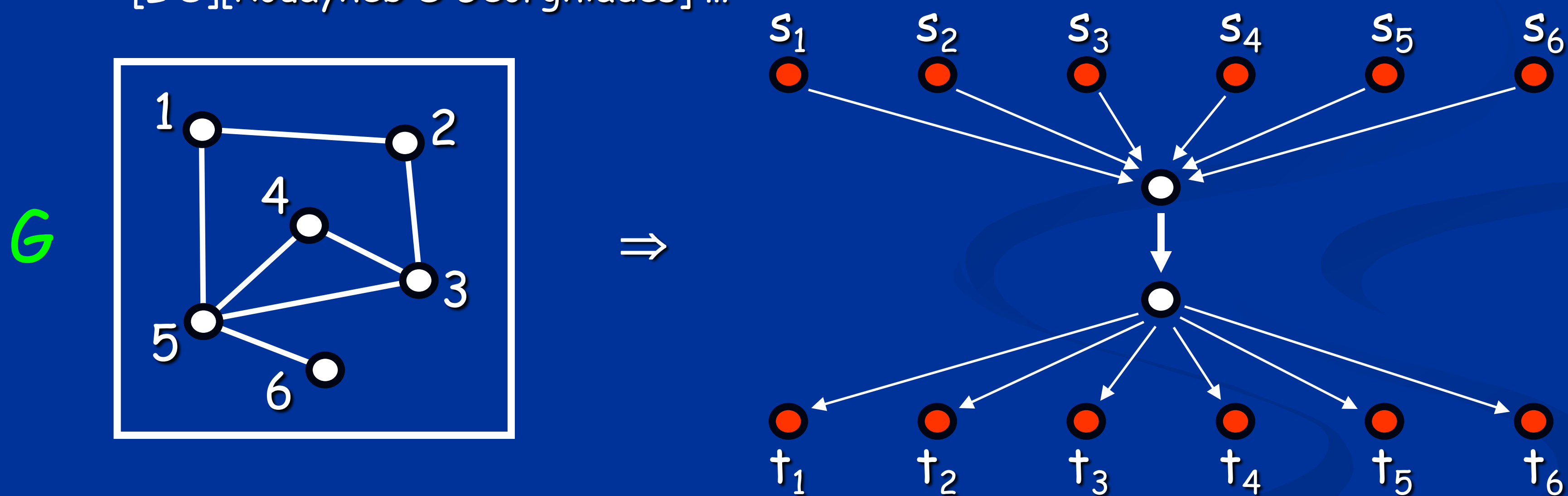

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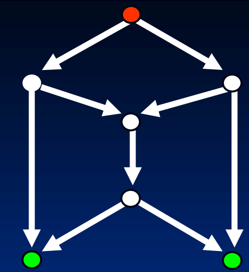
- Requirements: t_i wants information from s_i only.

Index Coding



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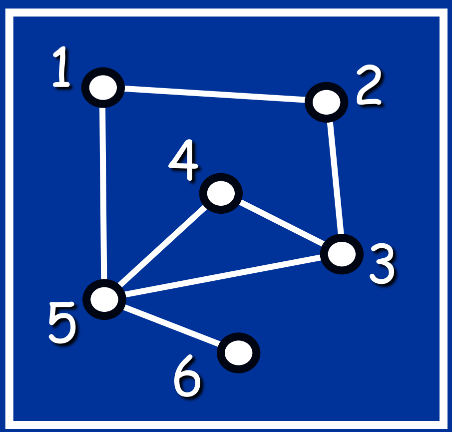

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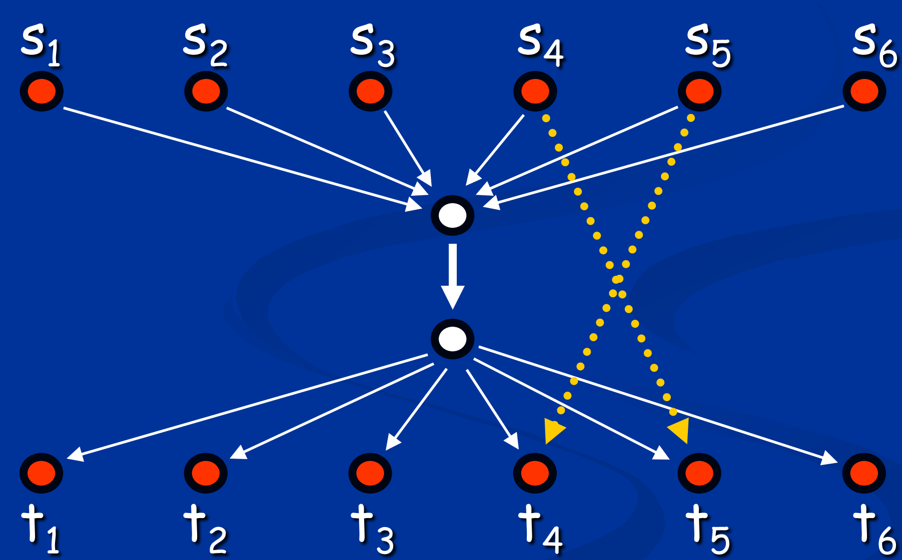
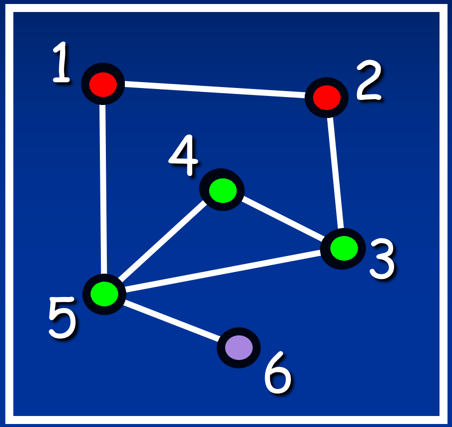
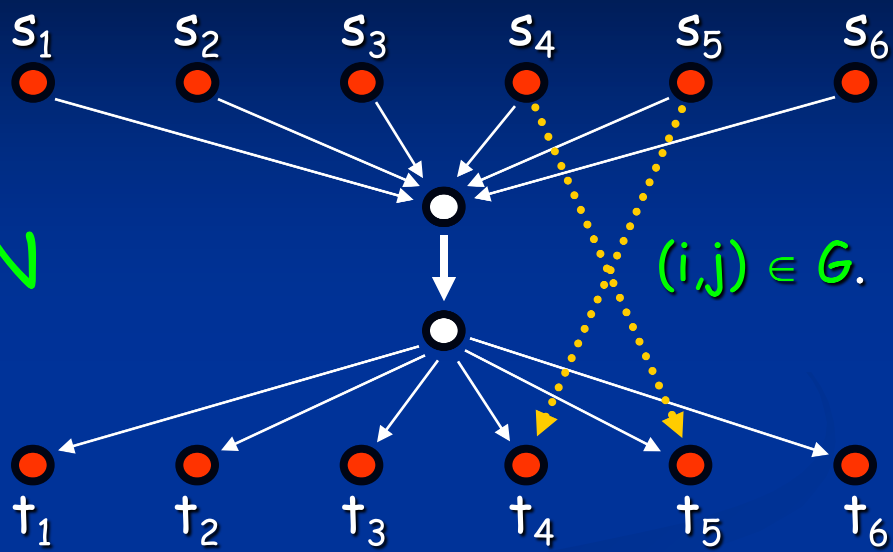


- Source/terminal for each vertex of G.
- Requirements: t_i wants information from s_i only.
- Side information: s_i connected to t_j ⇔ (i,j) edge in G.

