Newton polygons of exponential functions attached to Drinfeld modules of rank 2

(joint work with Yoonjin Lee)

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- Let $K = \mathbb{F}_q(T)$, $A = \mathbb{F}_q[T]$.

- Let $\infty = (\frac{1}{T})$ be the place at infinity of K with associated valuation function $v = v_\infty : K \to \mathbb{Z}$ such that $v_\infty(f) = -\deg(f)$ for f in A^*.

- Then $K_\infty = \mathbb{F}_q((\frac{1}{T}))$, $A_\infty = \mathbb{F}_q[[\frac{1}{T}]]$.

- Let C_∞ be the completion of an algebraic closure of K_∞ and denote also by v the extension of v from K to C_∞.

- Let the absolute value associated to v be given by $|x| = q^{-v(x)}$.

- Let $\phi_T = T + a_1\tau + a_2\tau^2$ be a Drinfeld A-module of rank 2 over K.

- By uniformization, there is an A-lattice $\Lambda_\phi = \Lambda_{\phi,\infty} \subseteq C_\infty$ of rank 2 and a surjective analytic function $e_\phi = e_{\phi,\infty} : C_\infty \to C_\infty$ satisfying

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda_\phi & \longrightarrow & C_\infty & \xrightarrow{e_\phi(z)} & C_\infty & \longrightarrow & 0 \\ \downarrow & & \downarrow & & {\scriptstyle a(z)}\downarrow & & {\scriptstyle \phi_a(z)}\downarrow & & \downarrow \\ 0 & \longrightarrow & \Lambda_\phi & \longrightarrow & C_\infty & \xrightarrow{e_\phi(z)} & C_\infty & \longrightarrow & 0 \end{array},$$

that is, e_ϕ has zero set equal to Λ_ϕ and $e_\phi(az) = \phi_a \circ e_\phi(z)$ for all $a \in A$ and is normalized so $e'_\phi(z) = 1$.

- The function $e_\phi(z)$ is called the exponential function attached to ϕ.

- It is uniquely determined by the above properties and can be written in the form $e_\phi(z) = \sum_{i=0}^{\infty} c_i \tau^i(z)$ where $\tau(z) = z^q$, $c_i \in C_\infty$, and $c_0 = 1$.

- We explicitly determine the Newton polygon and slopes of $e_\phi(z)$ for a general Drinfeld A-module ϕ of rank 2 defined over K.

- The method is mostly elementary but nonetheless reveals some interesting closed form patterns which might not be immediately apparent from the initial problem.

- The different cases of Newton polygons which arise depend on $v(j(\phi))$ where $j(\phi) = a_1^{q+1}/a_2$ is the j-invariant of $\phi_T = T + a_1\tau + a_2\tau^2$.

- The motivation is to study the field $K_{\phi,a}$ generated over K by the a-torsion points of ϕ.

- By work of Gardeyn, a natural object which arises in bounding the ramification over ∞ is the field $K_\infty(\Lambda_\phi)$ which contains the field generated by the a-torsion points of ϕ over K_∞ for all $a \in A$.

- Since Λ_ϕ is the zero set of the analytic function $e_{\phi,\infty}(z)$, the different of $K_\infty(\Lambda_\phi)/K_\infty$ can be bounded using information from the Newton polygon of $e_\phi(z)$.

- Using this explicit information about the Newton polygon of e_ϕ, we give explicit bounds on the ramification of $K_\infty(\Lambda_\phi)/K_\infty$ using the results of Gardeyn.

- Let $e_\phi = \sum_{i=0}^{\infty} c_i \tau^i$ be the exponential function associated to a Drinfeld module of rank 2 given by $\phi_T = T + a_1\tau + a_2\tau^2$.

- The exponential function is normalized so that $c_0 = 1$ and the following formulae determines its coefficient.

