On Wave Interactions in Strongly Stratified Flows

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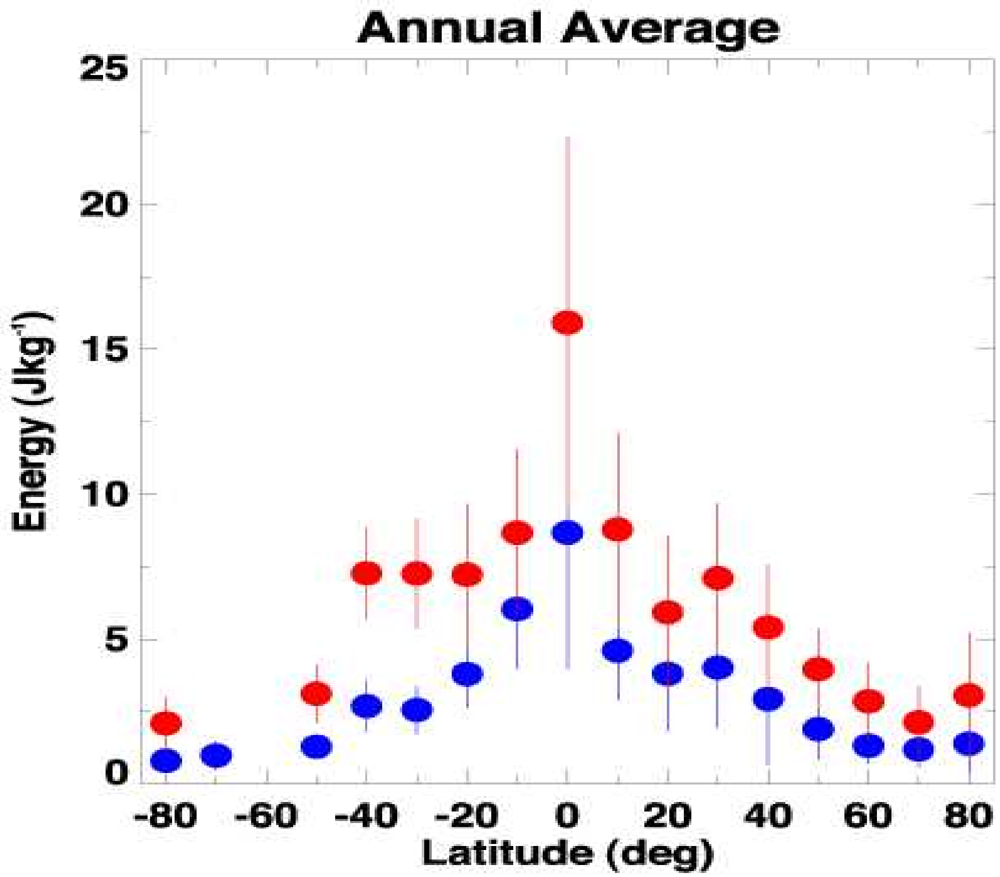
Wave Energy in the stratosphere (SPARC website; radiosonde data)



Annual average wave KE (red) and PE (blue) in the stratosphere (12-19 km at poles; 18-25 km at equator; spike = Kelvin waves).

Wave Interactions

Question:

Can we separate wave interactions

with goal of understanding their relative importance in strongly stratified flows?

Method:

We derive a PDE subsystem including only 3-wave interactions – the GGG model

then we compare GGG to full Boussinesq dynamics

Overall Goal: detailed understanding of all wave and vortical interactions in strongly stratified flows

Limitations of today's work:

- Dry dynamics
- When rotation is included, only f-plane dynamics

Next simplified moisture models

PhD student QIANG DENG

Dichotomous situation: theory/simulations of purely stratified flows

Which interactions determine the distribution of wave-mode energy in the forward transfer range?

- In periodic-box simulations,

  wave-vortical-wave interactions

  are important (Waite and Bartello, 2006)

- Weak-turbulence theories keeping only

  wave-wave-wave exact resonances

  reproduce some observed oceanic spectra (Lvov, Polzin & Tabak, 2004, McComas & Bretherton, 1977)

Here we delve deeper into wave-wave-wave interactions (including near- and non-resonant)

- Part I: $Bu = O(1)$, $N/f > 1$, $H/L < 1$

- Part II: Purely stratified flow with $Fr \ll 1$

The 3D rotating Boussinesq equations on an f-plane

Conservation laws for vertically stratified flow rotating about the vertical $\hat{\mathbf{z}}$-axis:

$$momentum: \quad \frac{D\mathbf{u}}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla P - N\theta\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u}$$

$$mass: \quad \nabla \cdot \mathbf{u} = 0$$

$$energy: \quad \frac{D\theta}{Dt} - Nw = \kappa\nabla^2\theta, \quad \theta = \frac{g}{N\rho_o}\rho'$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \qquad f = 2\Omega, \qquad Ro = \frac{U}{fL}$$

$$\rho = \rho_o - bz + \rho', \quad \rho' \ll \rho_o, |bz|, \quad N^2 = \frac{gb}{\rho_o}, \quad Fr = \frac{U}{NH}$$

Symbolically

The full equations:

$$0 \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \;\oplus\; ++ \;\oplus\; +- \;\oplus\; -- \qquad (1)$$

$$+ \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \;\oplus\; ++ \;\oplus\; +- \oplus\; -- \qquad (2)$$

$$- \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \;\oplus\; ++ \;\oplus\; +- \oplus\; -- \qquad (3)$$

where 0, +, - represent vortical and wave linear eigenmodes.

Restrictions of the sum:

QG (vortical mode interactions only):

$$0 \quad | 00$$

GGG (wave modes only):

$$+ \quad | \; ++ \;\oplus\; +- \;\oplus\; --$$

$$- \quad | \; ++ \;\oplus\; +- \;\oplus\; --$$

and everything in between, e.g.

PPG (add to QG interactions involving exactly 1 wave):