Reduction

G 

⇒ N 

Many beautiful connections between G and network N.

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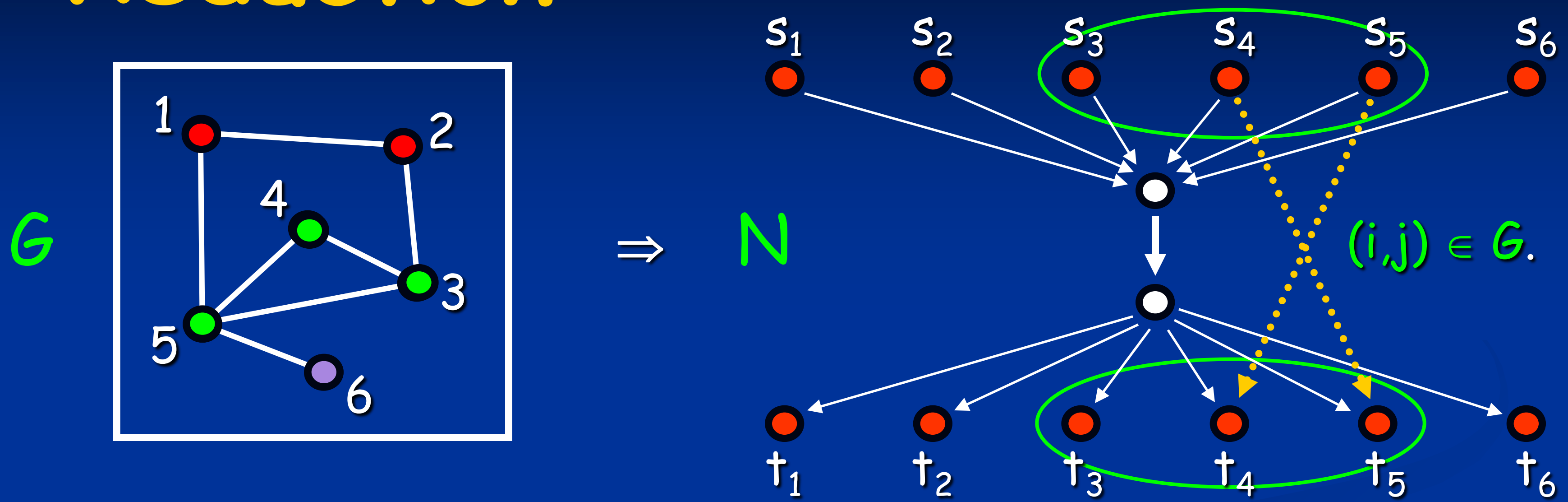

Many beautiful connections between G and network N.

- Simple NC for (s,t) pairs corresponding to any clique in G:
- Consider clique (3,4,5) and pairs (s_3,t_3), (s_4,t_4), (s_5,t_5).

Reduction

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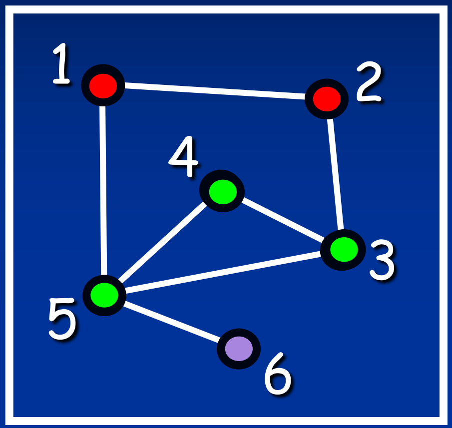

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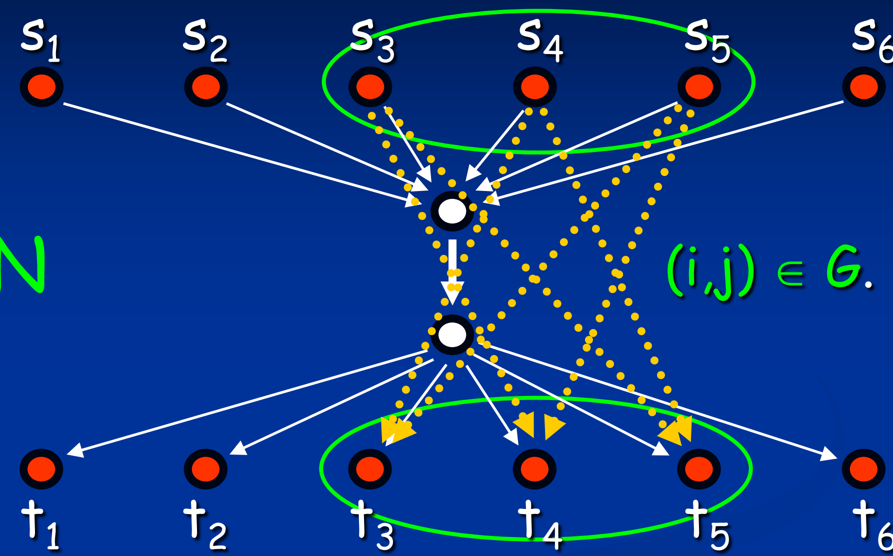

