Using wave-activity conservation laws to understand the generation of subgrid-scale energy and momentum

T. A. Shaw

with T. G. Shepherd
University of Toronto

BIRS workshop
April 29, 2009

Outline

- Introduction to wave-activity conservation laws.
- Application to the large-scale circulation of the atmosphere: Eliassen-Palm flux.
- Application to subgrid-scale parameterization:
  - → Three-dimensional wave-activity conservation laws for subgrid-scale energy and momentum
  - → Connection to resolved-scale energy and momentum budgets
  - → Understanding the generation of subgrid-scale energy and momentum.
- Conclusions

What is a wave-activity conservation law?

- A wave-activity conservation law is a relation of the form:

$$\frac{\partial A}{\partial t} + \nabla \cdot \mathbf{F} = 0$$

where A is the wave-activity density and $\mathbf{F}$ is its flux.

$\rightarrow$ It is a quantity that is conserved in the absence of forcing and dissipation.

- They are useful in the study of disturbances u' (waves, eddies) to some background flow U i.e. $u = U + u'$ with $A \sim O(u'^2)$.

$\rightarrow$ Their most general form follows from Hamiltonian geophysical fluid dynamics.

What is a wave-activity conservation law?

- Each conservation law can be associated with a symmetry in the background flow (spatial and temporal) according to Noether's theorem.

  $\rightarrow$ Temporal and spatial symmetries yield pseudoenergy and pseudomomentum wave activities.

- Disturbance energy and enstrophy are not wave activities because they are not conserved (there can be an exchange of background and disturbance energy and enstrophy).

  $\rightarrow$ Wave-activity conservation laws are very useful concepts when working in a modal decomposition since the individual modes are orthogonal in the appropriate sense (Held 1985 JAS).

Hamiltonian formulation

- Procedure for deriving conservation laws begins by casting the dynamics into the form

$$\xi_t = J \frac{\delta \mathcal{H}}{\delta \xi}$$

  where ξ is the state vector, $\mathcal{H}$ is the Hamiltonian and J is a skew-symmetric operator satisfying the Jacobi condition.

- When cast in terms of Eulerian variables fluid dynamical systems are non-canonical, i.e. there exists $\mathcal{C}$ such that

$$0 = J \frac{\delta \mathcal{C}}{\delta \xi}$$

  implying every invariant is only defined to within a Casimir (Shepherd 1990 Rev. Geophys.).

Hamiltonian formulation

- Given a steady background flow X we have

$$X_t = J\frac{\delta \mathcal{H}}{\delta \xi}\bigg|_{\xi=X} = 0 \quad \Rightarrow \quad \frac{\delta \mathcal{H}}{\delta \xi}\bigg|_{\xi=X} = -\frac{\delta \mathcal{C}^{\mathcal{E}}}{\delta \xi}\bigg|_{\xi=X}.$$

- We then define the quadratic pseudoenergy as

$$\mathcal{A}^{\mathcal{E}} \equiv \mathcal{H}(\xi) + \mathcal{C}^{\mathcal{E}}(\xi) - \mathcal{H}(X) - \mathcal{C}^{\mathcal{E}}(X).$$

- If $\nabla X = 0$ we define the pseudomomentum associated with the momentum invariant $\boldsymbol{\mathcal{M}}$:

$$\mathcal{A}^{\mathcal{M}} \equiv \mathcal{M}(\xi) + \mathcal{C}^{\mathcal{M}}(\xi) - \mathcal{M}(X) - \mathcal{C}^{\mathcal{M}}(X).$$

Application to large-scale circulations

- For the mid-latitude troposphere and middle atmosphere it is relevant to consider: $u = \overline{u}(y,z,t) + u'$ (where $\overline{}$ is a zonal average).

- The pseudomomentum wave-activity conservation law for the quasi-geostrophic Boussinesq equations is

$$\frac{\partial A}{\partial t} + \nabla \cdot \mathbf{F} = \mathrm{D}$$

Eliassen-Palm flux divergence

where

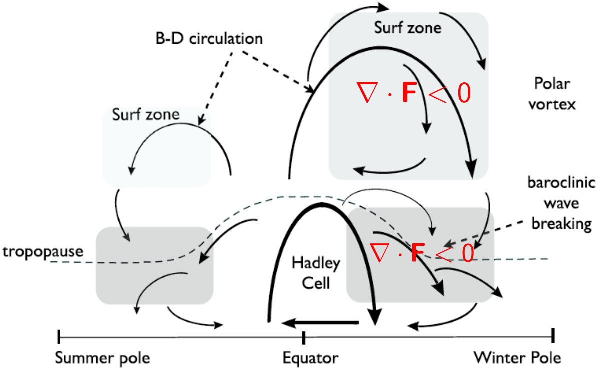
$$A = \frac{1}{2Q_y}\overline{q'^2}; \qquad \mathbf{F} = -\overline{u'v'}\,\hat{\mathbf{j}} + \frac{f_0}{N^2}\overline{v'b'}\,\hat{\mathbf{k}}; \qquad \mathrm{D} = \frac{1}{Q_y}\overline{S'q'}$$

where Q and q' are the background and disturbance potential vorticities and D includes both sources and sinks S'.