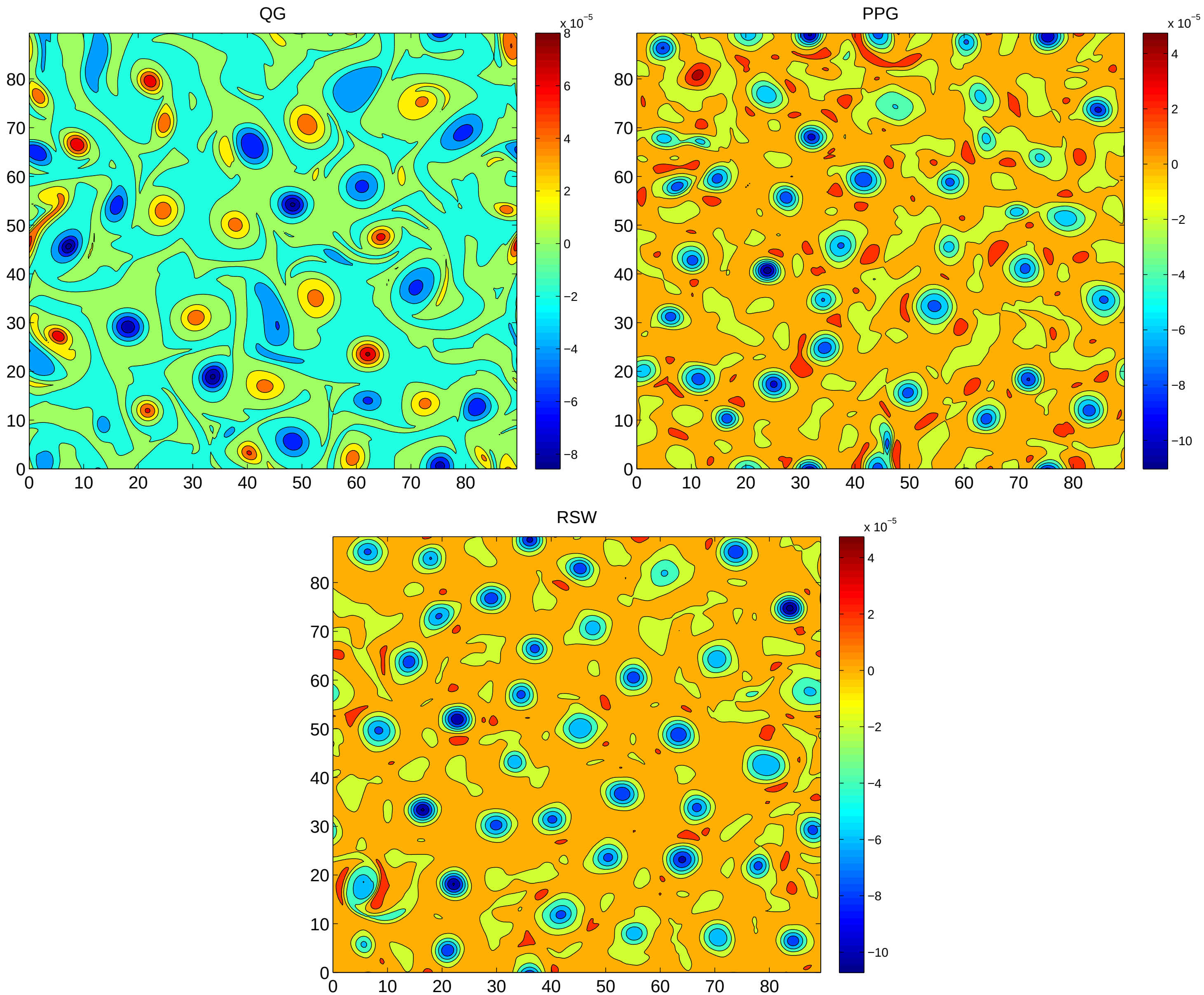
$$0 \quad | \; 00 \;\oplus\; 0+ \;\oplus\; 0- \qquad (ppg1)$$

$$+ \quad | \; 00 \qquad (ppg2)$$

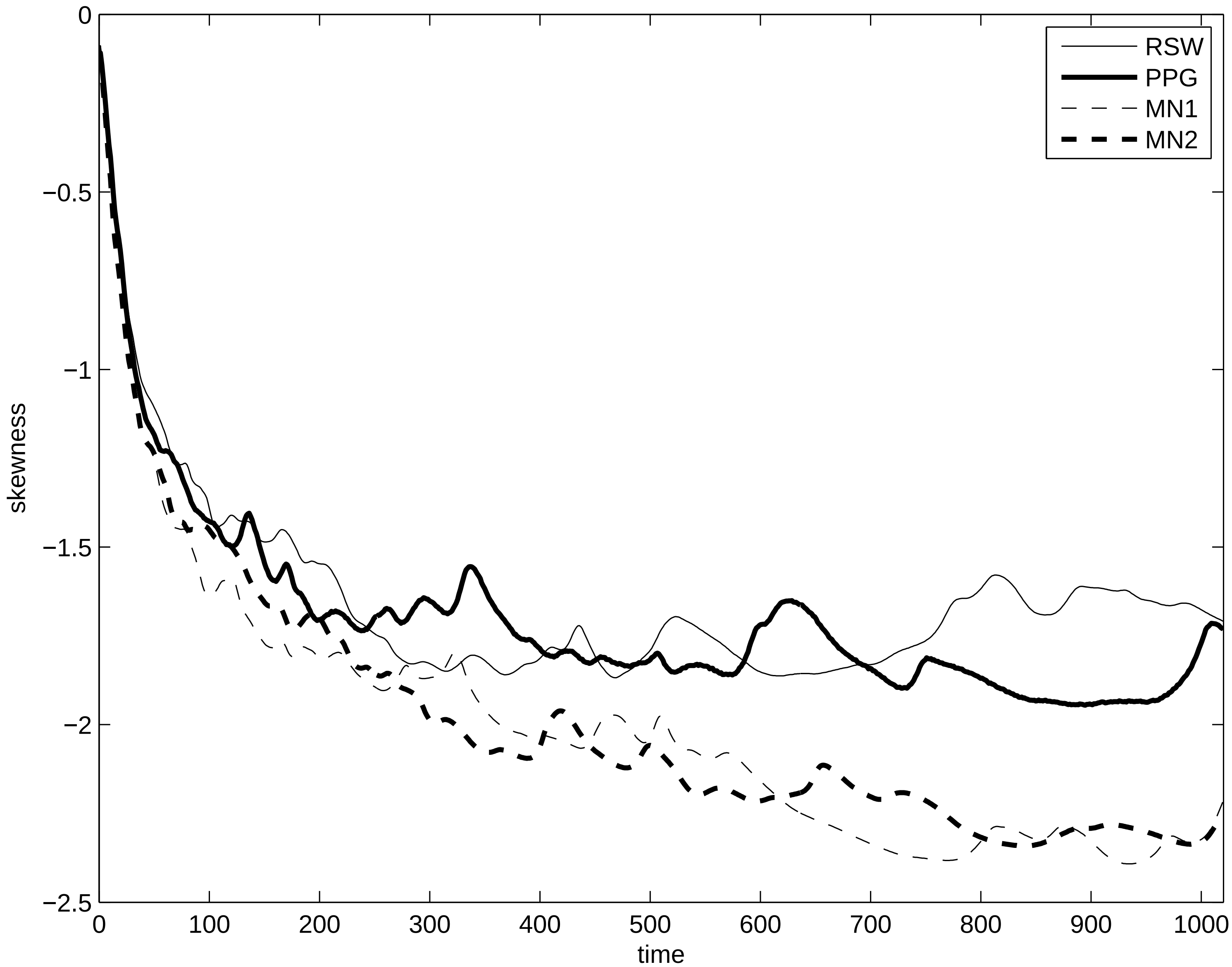
$$- \quad | \; 00 \qquad (ppg3)$$

Muraki, Snyder, Rotunno (1999), McIntyre & Norton (2000)

Succes in RSW decay, Ro=0.4, Fr = 0.25, divergence-free unbalanced i.



Compare to McIntyre & Norton PV inversion (MN1, MN2)



In physical space, change variables

Introduce a velocity potential and streamfunction:

$$u = \chi_x - \psi_y + \overline{u(z)}, \quad v = \chi_y + \psi_x + \overline{v(z)}, \qquad \overline{} = \text{horizontal avg}$$

and physical variables:

$$q = \nabla_h^2 \psi - \frac{f}{N}\frac{\partial \theta}{\partial z} \quad \text{linear potential vorticity}$$

$$R = \frac{N}{f}\theta + \frac{\partial \psi}{\partial z} \quad \text{geostrophic imbalance}$$

and an operator:

$$O = (\nabla_h^2 + \frac{f^2}{N^2}\partial_{zz}) \quad \text{with} \quad Q = O^{-1}q, \quad \mathcal{R} = O^{-1}R$$

===> an equivalent form of rotating Boussinesq:

$$\frac{\partial q}{\partial t} + \hat{z} \cdot \nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}) - \frac{f}{N} \partial_z [(\mathbf{u} \cdot \nabla)\theta] = 0$$

$$\frac{\partial f \nabla_h^2 R}{\partial t} - N^2\, Ow + N \nabla_h^2 [(\mathbf{u} \cdot \nabla)\theta] + f \partial_z (\hat{z} \cdot \nabla \times (\mathbf{u} \cdot \nabla \mathbf{u})) = 0$$

$$\frac{\partial \nabla^2 w}{\partial t} + f \nabla_h^2 R + \nabla_h^2 (\mathbf{u} \cdot \nabla w) - \partial_z (\nabla_h \cdot (\mathbf{u} \cdot \nabla \mathbf{u}_h)) = 0$$

$$\frac{\partial \overline{u(z)}}{\partial t} - f \overline{v(z)} + \overline{\partial_z (uw)} = 0, \qquad \frac{\partial \overline{v(z)}}{\partial t} + f \overline{u(z)} + \overline{\partial_z (vw)} = 0.$$

$$\theta = \tfrac{f}{N} (\nabla_h^2 \mathcal{R} - \partial_z Q), \quad \psi = Q + \tfrac{f^2}{N^2} \partial_z \mathcal{R}, \quad \chi = \nabla_h^{-2} \partial_z w$$

From here, we can get to...