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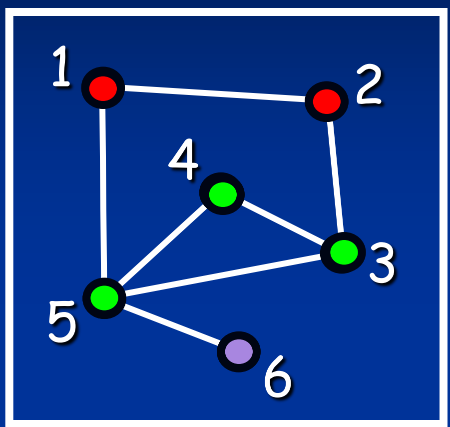
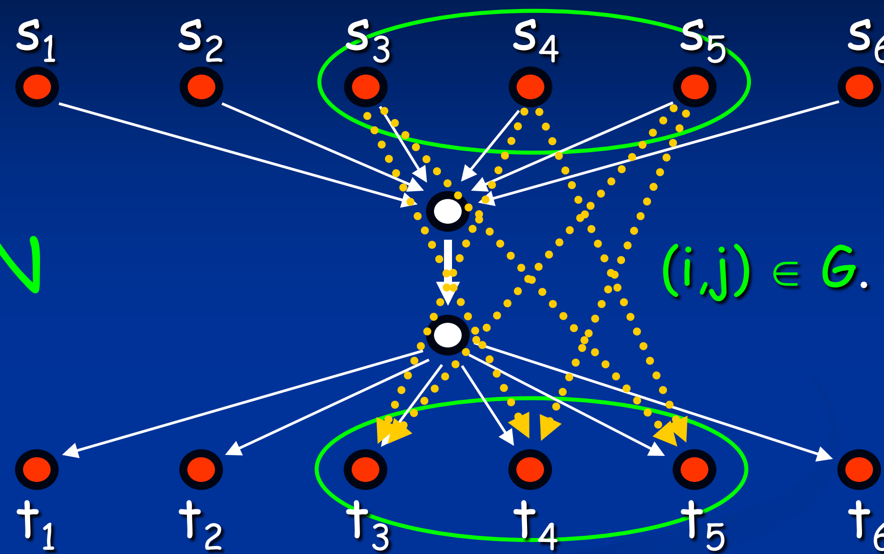
G 

$\Rightarrow$ N 

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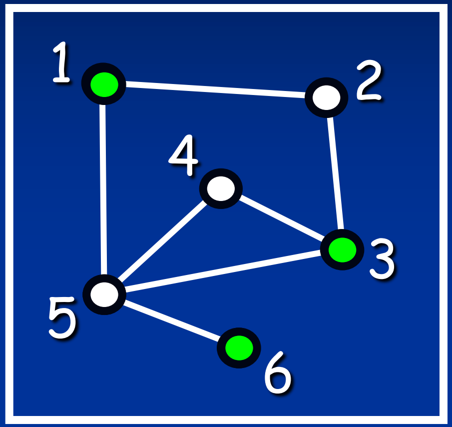
  - t_3 gets $s_4,\ s_5,\ s_3 \oplus s_4 \oplus s_5 \Rightarrow$ can deduce s_3! (same for t_4,t_5).
- Can do the same with each clique of G (using time sharing).
- This example: scalar linear capacity $\geq 1/3$.

Sandwich

G



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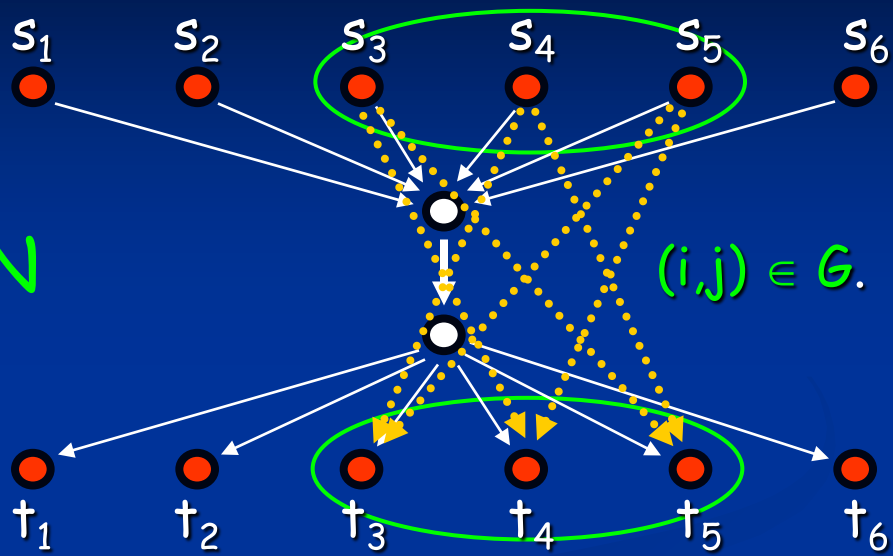
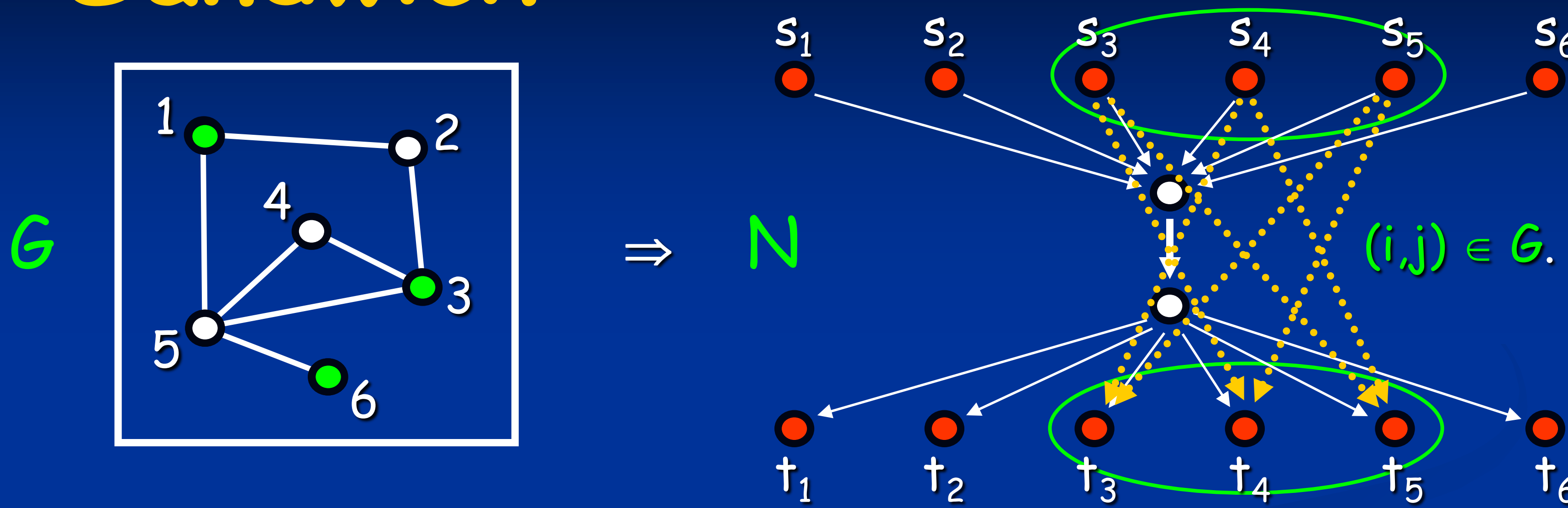


Sandwich

Clique cover: a cover of the vertices of G with cliques.