Application to large-scale circulations

- Eliassen-Palm flux divergence drives the background momentum (after taking the transformed Eulerian mean)

$$\frac{\partial \overline{u}}{\partial t} - f_0 \overline{v}^* = \nabla \cdot \mathbf{F}.$$



Flux convergences/ divergences drive poleward/equatorward motion.

Vallis (2006)

Application to subgrid-scale parameterization

- The solution to the equations of climate modelling requires the parameterization of processes with length scales smaller than the numerical discretization (subgrid-scale processes).

- The resolved and subgrid-scales are typically assumed to obey a scale separation in horizontal space and in time but are coupled in the vertical.

  $\rightarrow$ The resolved-scale flow is modelled by the hydrostatic primitive equations and the subgrid-scale is modelled by the anelastic or Boussinesq equations.

- In the wave-activity framework, the resolved-scale flow is the background and the subgrid-scale flow is the disturbance.

Application to subgrid-scale parameterization

- For the parameterization of subgrid-scales in climate models it is relevant to consider

$$\xi = \xi_r(z) + \xi_s(x, y, z, t)$$

  $\rightarrow$ The symmetries in the background flow imply three wave-activity conservation laws: pseudoenergy and two pseudomomenta associated with x and y symmetries.

- The dynamics of the resolved scale occur on longer spatial and temporal scales and can be incorporated using multiple scale asymptotics e.g. $\xi = \xi_r(\mathbf{x}_r, z, t_r) + \xi_s(\mathbf{x}_r, \mathbf{x}_s, z, t_r, t_s)$ where $\mathbf{x}_r = \epsilon^2 \mathbf{x}_s$.

  $\rightarrow$ Introduce average operator: $\overline{\xi} = \xi_r$.

Hamiltonian structure

- The subgrid-scales are assumed to be governed by the 3D anelastic equations.
- The anelastic equations can be cast in the Hamiltonian symplectic form with Hamiltonian and momentum invariants

$$\mathcal{H} = \int \left(\frac{\rho_r}{2} |\mathbf{v}|^2 + c_p \rho_r \pi_r \theta \right) \, dV \,, \quad \boldsymbol{\mathcal{M}} = \int \rho_r \mathbf{v} \; dV \,.$$

- Conservation of θ and q (the potential vorticity) lead to Casimir invariants which are functions of θ and q quantities.

  $\rightarrow$ The functional form of the Casimir invariants is central to the derivation of the wave activities.

Wave-activity conservation laws for the subgrid-scale

- The pseudoenergy and pseudomomentum densities for three-dimensional anelastic disturbances to a veering background flow are

$$
\begin{aligned}
A^{\mathcal{E}} &= \frac{\rho_r}{2}|\mathbf{v}_s|^2 + \frac{\rho_r}{2}\left[\frac{g}{\theta_r \theta_{r_z}} - \frac{\rho_r \mathbf{u}_r}{(\theta_{r_z})^2} \cdot \left(\frac{\mathbf{u}_{r_z}}{\rho_r}\right)_z\right](\theta_s)^2 - \frac{\rho_r \mathbf{u}_r}{\theta_{r_z}} \cdot \boldsymbol{\omega}_s^{\perp}\theta_s \\
\mathbf{A}^{\mathcal{M}_{x,y}} &= -\frac{1}{2}\frac{\rho_r^2}{(\theta_{r_z})^2}\left(\frac{\mathbf{u}_{r_z}}{\rho_r}\right)_z (\theta_s)^2 - \frac{\rho_r}{\theta_{r_z}}\boldsymbol{\omega}_s^{\perp}\theta_s
\end{aligned}
$$

with vertical fluxes:

$$
F_{(z)}^{\mathcal{E}} = c_p \rho_r \theta_r \pi_s w_s + \rho_r \mathbf{u}_r \cdot \mathbf{u}_s w_s, \qquad \mathbf{F}_{(z)}^{\mathcal{M}_{x,y}} = \rho_r \mathbf{u}_s w_s
$$

(Shaw & Shepherd 2008 JFM).