Reduced PDEs (viscous terms not included), e.g.,

QG results from keeping interactions involving only q

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_h \cdot \boldsymbol{\nabla}\right) q = 0, \qquad q = \left(\nabla_h^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\psi(\mathbf{x}, t)$$

$$\nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \mathbf{u}_h = \hat{\mathbf{z}} \times \boldsymbol{\nabla}\psi, \quad \theta = -\frac{f}{N}\frac{\partial \psi}{\partial z}$$

or the antithesis of QG

The energy-conserving GGG model results from eliminating all interactions involving q:

$$\frac{\partial q}{\partial t} = 0$$

$$\frac{\partial f \nabla_h^2 R}{\partial t} - N^2\, Ow + N\nabla_h^2[(\mathbf{u}' \cdot \nabla)\theta'] + f\partial_z(\hat{z} \cdot \nabla \times (\mathbf{u}' \cdot \nabla \mathbf{u}') = 0$$

$$\frac{\partial \nabla^2 w}{\partial t} + f\nabla_h^2 R + \nabla_h^2(\mathbf{u}' \cdot \nabla w) - \partial_z(\nabla_h \cdot (\mathbf{u}' \cdot \nabla \mathbf{u}'_h)) = 0$$

$$\frac{\partial \overline{u(z)}}{\partial t} - f\overline{v(z)} + \overline{\partial_z(u'w)} = 0$$

$$\frac{\partial \overline{v(z)}}{\partial t} + f\overline{u(z)} + \overline{\partial_z(v'w)} = 0$$

with definitions:

$$u' \equiv \chi_x - \frac{f^2}{N^2}\mathcal{R}_{zy} + \overline{u(z)}, \quad v' \equiv \chi_y + \frac{f^2}{N^2}\mathcal{R}_{zx} + \overline{v(z)}$$

$$w' \equiv w, \qquad \theta' \equiv \frac{f}{N}\nabla_h^2\mathcal{R}$$

to eliminate interactions involving q from the nonlinear terms.

Questions:

Can 3-wave interactions (resonant, near-resonant and non-resonant) support a forward transfer of energy with power-law scaling of the wave energy?

What is the power-law scaling? How does it compare with the family of solutions found by Lvov & Tabak? How does it compare with observations?

Can we learn more about generation of VSHF?
Embid & Majda (1998), Smith & Waleffe (2002), Laval, McWilliams & Dubrulle (2003), Waite & Bartello (2006)

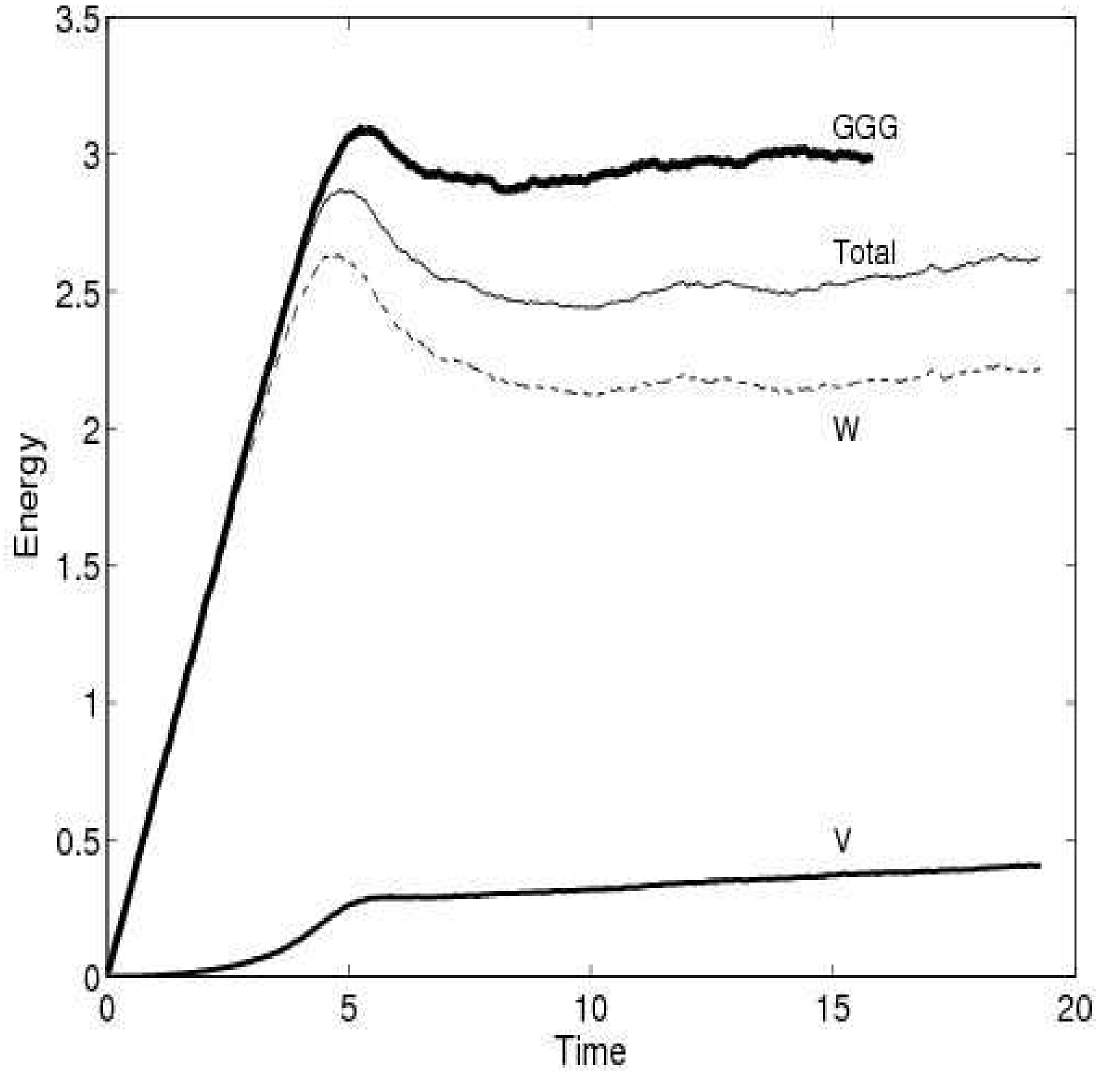
2nd question:

High-resolution simulations to determine wave-mode spectral scaling:

IBM Blue Gene/P at Argonne National Laboratory, DOE Innovative and Novel Computational Impact on Theory and Experiment (INCITE).