Independent set: a set of vertices of G with no edges.



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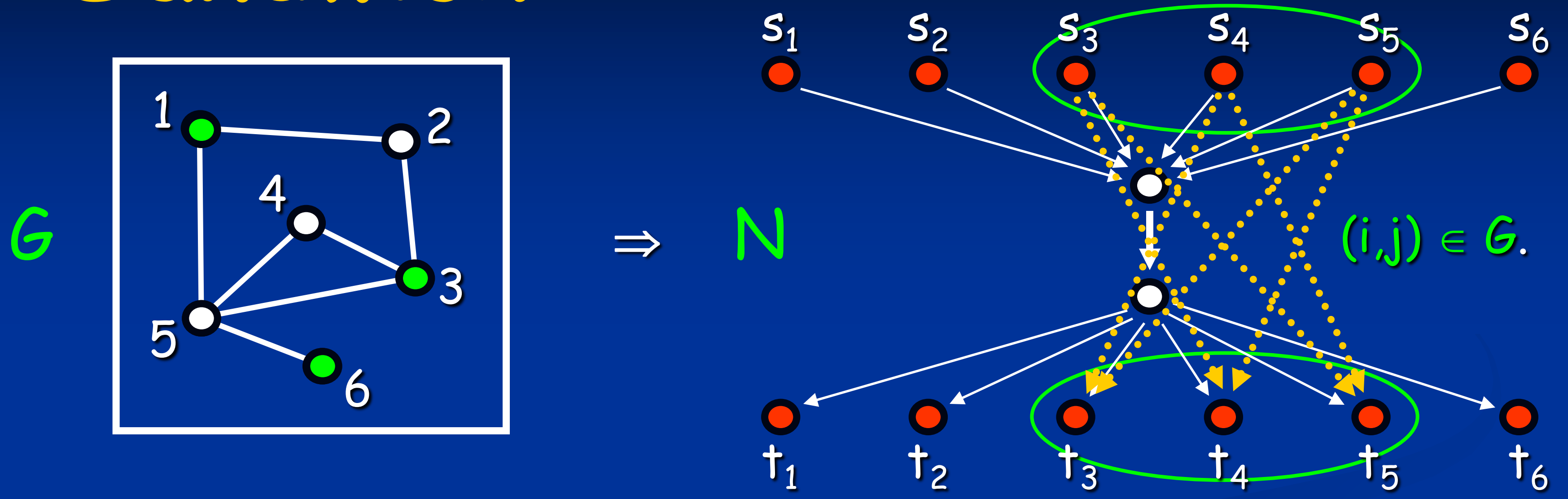
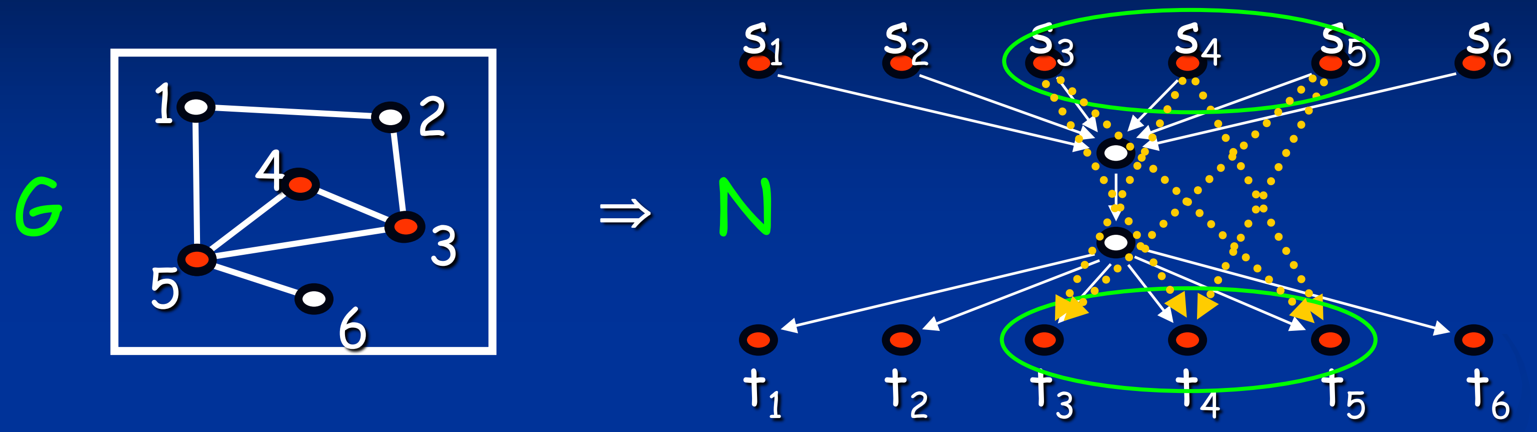


- Let ω be the size of a "clique cover" of G.
- Let α be the size of an "independent set" in G.
- We just saw: Scalar Linear capacity $\geq 1/\omega$.
- Also holds that: Scalar Linear capacity $\leq 1/\alpha$ [Haemers,Peeters].

$$1/\omega \leq C \leq 1/\alpha$$

Proof II for Scalar Linear



Proof Technique by reduction:

- Instance to hard problem ⇒ Network Coding instance
- Solution to hard problem ⇐ Solution to NC problem

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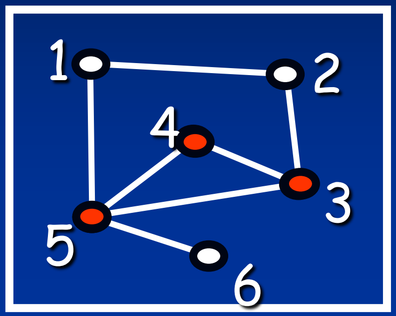

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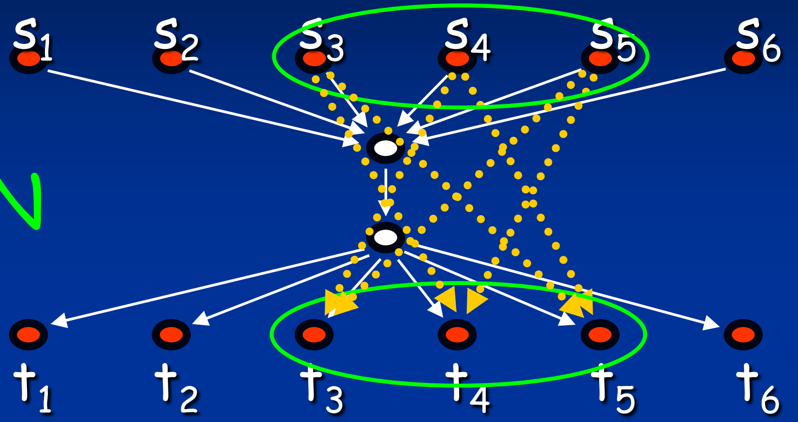