Averaged wave-activity conservation laws

- The averaged wave-activity conservation laws

$$\frac{\partial \overline{F^{\mathcal{E}}_{(z)}}}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\rho_r\theta_r}{\kappa}\overline{\pi_s w_s} + \rho_r \overline{\mathbf{u}_s w_s}\cdot\mathbf{u}\right) = \overline{\mathrm{D}^{\mathcal{E}}}$$

$$\frac{\partial \overline{\mathbf{F}^{\mathcal{M}_{x,y}}_{(z)}}}{\partial z} = \frac{\partial}{\partial z}(\rho_r\overline{\mathbf{u}_s w_s}) = \overline{\mathbf{D}^{\mathcal{M}_{x,y}}}.$$

- The source/sink terms $\mathrm{D}^{\mathcal{E}}$ and $\mathbf{D}^{\mathcal{M}_{x,y}}$ include diabatic heating (S^s_θ) and dissipation ($\mathbf{S_v}$) and satisfy the following constraint:

$$\overline{\mathrm{D}^{\mathcal{E}}} = \overline{\mathbf{v}_s\cdot\mathbf{S_v}} + \rho_r g\overline{\theta_s S^s_\theta}/(\theta_r\theta_{r_z}) + \mathbf{u}_r\cdot\overline{\mathbf{D}^{\mathcal{M}_{x,y}}}$$

Hamiltonian constraints

- Noether's theorem requires that $A^{\mathcal{E}} = cA^{\mathcal{M}}$ where c is the phase speed in the direction of symmetry.

  $\longrightarrow$ Generalized first Eliassen-Palm theorem.

- The relationship holds for the horizontally averaged vertical fluxes, $\overline{F^{\mathcal{E}}_{(z)}} = c\overline{F^{\mathcal{M}}_{(z)}}$, as well as the source/sink terms, and helps constrain the parameterization of subgrid-scale energy and momentum fluxes.

  $\longrightarrow$ Cannot tune the energy flux without a compensation in the momentum flux.

- Accounts for the non-local conservation of energy and momentum.

Contribution to the resolved scale

- To understand how the subgrid-scale wave-activity fluxes couple to the resolved-scale energy and momentum we use the theory of multiple scale asymptotics.

  $\rightarrow$ Systematic way to understand the interaction across scales when each scale is modeled by a different set of equations.

- Klein (2000 ZAMM), Majda & Klein (2003 JAS) have shown how the theory can be used to systematically derive balanced models in midlatitudes and the tropics.

- We use the freedom of multiple scale asymptotics to couple the dynamical equations of interest i.e. the hydrostatic primitive equations and the anelastic equations.

Contribution to resolved-scale energy and momentum

- The resolved-scale horizontal momentum equation:

$$\frac{\partial}{\partial t_r}(\rho_r \mathbf{u}_r) + \nabla_r \cdot (\rho_r \mathbf{v}_r \circ \mathbf{u}_r) + \mathbf{e}_z \times \rho_r \mathbf{u}_r + \frac{1}{\mathrm{M}^2}\nabla_p^H p_r = -\frac{\partial}{\partial z}(\rho_r \overline{\mathbf{u}_s w_s})$$

$$= \frac{\partial}{\partial z}\overline{\mathbf{F}_{(z)}^{\mathcal{M}_{x,y}}}$$

and total energy equation:

$$\frac{\partial}{\partial t_r}\left[\rho_r\left(K_r + \frac{1}{\mathrm{M}^2}\frac{1}{\kappa\gamma}T_r + \Phi_r\right)\right] + \nabla_r \cdot \left[\rho_r \mathbf{v}_r\left(K_r + \frac{1}{\mathrm{M}^2}\frac{1}{\kappa}T_r + \Phi_r\right)\right]$$

$$= S_\theta^r - \frac{\partial}{\partial z}\left(\frac{\rho_r T_r}{\kappa\theta_r}\overline{\theta_s w_s}\right) + \frac{\partial}{\partial z}(\mathbf{v}_s \cdot (\mathbf{d}_s)_z) - \frac{\partial}{\partial z}\left(\rho_r \mathbf{u}_r \cdot \overline{\mathbf{u}_s w_s} + \frac{\rho_r \theta_r}{\kappa}\overline{\pi_s w_s}\right)$$

$$= \frac{\partial}{\partial z}\overline{F_{(z)}^{\mathcal{E}}}$$

- Connection between resolved-scale and subgrid-scale represent generalized non-acceleration theorems (Shaw & Shepherd 2009 JAS).

Generation/Dissipation

- Regions of divergence/convergence of subgrid-scale fluxes can be thought of as source/sink regions for subgrid-scale energy and momentum.