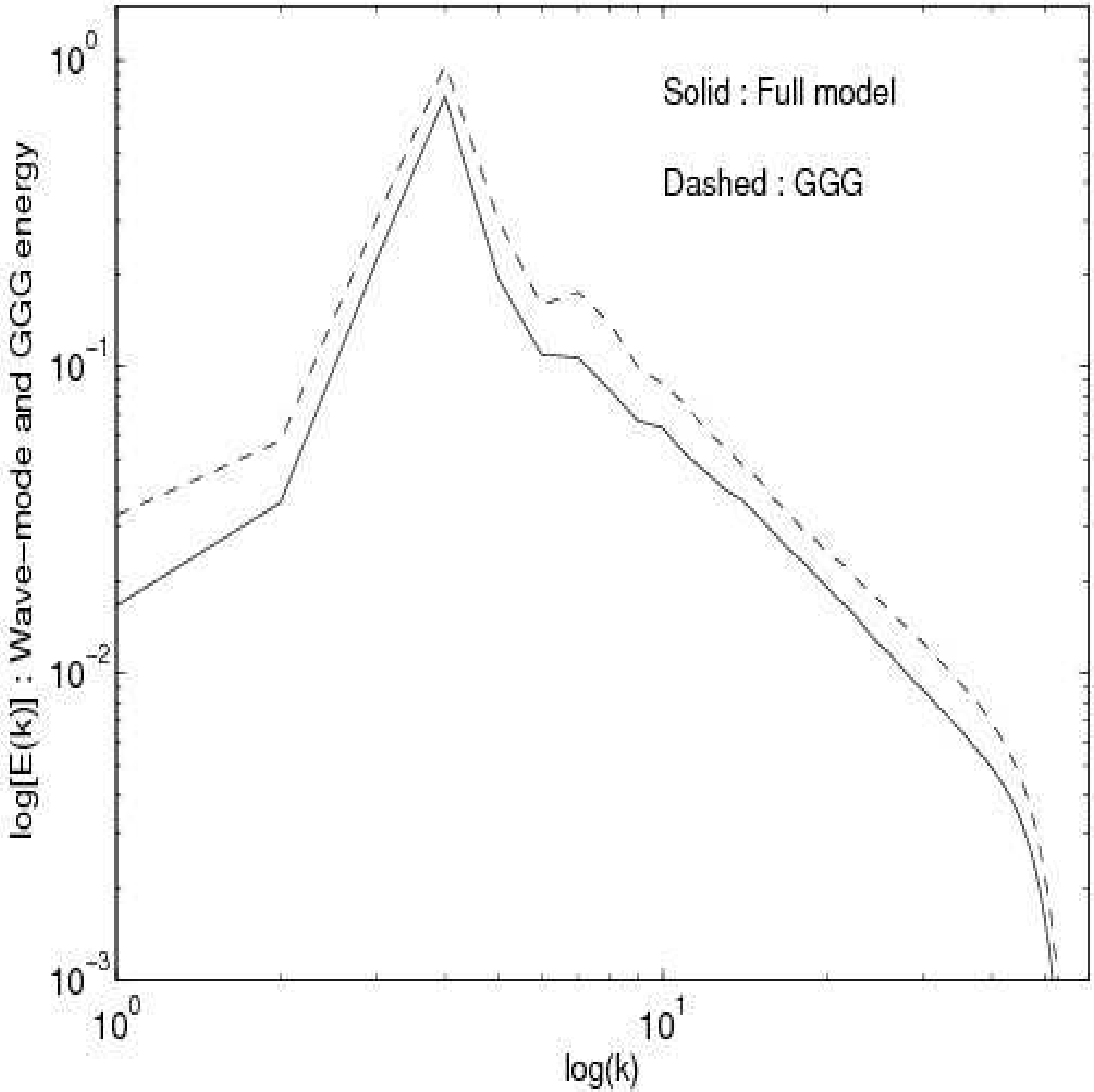
S. Kurien (LANL), M. Taylor (Sandia) & R. Balakrishnan (Argonne)

GGG vs. RBE Part I: $Bu = O(1)$ flows in $H/L < 1$



$H/L = 1/3$, $Ro = Fr = 0.1$, 162×486^2, identical IG-mode forcing.

Spectra at $t = 17.7$ sec.



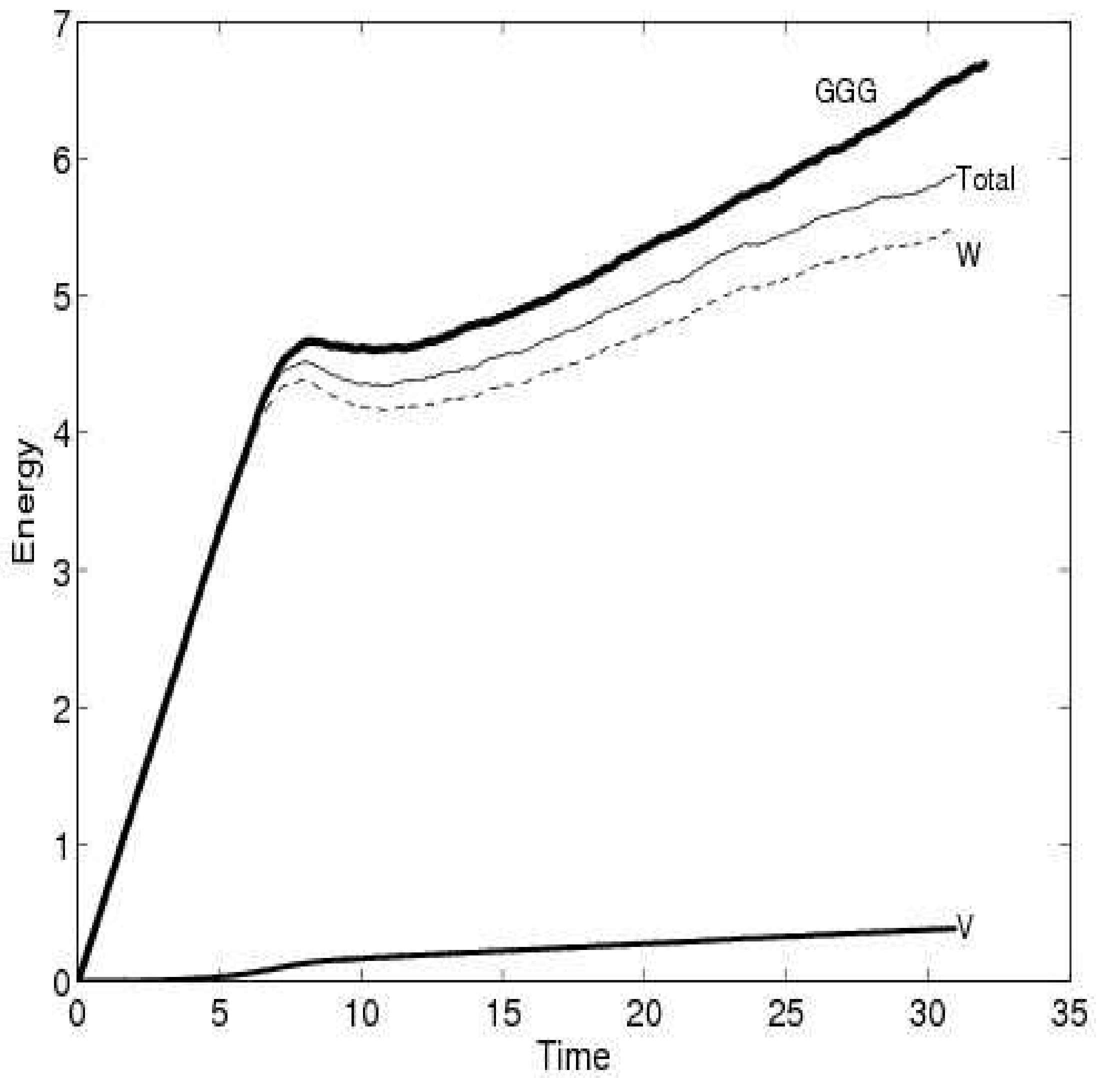
(W,W,W) scaling;   (W,V,W) help downscale transfer

$Bu = 1$, **moderate** Fr, Ro, H/L

3-wave (near-resonant) interactions are capable of forward energy transfer resulting in a forced-dissipative steady state.