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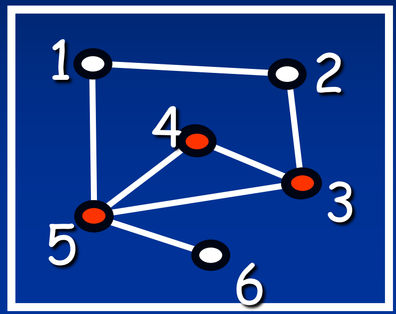
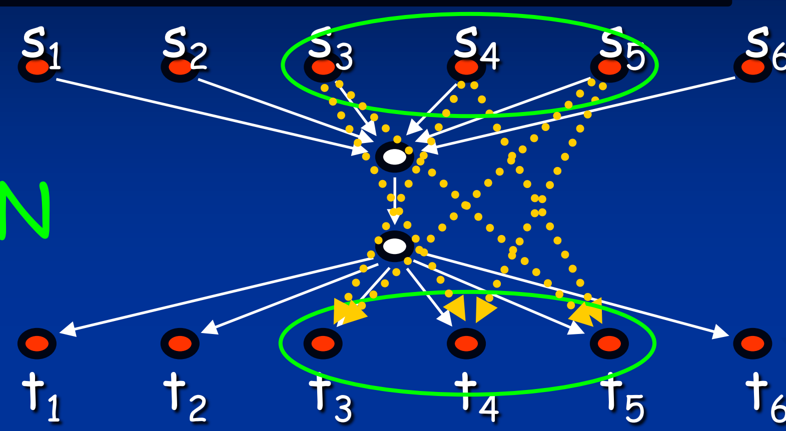
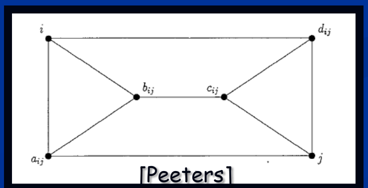
G 

⇒ N 

- Known: NP-hard to decide if G has clique cover of size 3 [Gary et al.].
- In the 90's [Peeters] presented a reduction taking any G to a graph f(G) such that
  - G has clique cover of size 3 iff f(G) does.
  - f(G) has clique cover of size 3 iff Index Coding instance for f(G) has Scalar Linear capacity 1/3.
- "Gadgets" in proof:



[Peeters]

i	*	0	0	*	0	0
a_{ij}	0	*	0	0	*	0
b_{ij}	0	0	*	0	0	*
c_{ij}	*	0	0	*	0	0
d_{ij}	0	*	0	0	*	0
j	0	0	*	0	0	*

i	*	0	0	0	0	*
a_{ij}	0	*	0	*	0	0
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c_{ij}	0	*	0	*	0	0
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j	*	0	0	0	0	*

[Peeters]

Proof Technique by reduction:

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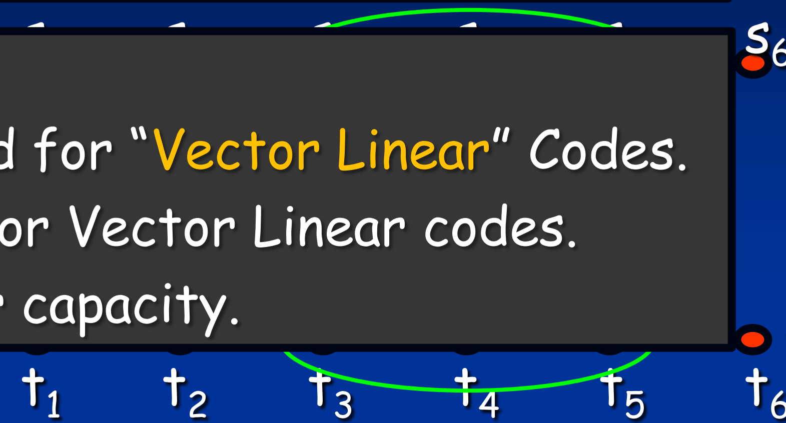
- Solution to hard problem ⇐ Solution to NC problem



- Remark:
  - Both reductions do not hold for "Vector Linear" Codes.
  - Later: will show hardness for Vector Linear codes.

Now: estimation of Scalar Linear capacity.

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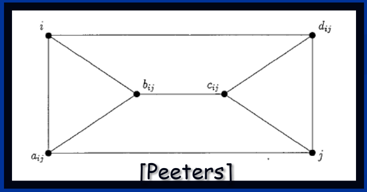
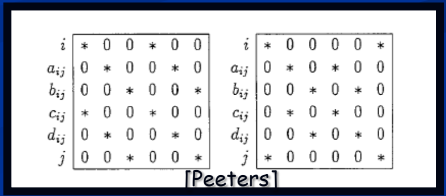
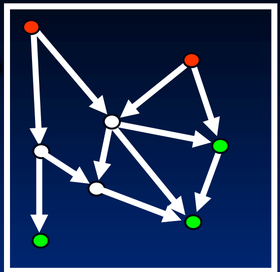




What about approximately finding capacity?



- Up to now: Finding Scalar-Linear NC that obtains capacity is NP-hard.
- Question: Is it easy to find a Scalar Linear NC that enables communication at rate 50% the capacity?

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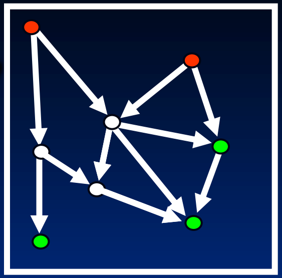

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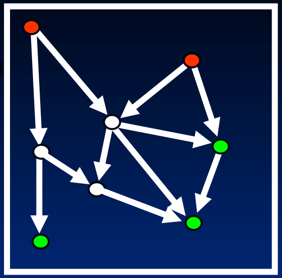

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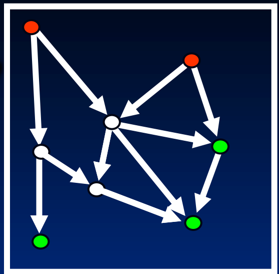


- Up to now: Finding Scalar-Linear NC that obtains capacity is NP-hard.
- Question: Is it easy to find a Scalar Linear NC that enables communication at rate 0.001% the capacity?
- NO! "Hard" to find a Scalar Linear NC that enables communication within any constant factor of capacity.
- Main idea: Use Index Coding and connection to the clique cover [LSprintson].
  - Previous two constructions do not extend when trying to find NC that approximately meet capacity.