- (T may be replaced by $a \in A$ transcendental over $\mathbb{F}_q$ to obtained similar formulae)

$$c_1 = \frac{a_1 c_0^q}{T^q - T},$$

$$c_i = \frac{a_1 c_{i-1}^q + a_2 c_{i-2}^{q^2}}{T^{q^i} - T} \text{ for } i \geq 2.$$

- Let $d_i = \frac{v(c_i)}{q^i}$. Then we have the following formula for the d_i's.

$$d_0 = 0,$$
$$d_1 = \left(\frac{v(a_1)}{q} + 1\right) + d_0,$$
$$d_i \geq \min\left(\frac{v(a_1)}{q^i} + d_{i-1}, \frac{v(a_2)}{q^i} + d_{i-2}\right) + 1 \text{ for } i \geq 2,$$

where equality holds if the values in the minimum are distinct.

- If $\frac{v(a_1)}{q^i}+d_{i-1} \neq \frac{v(a_2)}{q^i}+d_{i-2}$, then we note that at each step in the recursion sequence, there are two choices: the new term d_i to be computed is either derived by a formula involving the previous term, called a *type* I term, or the term before the previous term, called a *type* II term.

- When $\frac{v(a_1)}{q^i}+d_{i-1} = \frac{v(a_2)}{q^i}+d_{i-2}$, we say that d_i is of *type* III (We also call this an *exceptional* case).

- A run of type I (or type II) is a subsequence of consecutive terms in the recursion sequence all of which are type I (or type II).

- Our strategy is to regard the sequence as being grouped into runs: starting with a run of type I, then a run of type II, then a run of type I, etc.

- With this point of view, we determine the exact conditions which tell us when we switch from a run of one type to a run of another type.

- The pattern of types are only dependent on the $\overline{K}$-isomorphism class of ϕ.

- We begin with run of a type I, that is $d_i = \left(\frac{v(a_1)}{q^i} + 1\right) + d_{i-1}$ starting from $i = 1$.

- When do we switch over in the sense that d_{i+1} has type **II** or **III** ?

- This happens when

$$\frac{v(a_1)}{q^{i+1}} + d_i \geq \frac{v(a_2)}{q^{i+1}} + d_{i-1},$$

equivalently when

$$\frac{v(a_1)}{q^{i+1}} + \frac{v(a_1)}{q^i} + 1 \geq \frac{v(a_2)}{q^{i+1}}. \tag{1}$$

- We also note that Eq. (1) is equivalent to

$$\frac{v(a_2)}{q+1} - v(a_1) \leq \frac{q^{i+1}}{q+1}.$$

- Let m be the least integer $m \geq 1$ such that

$$\frac{v(a_2)}{q+1} - v(a_1) \leq \frac{q^{m+1}}{q+1}. \tag{2}$$

Then $d_1, \ldots, d_m$ is a run of type I and d_{m+1} is of of type II (or III) if we have strict inequality (or equality) in Eq. (2).

- It follows that once Eq. (1) holds for $i = m$, it will hold for any $i \geq m$.

- Once we switch to a type II or III term, the conditions become more involved.

Proposition. Let m be the least integer $m \geq 1$ such that

$$\frac{v(a_2)}{q+1} - v(a_1) \leq \frac{q^{m+1}}{q+1}. \tag{3}$$

Then $d_1, \ldots, d_m$ is a run of type I. For the type of d_{m+n} with $n \geq 1$, there are three cases: d_{m+n} is of type I, II or III. We determine the type of d_{m+n+1} for $n \geq 1$ as follows:

(i) If d_{m+n} is of type II, and $m+n-i$ is the largest integer $< m+n$ such that d_{m+n-i} is of type I with $d_{m+n-i+1}, d_{m+n-i+2}, \ldots, d_{m+n}$ a run of type II, then

d_{m+n+1} has type I when

$$\frac{v(a_2)}{q+1} - v(a_1) > \frac{q^{m+n+1}}{q^{i+1}+1} \qquad \text{if } i \text{ is even,} \tag{4}$$

$$\frac{v(a_2)}{q+1} - v(a_1) < 0 \qquad \text{if } i \text{ is odd,} \tag{5}$$

d_{m+n+1} has type II when

$$\frac{v(a_2)}{q+1} - v(a_1) < \frac{q^{m+n+1}}{q^{i+1}+1} \qquad \text{if } i \text{ is even,} \tag{6}$$

$$\frac{v(a_2)}{q+1} - v(a_1) > 0 \qquad \text{if } i \text{ is odd,} \tag{7}$$

and d_{m+n+1} has type III when

$$\frac{v(a_2)}{q+1} - v(a_1) = \frac{q^{m+n+1}}{q^{i+1}+1} \qquad \text{if } i \text{ is even,} \tag{8}$$

$$\frac{v(a_2)}{q+1} - v(a_1) = 0 \qquad \text{if } i \text{ is odd.} \tag{9}$$

(ii) If d_{m+n} is of type III and $\frac{v(a_2)}{q+1} - v(a_1) < \frac{q^{m+n+1}}{q+1}$, then d_{m+n+1} is of type II.