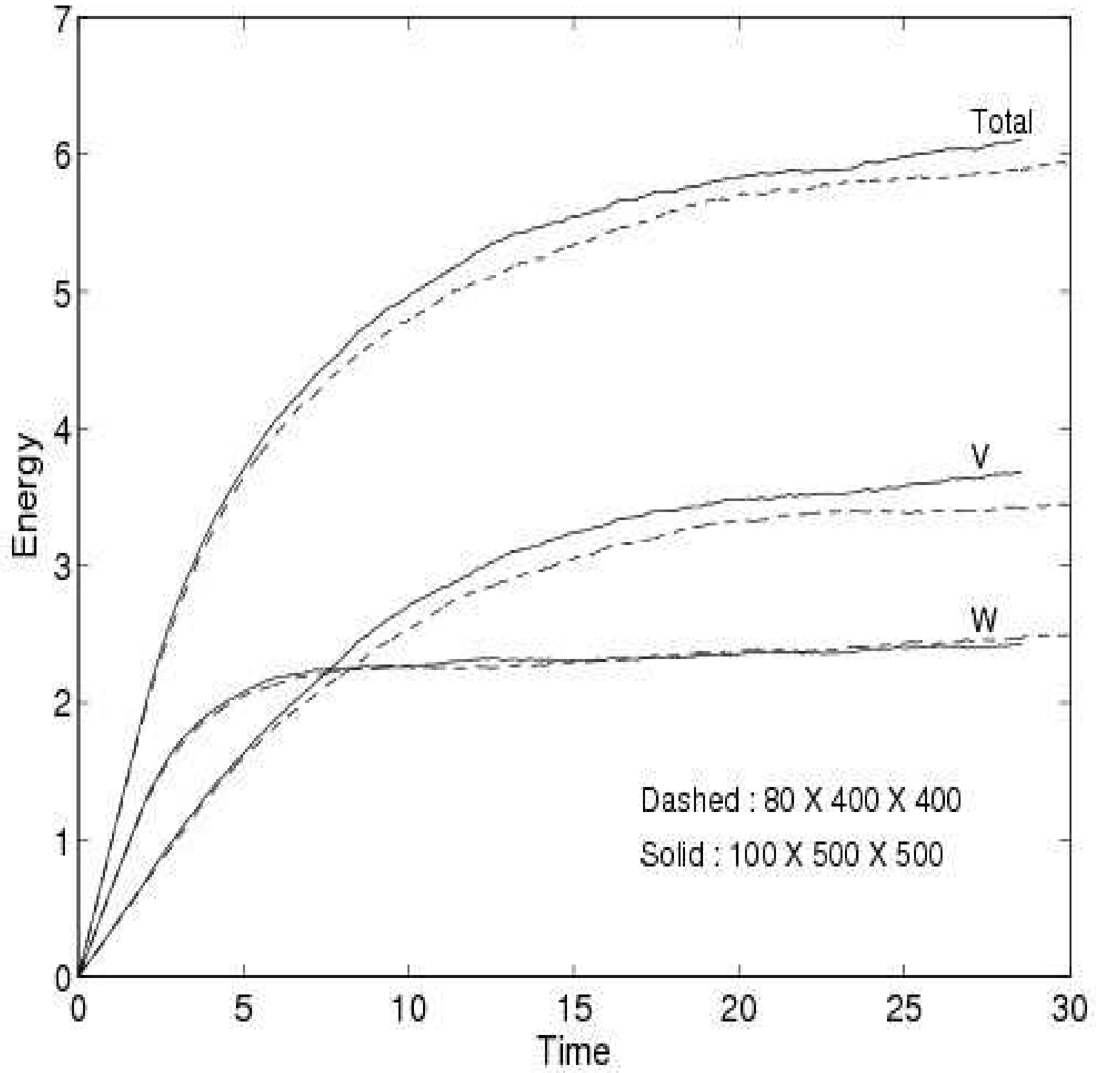
3-wave (near-resonant) interactions will be important for determining wave-mode spectra under the influence of unbalanced high-frequency forcing.

Smaller $Fr = Ro = 0.05$, $H/L = 1/5$, 100×500^2



RBE and GGG with IG-mode forcing: Under-resolved!

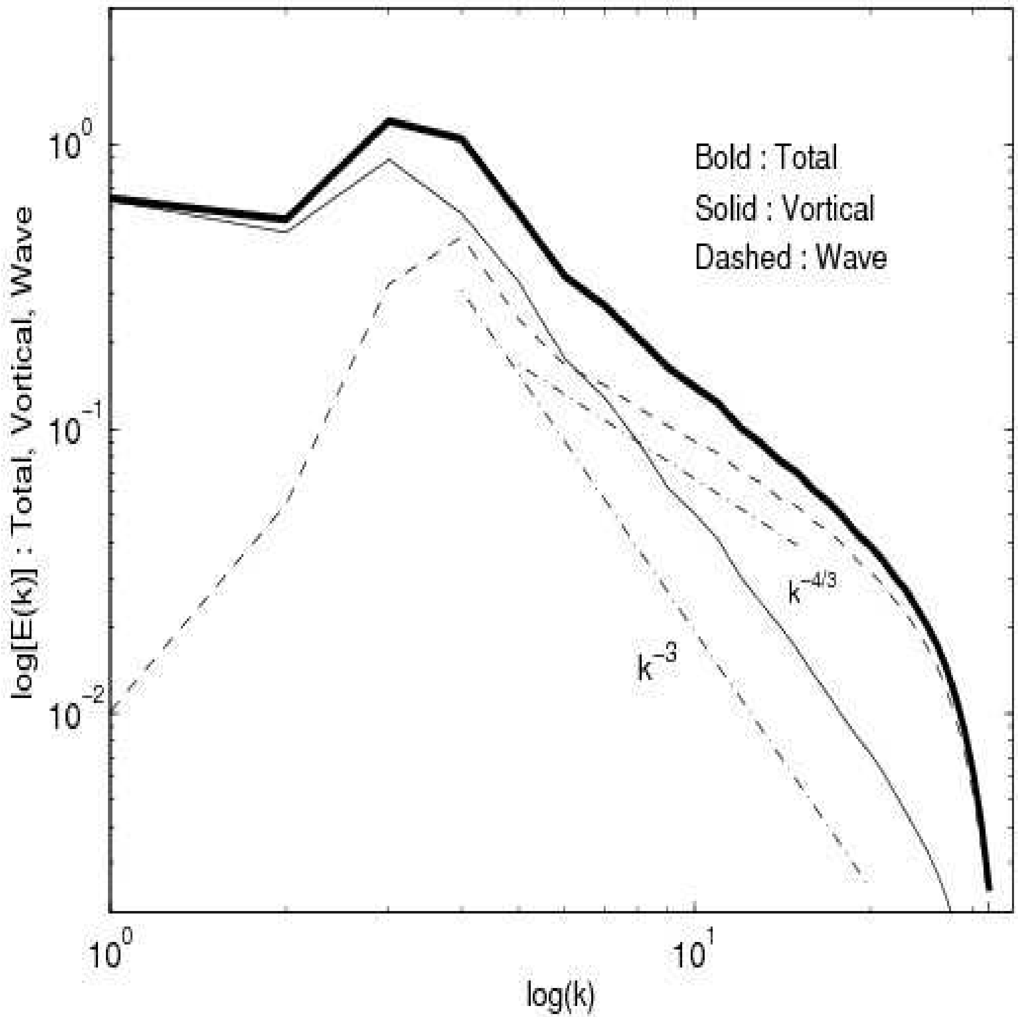
Smaller $Fr = Ro = 0.05$, $H/L = 1/5$, 100×500^2



RBE with all modes forced

$Fr = Ro = 0.05$, $H/L = 1/5$, 100×500^2, **all modes forced**

Calculations appear converged, but are wave modes under-resolved? Scalings suspect...