Structure

Will show: "Hard" to find a Network Code that enables communication within any constant factor of capacity.

Proof Structure (reduction):

Polynomial algorithm for approximately solving the NC problem $\Rightarrow$ Poly. alg. for long standing open problem in field of Graph Coloring.

Plan:

- Present "Hard" problem in Graph Coloring.
- Sketch proof (correctness of reduction).

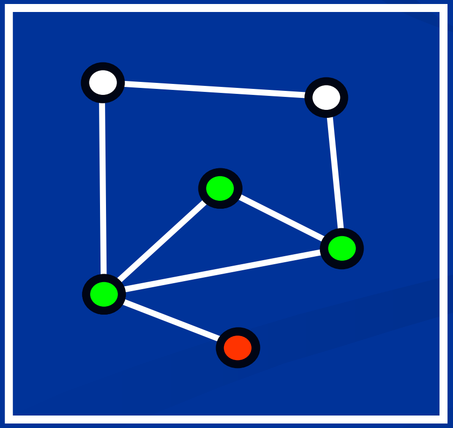
What is the "hard" problem?

A k-clique cover of G is an assignment of k colors to vertices of G such that each color class is a clique.

- $\omega(G)$ the minimum k such that G has k-clique cover.

Question: Can one efficiently find a minimum clique cover of G?
Answer: NO! Problem NP-Hard (implies solving SAT).

G



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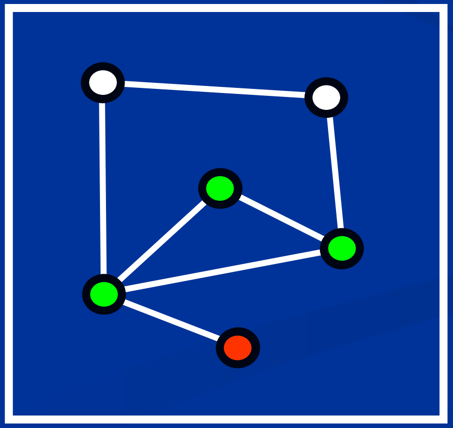

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Answer: "NO". Long standing open problem for $\theta \leq \log(n)$, provably "hard" when θ is a universal constant.

What is the “hard” problem?

Will show:

Given undirected G that has a clique cover of size 3, one can efficiently construct a Network Coding instance such that:

- Any Scalar Linear Network Code over F_q with rate $\geq \rho C$ implies a clique cover of G with $q^{3/\rho}$ colors (if q,ρ are constant $\Rightarrow$ “hard”).

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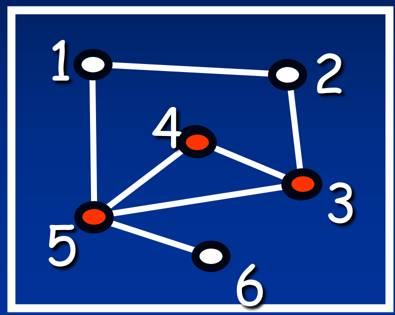
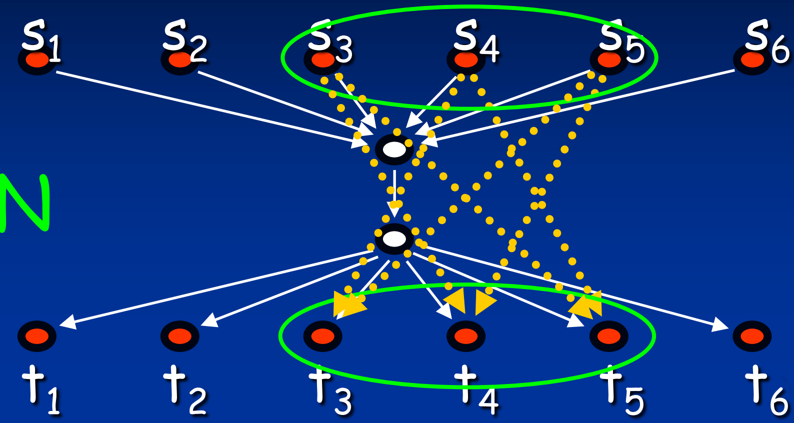


⇒ N



- Let G be a graph which can be covered by 3 cliques.
- Implies C(N) is at least 1/3 (time sharing between cliques).
- Key Lemma: From any scalar linear solution to N over F_q with rate $= 1/m \geq \rho \cdot C(N) \geq \rho/3$ one can find "not to many" cliques in G, that eventually cover all of G.
- Number of cliques $\leq q^{3/\rho}$: We have a "small" clique cover of G!!

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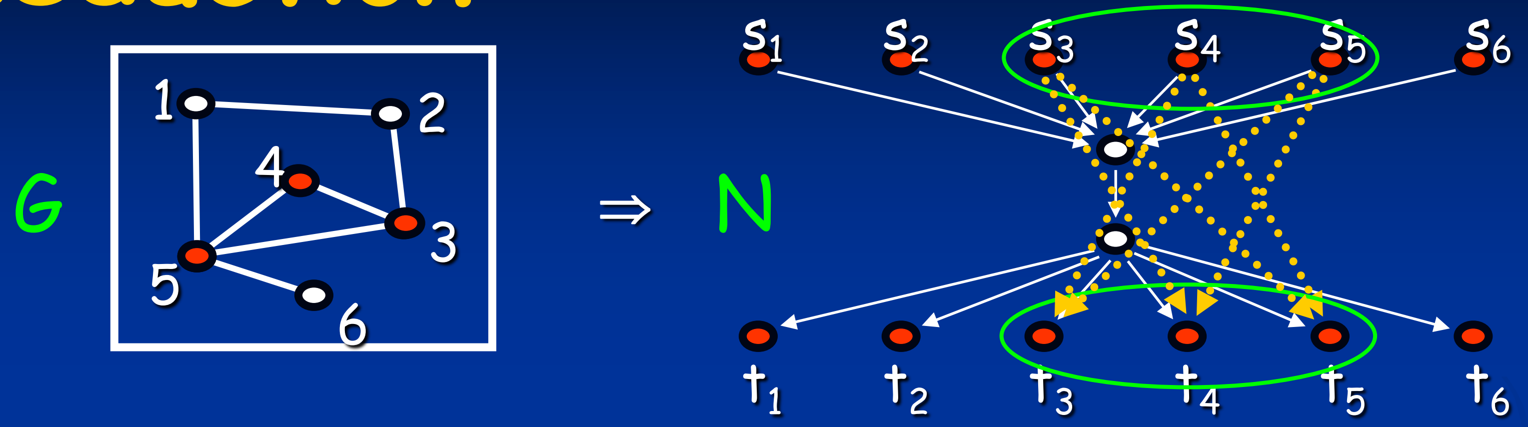