(iii) Assume d_{m+n} is of type II, and $m+n-i$ is the largest integer $< m+n$ such that d_{m+n-i} is of type III with

$$d_{m+n-i+1}, d_{m+n-i+2}, \ldots, d_{m+n}$$

a run of type II.

If i is even, then d_{m+n+1} has type II if $\frac{v(a_2)}{q+1} - v(a_1) < \frac{q^{m+n+1}}{q^{i+1}+1}$, and d_{m+n+1} has type II or III if $\frac{v(a_2)}{q+1} - v(a_1) = \frac{q^{m+n+1}}{q^{i+1}+1}$.

If i is odd, then d_{m+n+1} has type I if $\frac{v(a_2)}{q+1} - v(a_1) < 0$, and d_{m+n+1} has type I or III if $\frac{v(a_2)}{q+1} - v(a_1) = 0$.

(Note (ii) and (iii) are only sufficient conditions.)

Theorem. Let $\phi_T = T + a_1\tau + a_2\tau^2$ be a Drinfeld A-module defined over K of rank 2. Let $e_\phi(z) = \sum_{i=0}^{\infty} c_i\tau^i(z)$ be its associated exponential function and let $d_i = v(c_i)/q^i$. We have the following cases for the types of the sequence $d_1, d_2, \ldots$

Let m be the smallest integer $m \geq 1$ such that $\frac{v(a_2)}{q+1} - v(a_1) \leq \frac{q^{m+1}}{q+1}$.

Case 1: $\frac{v(a_2)}{q+1} - v(a_1) < 0$.
Then the sequence $d_1, d_2, d_3, d_4, \ldots, d_j, \ldots$ has type $\mathrm{I}, \mathrm{II}, \mathrm{I}, \mathrm{II}, \ldots$, that is, d_j has type I if j is odd, and d_j has type II if j is even.

Case 2: $\frac{v(a_2)}{q+1} - v(a_1) > 0$.

(i) If $\frac{v(a_2)}{q+1} - v(a_1) < \frac{q^{m+1}}{q+1}$, then $d_1, \ldots, d_m$ is a run of type I, and d_{m+i} has type II for any $i \geq 1$.

(ii) Assume $\frac{v(a_2)}{q+1} - v(a_1) = \frac{q^{m+1}}{q+1}$.

Let $\delta_{m+n-i} \geq 0$ be defined as follows: $d_{m+n} = d_{m+n-1} + \frac{v(a_1)}{q^{m+n}} + 1 + \delta_{m+n}$.

If there exists k such that k is the smallest integer ≥ 1 with $\delta_{m+(2k-1)} \neq \frac{q-1}{q^{2k}}$, then the sequence

$$d_1, \ldots, d_m, d_{m+1}, d_{m+2}, \ldots, \\ d_{m+(2k-1)}, d_{m+2k}, d_{m+(2k+1)}, d_{m+(2k+2)}, d_{m+(2k+3)}, \ldots \quad (10)$$

has types $\mathrm{I}, \ldots, \mathrm{I}, \mathrm{III}, \mathrm{II}, \ldots, \mathrm{III}, \mathrm{II}, \mathrm{I}/\mathrm{II}, \mathrm{II}, \mathrm{II}, \ldots$, where d_1 through d_m have type I and d_{m+j} has the following types for $j \geq 1$:

type III if $j = 1, 3, 5, \ldots, (2k-1)$,

type I or II if $j = 2k+1$,

type II otherwise.