Part II. Purely Stratified Flow

$$\frac{\partial \nabla_h^2 \psi}{\partial t} + \hat{z} \cdot (\nabla \times (\mathbf{u} \cdot \nabla)\mathbf{u}) = 0$$

$$\frac{\partial N \nabla_h^2 \theta}{\partial t} - N^2 \nabla_h^2 w + N \nabla_h^2 (\mathbf{u} \cdot \nabla \theta) = 0$$

$$\frac{\partial \nabla^2 w}{\partial t} + N \nabla_h^2 \theta + \nabla_h^2 (\mathbf{u} \cdot \nabla w) - \partial_z \nabla_h \cdot (\mathbf{u} \cdot \nabla \mathbf{u_h}) = 0$$

$$\frac{\partial \overline{u(z)}}{\partial t} + \overline{\partial_z (uw)} = 0$$

$$\frac{\partial \overline{v(z)}}{\partial t} + \overline{\partial_z (vw)} = 0$$

$$\frac{\partial \overline{\theta(z)}}{\partial t} + \overline{\partial_z (\theta w)} = 0$$

From here, we can get to

Reduced PDEs (viscous terms not included), e.g.,

Interactions among slow vortical modes (excluding waves):

$$\frac{\partial \nabla_h^2 \psi}{\partial t} - J(\nabla_h^2 \psi, \psi) = 0$$

is conservation of vertical vorticity.

GGG = waves only

$$\frac{\partial \nabla_h^2 \psi}{\partial t} = 0$$

$$\frac{\partial N \nabla_h^2 \theta}{\partial t} - N^2 \nabla_h^2 w + N \nabla_h^2 (\mathbf{u}' \cdot \nabla \theta') = 0$$

$$\frac{\partial \nabla^2 w}{\partial t} + N \nabla_h^2 \theta + \nabla_h^2 (\mathbf{u}' \cdot \nabla w) - \partial_z \nabla_h \cdot (\mathbf{u}' \cdot \nabla \mathbf{u}'_{\mathbf{h}}) = 0$$

$$\frac{\partial \overline{u(z)}}{\partial t} + \overline{\partial_z (u'w)} = 0, \quad \frac{\partial \overline{v(z)}}{\partial t} + \overline{\partial_z (v'w)} = 0, \quad \frac{\partial \overline{\theta(z)}}{\partial t} = 0$$

$$u' = \chi_x + \overline{u(z)}, \quad v' = \chi_y + \overline{v(z)}, \quad w' = w, \quad \theta' = \theta - \overline{\theta(z)}$$

What can we learn about response to unbalanced, high-frequency forcing?

Waite & Bartello (2006)

An additional issue: growth of the VSHF

(VSHF, V, W) vs. (VSHF, W, W)

Note: we never force VSHF directly

Sam Stechmann's talk on convectively coupled gravity waves in shear

Back to physical space:

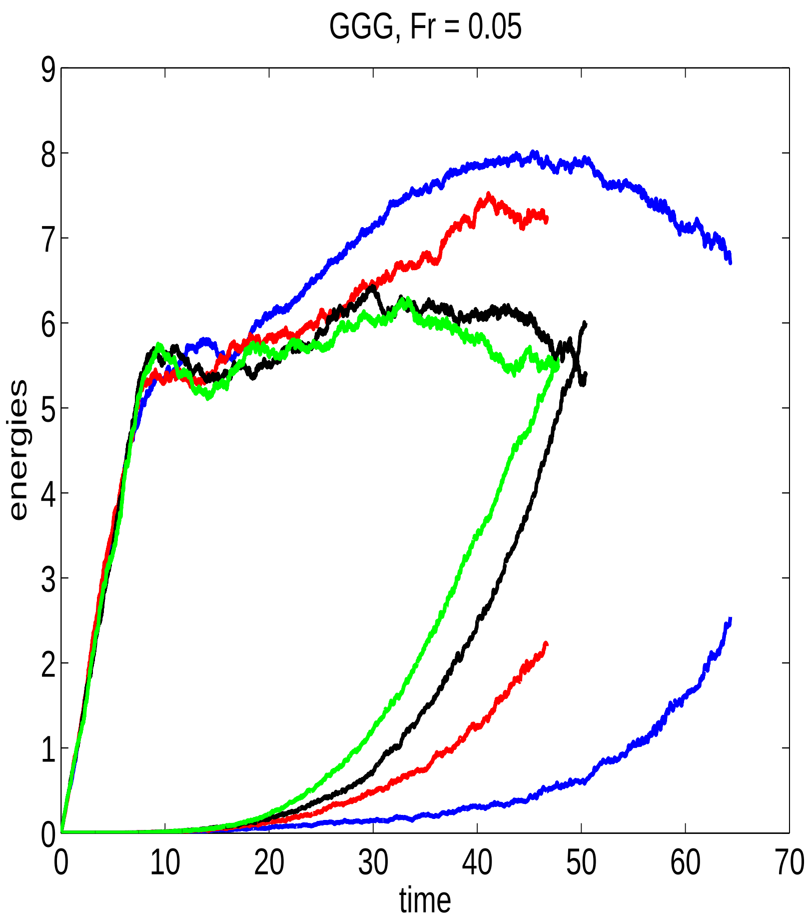
Generation of VSHF in GGG: (VSHF,$\pm, \pm$):

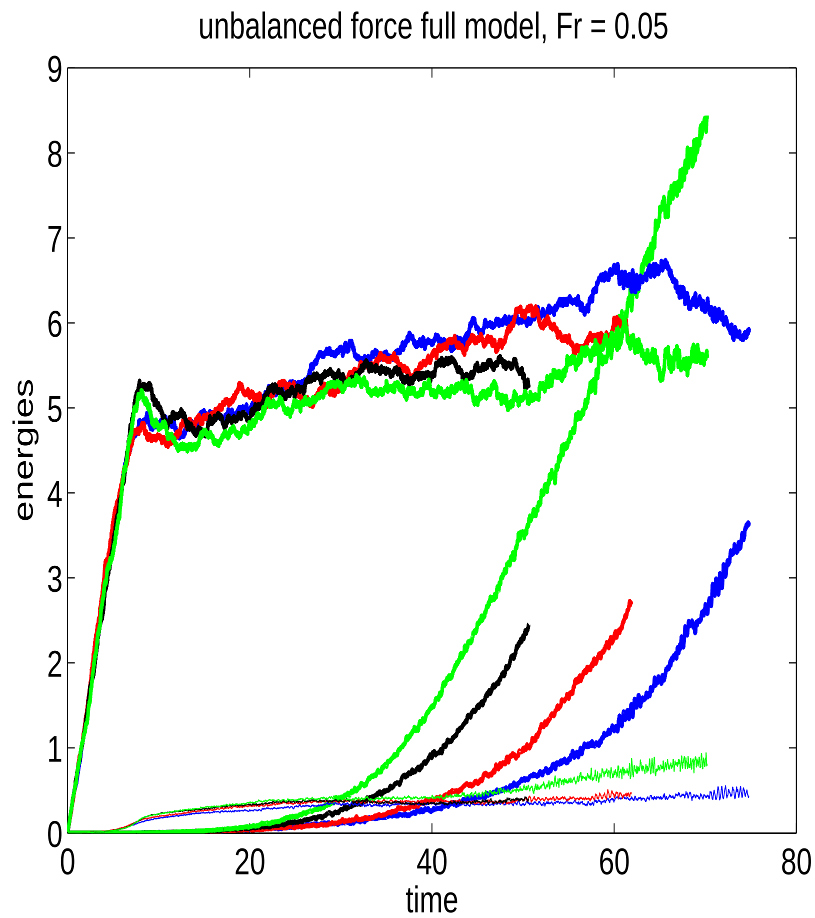
$$\partial_t \overline{u} = -\overline{\partial_z \chi_x w}, \quad \partial_t \overline{v} = -\overline{\partial_z \chi_y w}$$