  → Many types of subgrid-scale momentum transfers: gravity wave drag, convective momentum transport, boundary layer transport.

  → All parameterizations close the quadratic terms differently.

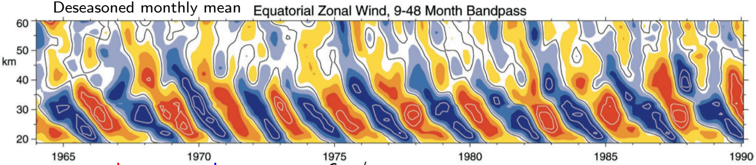
- The wave-activity source/sink terms provide a way to understand the generation and dissipation of subgrid-scale energy and momentum.

Generation of gravity waves

- Climate models require the parameterization of the transfer of momentum by small-scale gravity waves to obtain a reasonable modelled climate.

  $\longrightarrow$ Gravity waves are important for both the quasi-biennial oscillation and the upper atmospheric circulation.

Deseasoned monthly mean



eastward westward contour: 6m/s

Baldwin et al. (2001)

- Vertically propagating gravity waves are generated by convection, orography, flow over fronts, instability, etc.

Generation of gravity waves

- GWD parameterizations require as input the momentum flux as a function of wave phase speed and the horizontal and vertical wave numbers, all of which are poorly constrained by observations.

- The spectrum of waves is launched into the troposphere from a specific altitude (usually the tropopause) and is typically independent of latitude.

- Most parameterization do not account for changes in the wave-generation region (i.e. the troposphere).

  → There exists a few parameterizations which try to couple the wave spectrum to the modelled convection, however they assume weak shear (WKB) conditions.

Generation of gravity waves

- We can distinguish between generation due to mechanical processes and generation due to diabatic processes.

  i) Mechanical generation (flow over convective complexes, fronts, shear-flow instability):

$$\int_0^{z_T} (u_r - c)\overline{\mathrm{D}^{\mathcal{M}_x}}\, dz = 0$$

  → In the source region, the momentum flux convergence is single-signed.

  → The phase speed c must lie within the range of u_r.

  The dual requirements of energy and momentum conservation constrain the source spectrum.

Generation of gravity waves

- If we assume hydrostatic waves (a common assumption in GWD parameterizations) then we have

$$m = \frac{N}{c - u_r}$$

  with m the vertical wave number.

- If c spans the range of available $|u_r| = [0, u_{r_{max}}]$ this also restricts the vertical wave number:

$$|m| > \frac{N}{u_{r_{max}}}$$

  If N $= 0.02$ s^{-1} and $u_{r_{max}} = 30$ ms^{-1} this implies $|m| > 1.5$ m^{-1}.

  $\rightarrow$ Provides a more systematic choice for the small-m limit.

Generation of gravity waves

ii) Diabatic generation ($S_\theta^s \neq 0$):

In this case we have two relationships to exploit:

$$\begin{aligned}
\frac{\partial}{\partial z}(\rho_r \overline{u_s w_s}) = \mathrm{D}^{\mathcal{M}_x} &= -\frac{\rho_r}{\theta_{r_z}}\left[\frac{\rho_r}{\theta_{r_z}}\left(\frac{u_{r_z}}{\rho_r}\right)_z \overline{\theta_s S_\theta^s} - \overline{\omega_{s_{(y)}} S_\theta^s}\right] \\
(u_r - c)\mathrm{D}^{\mathcal{M}_x} &= -\frac{g\rho_r}{\theta_r \theta_{r_z}} \overline{\theta_s S_\theta^s}
\end{aligned}$$

$\rightarrow$ Relationships follow from pseudomomentum wave-activity conservation law and the relationship between pseudoenergy and pseudomomentum source/sink terms.

Momentum flux at any level is equal to

$$\rho_r \overline{u_s w_s} = -\int_0^z \frac{\rho_r}{\theta_{r_z}}\left[\frac{\rho_r}{\theta_{r_z}}\left(\frac{u_{r_z}}{\rho_r}\right)_z \overline{\theta_s Q_\theta^s} - \overline{\omega_{s_{(y)}} S_\theta^s}\right] dz.$$

Generation of gravity waves

ii) Diabatic processes (cont'd):

If $\overline{\omega_{s_{(y)}} S_\theta^s}$ is non-negligible then we can regress it onto $\overline{\theta_s S_\theta^s}$ and obtain:

$$\int_0^{z_T} \frac{\theta_r}{g} \left\{ (u_r - c) \left[\frac{\rho_r}{\theta_{r_z}} \left(\frac{u_{r_z}}{\rho_r} \right)_z - \alpha \right] - \frac{g}{\theta_r} \right\} \overline{\mathrm{D}^{\mathcal{M}_x}}\, dz = 0$$

else if $\overline{\omega_{s_{(y)}} S_\theta^s}$ is negligible then we have:

$$\int_0^{z_T} \frac{\theta_r}{g} \left[(u_r - c) \frac{\rho_r}{\theta_{r_z}} \left(\frac{u_{r_z}}{\rho_r} \right)_z - \frac{g}{\theta_r} \right] \overline{\mathrm{D}^{\mathcal{M}_x}}\, dz = 0$$

$\longrightarrow$ In this case c is constrained by the stratification and the vertical wind shear.