If there is no such k, that is, $\delta_{m+(2k-1)} = \frac{q-1}{q^{2k}}$ for any $k \geq 1$, then the sequence

$$d_1, \ldots, d_m, d_{m+1}, d_{m+2}, \ldots, d_{m+(2k-1)}, d_{m+2k}, d_{m+(2k+1)}, d_{m+(2k+2)}, \ldots$$

has types $\mathrm{I}, \ldots, \mathrm{I}, \mathrm{III}, \mathrm{II}, \ldots, \mathrm{III}, \mathrm{II}, \mathrm{III}, \mathrm{II}, \ldots$, where d_1 through d_m have type I and d_{m+j} has the following types for $j \geq 1$ with $k \geq 1$:

$$\begin{array}{lll} \text{type} & \mathrm{III} & \text{if} \quad j = 2k-1, \\ \text{type} & \mathrm{II} & \text{if} \quad j = 2k. \end{array}$$

Case 3: $\frac{v(a_2)}{q+1} - v(a_1) = 0$.
The sequence $d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, \ldots$ has types:
$\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{II}, \ \mathrm{I}/\mathrm{III}, \ \mathrm{II}, \ \mathrm{I}/\mathrm{III}, \mathrm{II}, \ldots$, where d_1 has type I, d_3 has type III, for $k \geq 1$, d_j has type II if $j = 2k$ and type I or III if $j = 2k+3$. In detail, if d_{2k+3} has type I (respectively, III) with $k \geq 1$, then d_{2k+5} has type III (respectively, I or III).

Theorem. Let $\phi_T = T + a_1\tau + a_2\tau^2$ be a Drinfeld A-module defined over K of rank 2. Let $e_\phi(z) = \sum_{i=0}^{\infty} c_i\tau^i(z)$ be its associated exponential function. As before, let $v_i = v(c_i)$ and $P_i = (q^i, v_i)$ for $i \geq 0$. Let $s_i = \frac{v_i - v_{i-1}}{q^i - q^{i-1}}$ be the slope of the line segment from the point $P_{i-1} = (q^{i-1}, v_{i-1})$ to $P_i = (q^i, v_i)$. Let m be the smallest integer $m \geq 1$ such that $\frac{v(a_2)}{q+1} - v(a_1) \leq \frac{q^{m+1}}{q+1}$. Then the Newton polygon of $e_\phi(z)$ is determined as follows.

Case 1: If $\frac{v(a_2)}{q+1} - v(a_1) < 0$, then the Newton polygon consists of the points P_0, P_2, $P_4, \ldots, P_{2k-2}, P_{2k}, \ldots$ with slopes such that $S_{k+1} = S_k + 1$ for each $k \geq 1$, where S_k denotes the slope of the line segment joining P_{2k-2} and P_{2k} for $k \geq 1$ and $S_1 = \frac{v(a_2)+q^2}{q^2-1} + v_0$.

Case 2: $\frac{v(a_2)}{q+1} - v(a_1) > 0$.

(i) If $\frac{v(a_2)}{q+1} - v(a_1) < \frac{q^{m+1}}{q+1}$, then all the points P_i with $i \geq 0$ form the Newton Polygon with slopes such that $s_1 < s_2 < \ldots < s_m < s_{m+1} < \ldots$, $s_{i+1} = s_i + 1$ for each $1 \leq i \leq m-1$, $s_{m+1} - s_m = \frac{v(a_2)-(q+1)v(a_1)-q^m}{q^m(q-1)}$, $s_{i+1} = s_{i-1} + 1$ for each $i \geq m+1$, and $s_1 = \frac{v(a_1)+q}{q-1} + v_0$.

(ii) If $\frac{v(a_2)}{q+1} - v(a_1) = \frac{q^{m+1}}{q+1}$, then the Newton polygon consists of the points $P_0, P_1, \ldots, P_m, P_{m+2}, P_{m+4}, \ldots, P_{m+2i}, \ldots$ with slopes satisfying the following: Let S_j denote the slope of the line segment joining $P_{m+(2j-2)}$ and P_{m+2j} for $j \geq 1$. Then $s_{i+1} = s_i + 1$ for each $1 \leq i \leq m-1$, $S_1 = s_1 + m$, $S_{j+1} = S_j + 1$ for each $j \geq 1$, and $s_1 = \frac{v(a_1)+q}{q-1} + v_0$.