In the full model we also have: (VSHF, 0, $\pm$)

$$\partial_t \overline{u} = -\overline{\partial_z \psi_x w}, \quad \partial_t \overline{v} = -\overline{\partial_z \psi_y w}$$

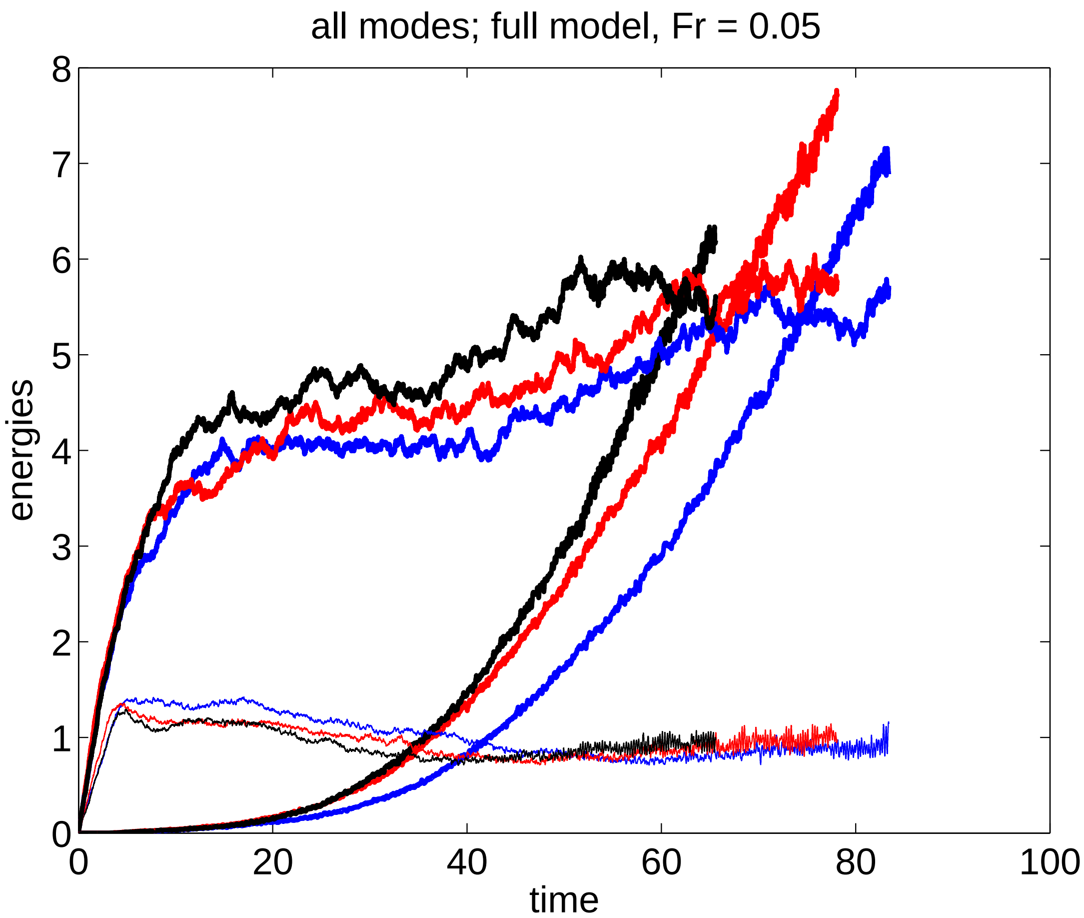
Energies in time





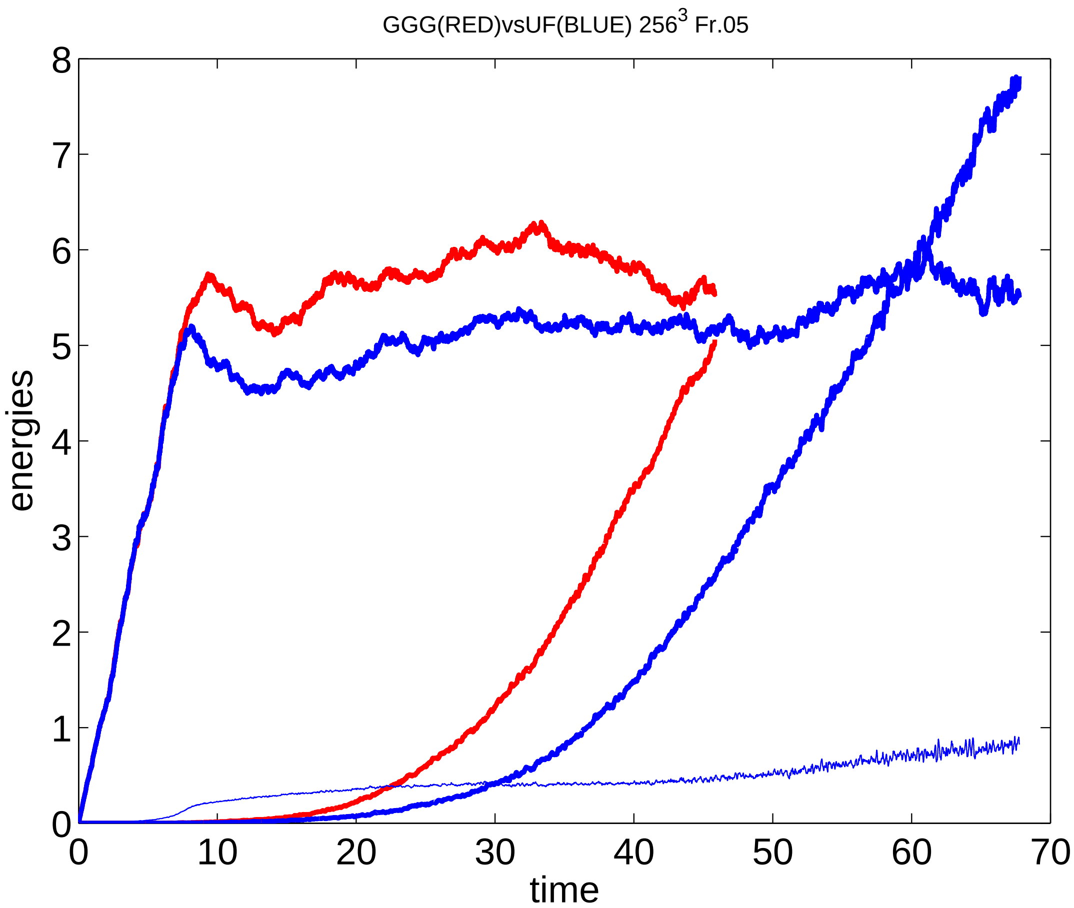
blue 128^3, red 162^3, black 192^3, green 256^3