Generation/Dissipation

- Wave-activity conservation laws provide constraints on c for the generation of gravity wave momentum flux by mechanical and diabatic sources.

  $\longrightarrow$ Relationships exists even without a phase speed.

- Constraints can be validated using cloud resolving model simulations and then used in GWD parameterizations.

  $\longrightarrow$ Work in progress.

Application to other parameterizations

- Convective momentum transport is another important source of subgrid-scale momentum transfer in climate models.
- Wave-activity conservation laws could provide useful relationships in this case as well.
- Current CMT parameterizations (Schneider & Lindzen 1976 JGR, Kershaw et al. 1997 QJRMS) only account for pseudomomentum changes.

  $\rightarrow$ They do not account for the pseudoenergy changes nor the relationships between the two.

- Exploring the wave-activity constraints for CMT is work in progress.

Conclusions

- Wave-activity conservation laws provide a way to understand the exchange of energy and momentum between a background flow and a disturbance.

- The Eliassen-Palm wave-activity flux is a classic example and has been crucial to theoretical analysis of large-scale circulations.

- Wave-activity conservation laws can also be applied to the problem of subgrid-scale parameterization in climate modelling using a shear-stratified background flow.

  $\longrightarrow$ Provides a concise understanding of the exchange of energy and momentum between the resolved and subgrid-scales.

Conclusions

- Wave-activity conservation laws can be used to understand the generation of subgrid-scale energy.
- In the context of the generation of gravity waves we can distinguish between generation by mechanical and diabatic sources.
  - → Mechanical generation restricts the phase speed according to the background wind.
  - → Wind shear and stratification affect the phase speed in the presence of a diabatic source.
- The conservation laws could potentially be used to understand regions of momentum flux convergence/divergence due to convective momentum transport.

Hamiltonian Constraints

- According to Noether's theorem the pseudoenergy and x-pseudomomentum are related to time and space symmetries

$$\xi_t = J\frac{\delta \mathcal{A}^{\mathcal{E}}}{\delta \xi}, \quad -\xi_x = J\frac{\delta \mathcal{A}^{\mathcal{P}_x}}{\delta \xi}$$

which for a disturbance propagating in the $\hat{\mathbf{x}}$ direction with phase speed c i.e. $\xi_t + c\xi_x = 0$ implies

$$\mathcal{A}^{\mathcal{E}} = c\mathcal{A}^{\mathcal{P}_x}$$

assuming the symplectic operator J is non-singular.

CMT parameterizations

- According to Gregory et al. (1997 QJRMS) and Schneider & Lindzen (1976), cumulus momentum transport is expressed as

$$\mathbf{F}_c = -\frac{\partial}{\partial z}(\rho_0 \overline{\mathbf{u}' w'}) \approx \frac{\partial}{\partial z}\left[M_u(\mathbf{u}_u - \overline{\mathbf{u}}) + M_d(\mathbf{u}_d - \overline{u})\right]$$

where u and d refer to up and downdraft averaged quantities.

- The in-cloud velocities are calculated from the momentum equation

$$\begin{aligned}
\frac{\partial}{\partial z}(M_u \mathbf{u}_u) &= E_u \overline{\mathbf{u}} - D_u \overline{\mathbf{u}}_u + \mathbf{P}_G^u \qquad \text{with} \qquad \mathbf{P}_G^u = -C_u M_u \frac{\partial \mathbf{u}}{\partial z} \\
\frac{\partial}{\partial z}(M_d \mathbf{u}_d) &= E_d \overline{\mathbf{u}} + \mathbf{P}_G^d \qquad \text{with} \qquad \mathbf{P}_G^d = -C_d M_d \frac{\partial \mathbf{u}}{\partial z}
\end{aligned}$$

- These CMT parameterizations close the pseudomomentum but do not account for the pseudoenergy.

  $\longrightarrow$ Enthalpy tendency $= \mathrm{D}^{\mathcal{E}} - \mathbf{u} \cdot \mathbf{D}^{\mathcal{P}}$.