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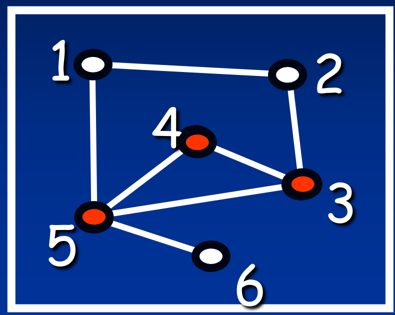
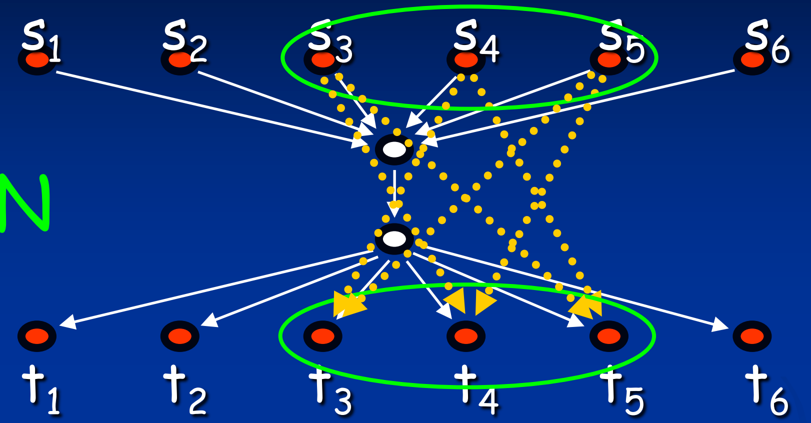
G 

⇒ N 

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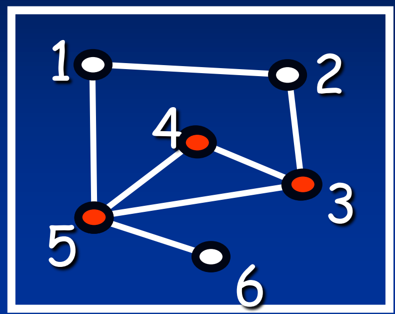

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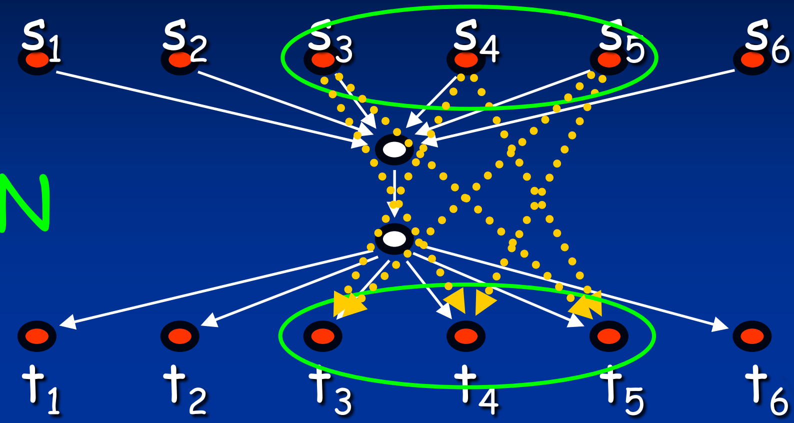

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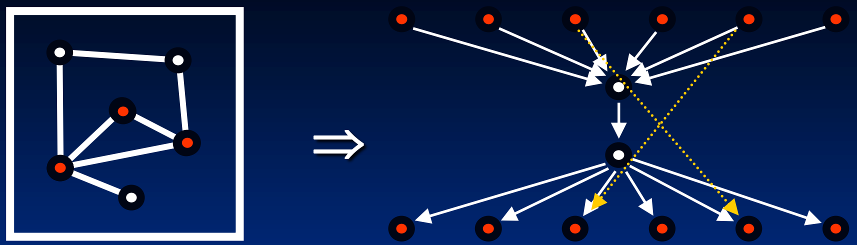

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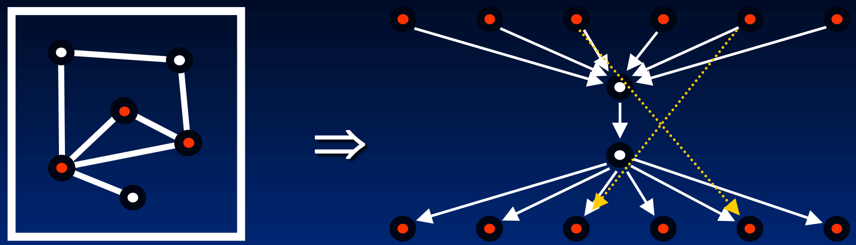

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- Consider linear combinations transmitted over bottleneck: $v_1, v_2, v_3, \ldots, v_m$.
  - $v_i = s_1 + s_3 + 2s_5 + s_7$
- NC is feasible, thus each terminal can decode using $\{v_i\}$ and .
- For each terminal let z_i be the combination of v_i's used before side information is used for decoding: $z_i = s_3 + s_5 + 2s_8 + s_9$

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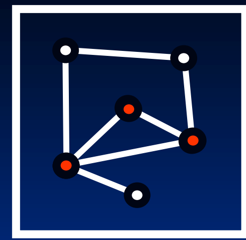

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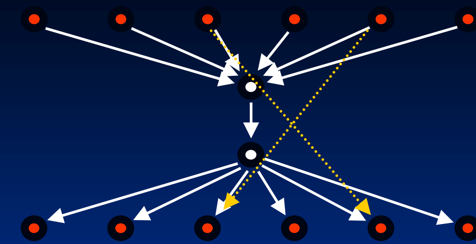

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- $\{z_i\}$ includes at most $q^m \leq q^{3/\rho}$ different equivalence classes.
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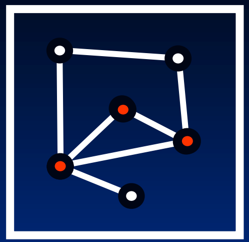

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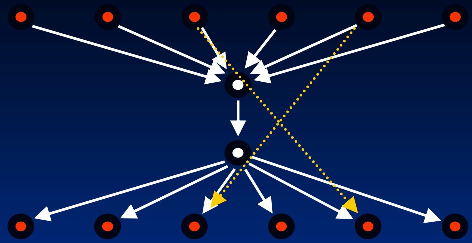

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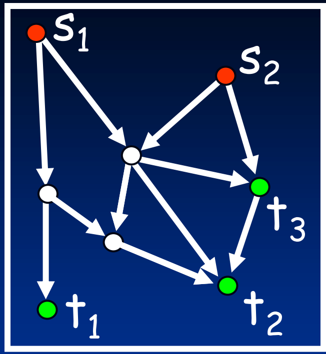
Recall: scalar Linear



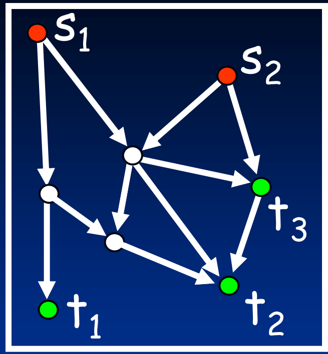
e.g. [Dougherty Freiling Zeger].

