Inverse Problem of Acousto-Optic Imaging

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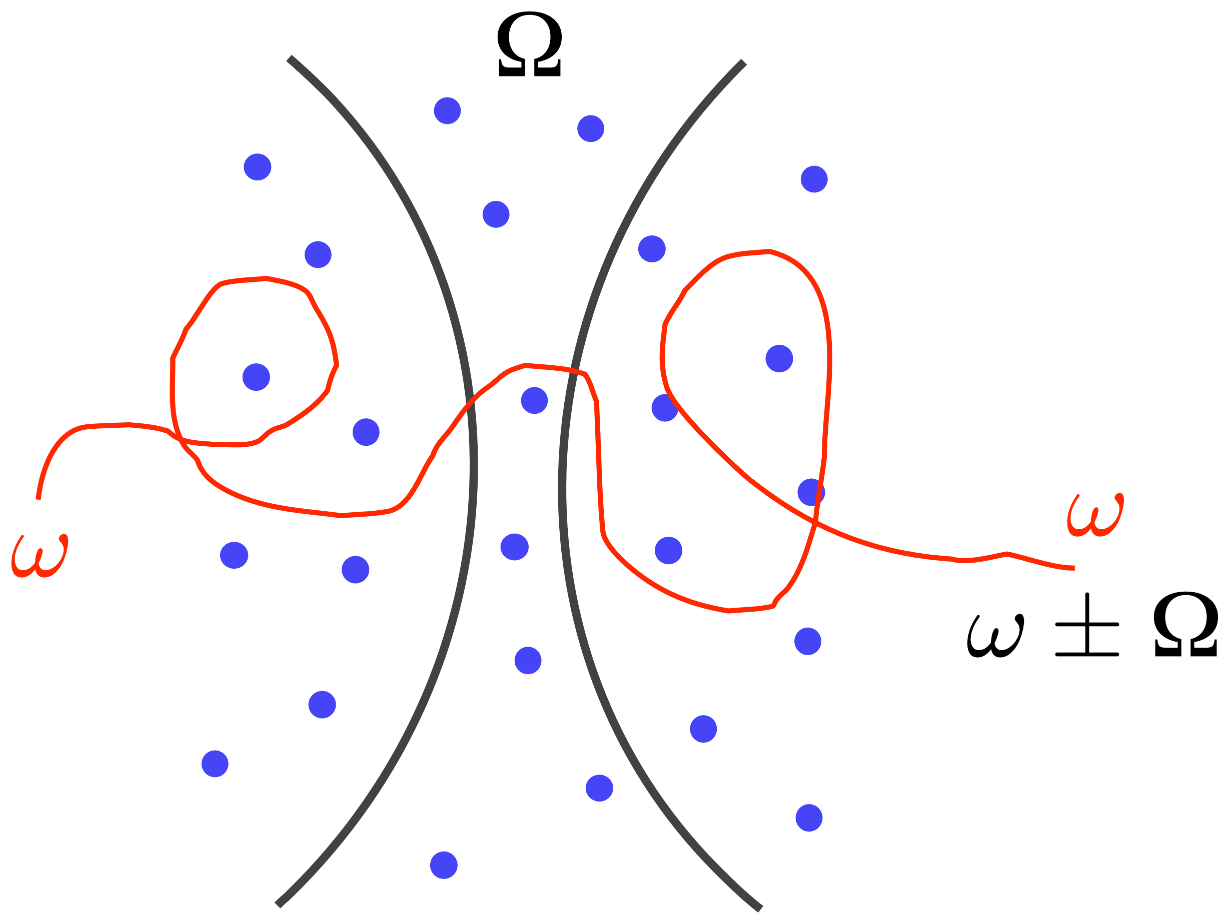
Joint work with Guillaume Bal

Inverse problems with interior control

A generic feature of many inverse problems in imaging is severe ill-posedness. Such ill-posedness typically results in reconstructed images with low spatial resolution. We will show that improvements in resolution are possible, in principle, if “internal degrees of freedom” of the scatterer are experimentally manipulated.