Case 3: If $\frac{v(a_2)}{q+1} - v(a_1) = 0$, then the Newton polygon consists of the points P_0, P_2, $P_4, \ldots, P_{2k-2}, P_{2k}, \ldots$ with slopes such that $S_{k+1} = S_k + 1$ for each $k \geq 1$ and $S_1 = \frac{v(a_2)+q^2}{q^2-1} + v_0$, where S_k denote the slope of the line segment joining P_{2k-2} and P_{2k} for $k \geq 1$.

- We say $\Lambda \subseteq C_\infty$ is an A-lattice of rank 2 if $\Lambda = A\lambda_1 + A\lambda_2$ with $\lambda_1, \lambda_2 \in C_\infty$ being K_∞-linearly independent, and we refer to $\{\lambda_1, \lambda_2\}$ as an A-basis for Λ.

- First of all we note that if $\lambda \in \Lambda$ and $\lambda = \kappa_1\lambda_1 + \kappa_2\lambda_2$ with $\kappa_1, \kappa_2 \in K$, then in fact, $\kappa_1, \kappa_2 \in A$.

- Let $B_\kappa = \{\lambda \in \Lambda : |\lambda| \leq \kappa\}$ for $\kappa \in \mathbb{R}$. We define ν_i to be the infimum of the set of κ such that B_κ contains i number of K-linearly independent elements.

- An A-basis $\{\lambda_1, \lambda_2\}$ for Λ arising as in the following lemma is called a *minimal A-basis* for Λ, and because the hypotheses can always be satisfied (as B_κ is finite for each κ), minimal A-bases for Λ exist.

Lemma 1. *Let $\lambda_i \in \Lambda$ be elements such that $\{\lambda_1, \lambda_2\}$ are K-linearly independent and $|\lambda_i| = \nu_i$ for each $i = 1, 2$. Then $\{\lambda_1, \lambda_2\}$ is an A-basis for Λ.*

Theorem. (Gardeyn). Let $\{\lambda_1, \lambda_2\}$ be a minimal A-basis for Λ_ϕ. Let

$$\begin{aligned} \mathcal{D}_\infty^{\Lambda_\phi} &= 2 \cdot q \cdot (v(\lambda_1) - v(\lambda_2)) \frac{\nu_2}{\nu_1} \\ &= 2 \cdot q \cdot (v(\lambda_1) - v(\lambda_2)) \, q^{v(\lambda_1) - v(\lambda_2)}. \end{aligned}$$

Then

$$v(\mathcal{D}(K_\infty(\Lambda_\phi)/K_\infty)) \leq 1 + \mathcal{D}_\infty^{\Lambda_\phi}.$$

The zero set of e_ϕ is precisely Λ_ϕ. From the Newton polygon of e_ϕ, it is possible to derive information about the valuations of the zeros of e_ϕ and to determine a minimal A-basis for Λ_ϕ.

Theorem. *Let $\phi_T = T + a_1\tau + a_2\tau^2$ be a Drinfeld A-module of rank 2 over K, and let $\mathcal{D}(K_\infty(\Lambda_\phi)/K_\infty)$ be the different of $K_\infty(\Lambda_\phi)/K_\infty$. Let m be the smallest integer $m \geq 1$ such that $\frac{v(a_2)}{q+1} - v(a_1) \leq \frac{q^{m+1}}{q+1}$. Then we have the following upper bound for the different of $K_\infty(\Lambda_\phi)/K_\infty$:*

$$v(\mathcal{D}(K_\infty(\Lambda_\phi)/K_\infty)) \leq$$
$$1 + \begin{cases} 2q^2 & \textit{if } v(a_2) - (q+1)v(a_1) \leq 0 \\ 2mq^{m+1} & \textit{if } v(a_2) - (q+1)v(a_1) = q^{m+1} \\ 2\delta q^{\delta+1} & \textit{if } 0 < v(a_2) - (q+1)v(a_1) < q^{m+1} \end{cases}$$

where $\delta = \frac{v(a_2)-(q+1)v(a_1)}{q^m(q-1)} - \frac{1}{q-1} + m - 1$.