- General acyclic NC instance.
- Requirement matrix: $R=[r_{ij}]$
  - $r_{ij} \in \{0,1\}$, $r_{ij}=1 \Leftrightarrow$ terminal t_j wants info. of source s_i.
- Alphabet Σ of size q (here Σ will be a field).
- Scalar Linear: Each source has single character from Σ.
- Objective: design linear network code which is feasible.

- A Network Code enables communication at rate $1/m$:
  - Each link can carry m characters from Σ during transmission.

Capacity: C = maximum rate feasible linear NC.

General capacity



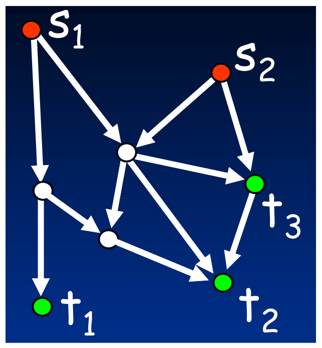
A NC scheme is (k,m) feasible:

- Each source has message of size k characters from Σ.
- Each link can carry m characters from Σ during transmission.
- All requirements satisfied.

- Rate = k/m.

Capacity: C = maximum (k/m) over (k,m) feasible NCs.

Capacity



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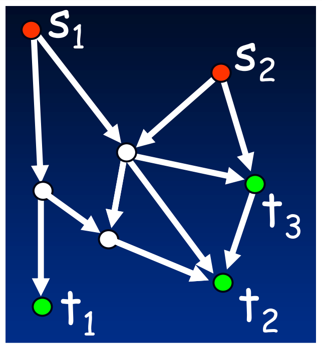
- Scalar Linear NC:
  - Encoding functions are linear AND k=1.
- Vector Linear of dimension k*:
  - Encoding functions are linear AND k =k*.
- General NC:
  - No restrictions!

Capacity



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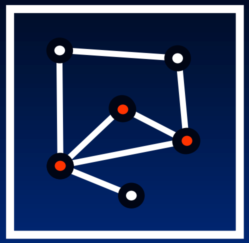
Single number characterization of the Network Coding instance.

Vector Linear



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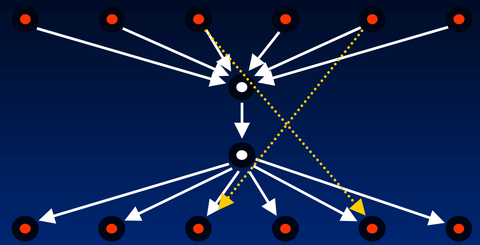

Up to now:

- Exact Scalar linear: NP-Hard.
- Approximate Scalar Linear: "Hard" for const. approx.

- What about Vector Linear?
- Same construction and proof actually hold for Vector Linear.

- Let G be a graph which can be covered by 3 cliques.
- Implies $C(N)$ is at least $1/3$ (time sharing between cliques).
- Key Lemma: From any k-dim. vector linear solution to N over F_q with rate = $k/m \geq \rho \cdot C(N) \geq \rho/3$ one can find "not to many" cliques in G, that cover all of G.
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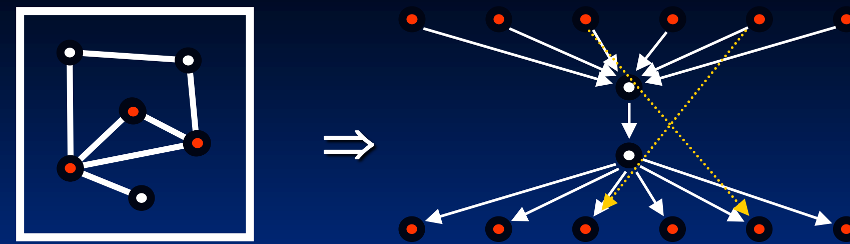

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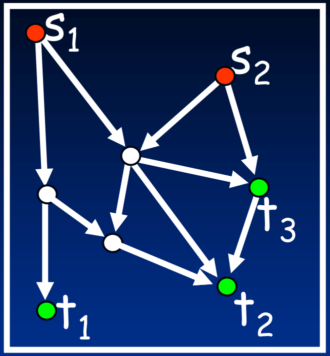
General codes



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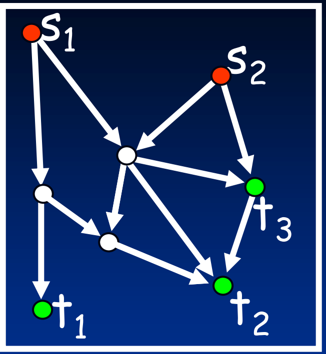
- What about General Network Coding?

General codes



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- Define the notion of a "rate vector".

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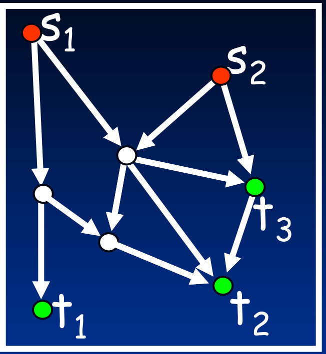

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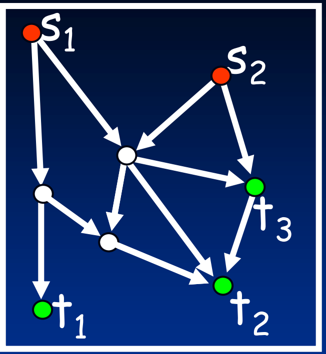

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- General coding is necessary to obtain capacity [Dougherty Freiling Zeger].

Summary

- Exact Scalar linear:
  - NP-Hard.
- Approximate Scalar Linear:
  - "Hard" for constant approximation and field size.
- Approximate Vector Linear:
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- General codes:
  - Imply an oracle for the set of entropic vectors.
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Open problems:
- Feasibility of General NC
  - Easy or undecidable?
  - Prove NP-Hard.
- Feasibility for Vector Linear.
  - Prove NP-Hard for any dim.

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The "positive" angle:

- Special networks:
  - Topology: Ring, line, star, planar
  - Few source/sink pairs.
- Loose inner and outer bounds that are computable.

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Thanks!