Full model, all modes forced (except VSHF)



blue 128^3, red 162^3, black 192^3

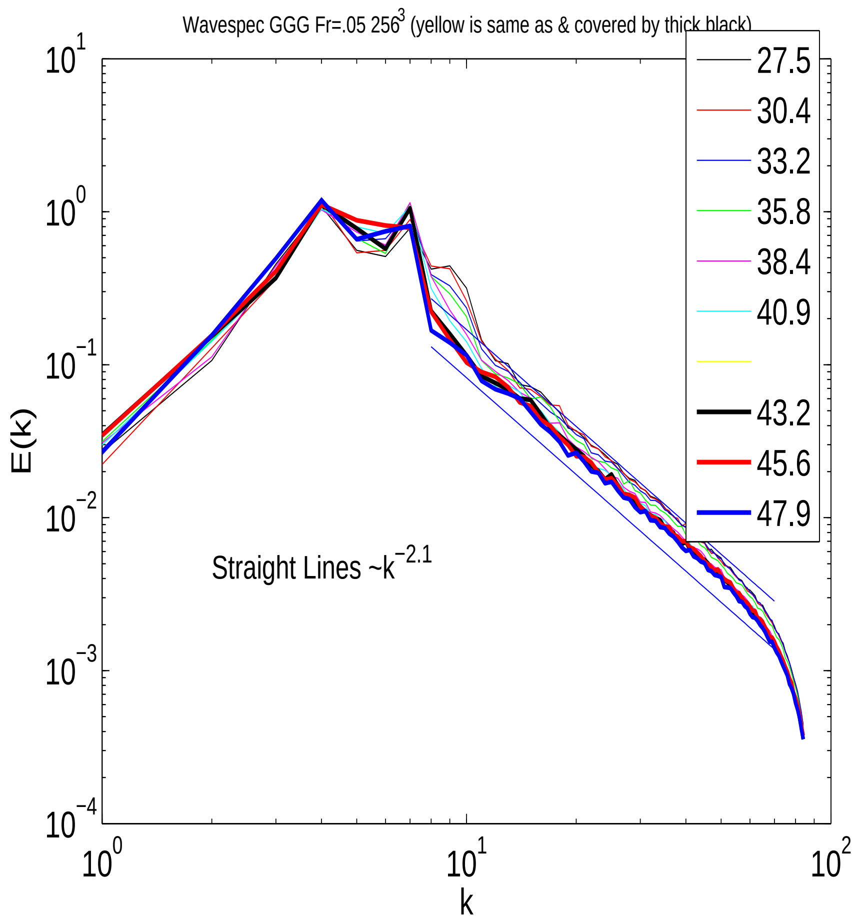
Compare growth of VSHF

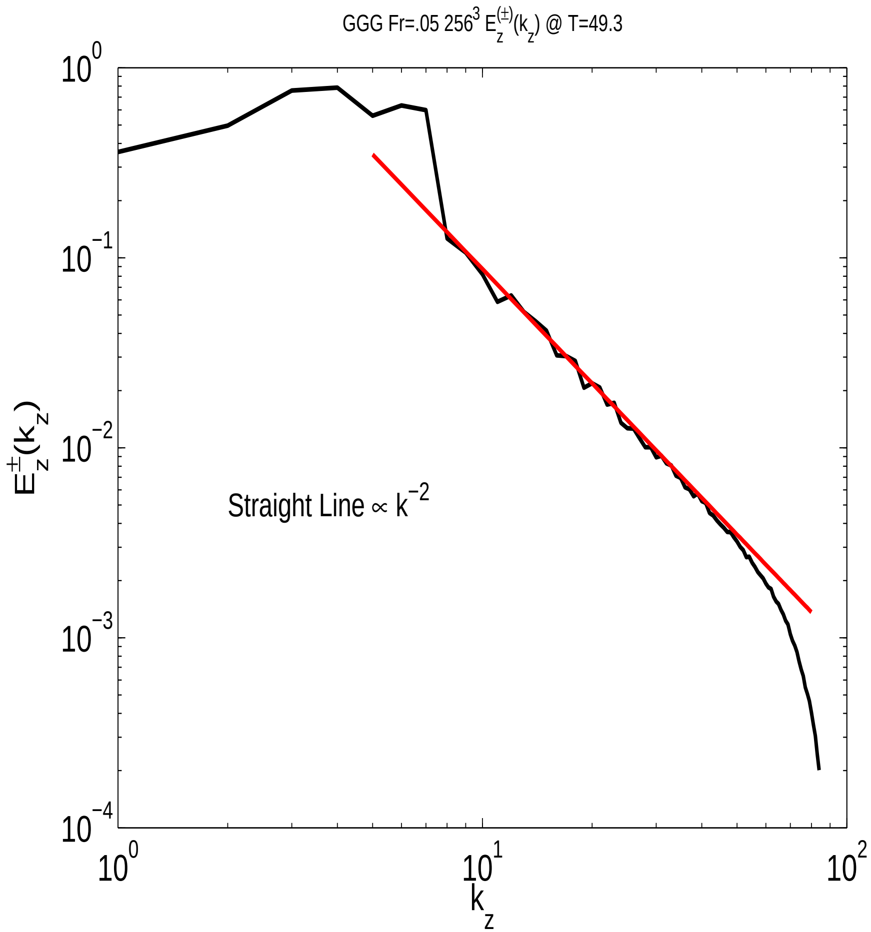


red GGG; blue unbalanced forcing, full model

clearly $\partial_t \overline{u} = -\overline{\partial_z \chi_x w}, \quad \partial_t \overline{v} = -\overline{\partial_z \chi_y w}$ most important

GGG Spectra (with caution), $Fr = 0.05$, 256^3





Garrett-Munk $E_z(k_z) \propto k_z^{-2}$

Future work on waves in simplified moisture models

KM(2006), MMX (2008), MX (2009)

$$\frac{D\mathbf{u}}{Dt} = ... + \frac{1}{Fr}(T + \bar{\epsilon} q_v + q_l)\hat{\mathbf{k}}$$

$$\frac{DT}{Dt} = ... + \delta^+ \; A \; F_c \; (q_v - q_s) - \delta^- \; A \; F_e \; q_l$$

$$\frac{Dq_v}{Dt} = -\delta^+ \; F_c \; (q_v - q_s) + \delta^- \; F_e \; q_l$$

$$\frac{Dq_l}{Dt} + F_v \; \frac{\partial q_l}{\partial z} = \delta^+ \; F_c \; (q_v - q_s) - \delta^- \; F_e \; q_l$$

$q_v, q_l =$ mixing ratios of water vapor and liquid water
$q_s(z) =$ saturation profile, $F_v = V_t/U, \;\; F_c = \tau_a/\tau_c, \;\; F_e = \tau_a/\tau_e$
$\delta^+ = H(q_v - q_s)$, $\quad \delta^- = H(q_s - q_v), \;\; A = L/(C_m T_o)$

Conclusions

Keeping fast inertia-gravity waves can qualitatively and quantitatively change flow dynamics (e.g. asymmetries, shear flow growth)

3-wave interactions set up their own forced-dissipative statistically steady state for $Bu = O(1)$ (when they are resolved!)

3-wave (near-resonant) interactions are generators of shear $Bu \gg 1$

Spectra ?