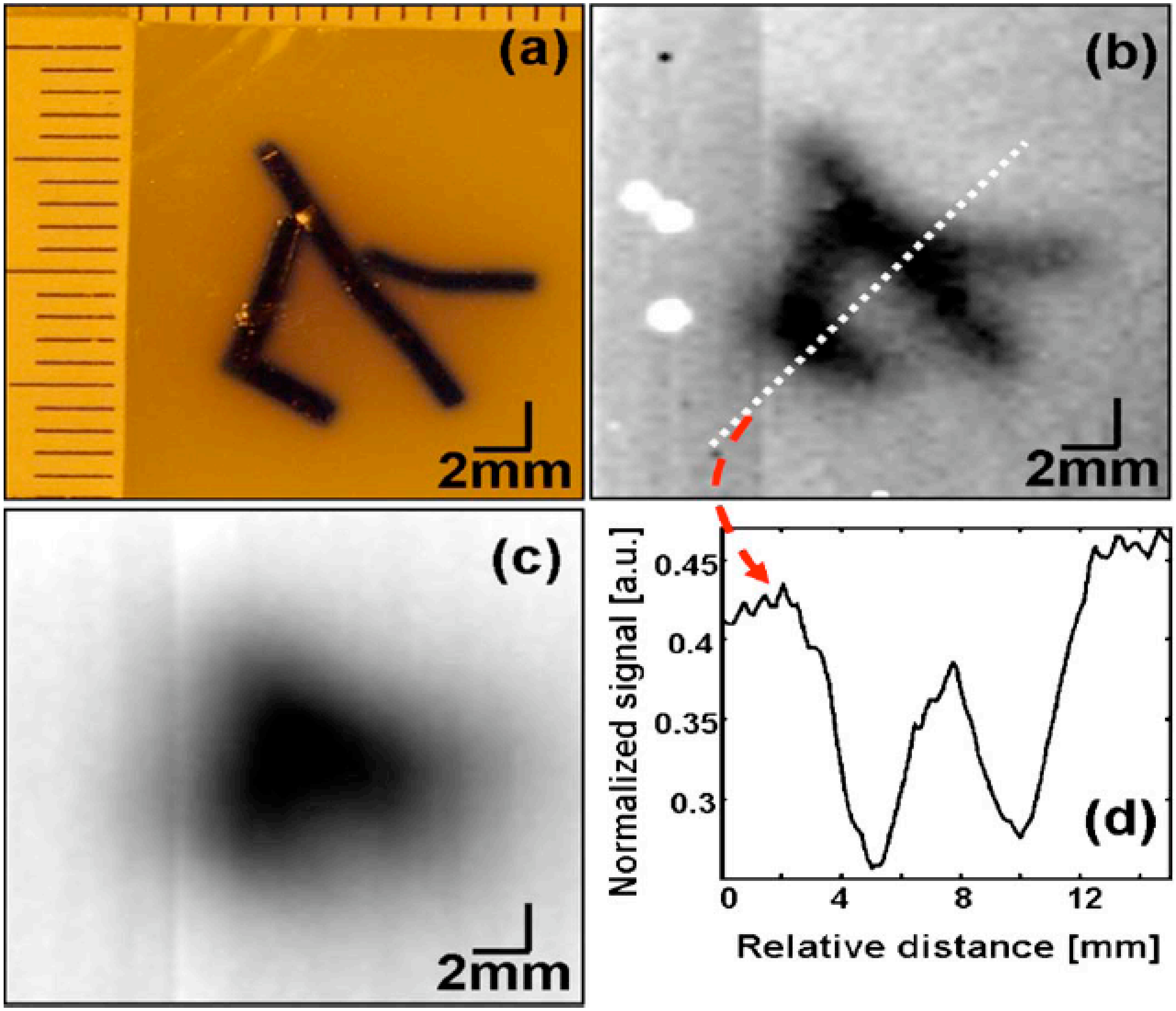
- Biomechanical imaging
- Current-density impedance imaging
- Acousto-electric impedance tomography
- Deconstructing the Born series
- Acousto-optic imaging
- Many more...

Acousto-optic imaging



$$I(\omega \pm \Omega) \ll I(\omega)$$

Acousto-optic images

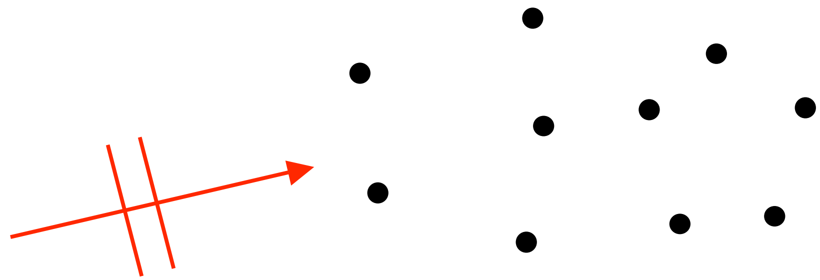


Courtesy of Lihong Wang

Overview

- In acousto-optic imaging (AOI) the images are not tomographic.
- We show that tomographic images can be reconstructed from elastically scattered light that is modulated by a standing acoustic wave.
- We take advantage of the separation of scales: $c_s \ll c$.
- Neither interferometric measurements of tagged photons nor the use of a focused acoustic wave field is required.

Model I



Consider a fluid suspension of scattering particles on which an acoustic wave is incident. The acoustic wave leads to spatial modulation of the number density of scatterers and the speed of light in the medium. If the amplitude of the pressure wave is sufficiently small, each particle will oscillate independently about its local equilibrium position. The equation of motion of a particle with velocity $\mathbf{u}$ is of the form

$$\rho \frac{d\mathbf{u}}{dt} = \frac{4\pi\eta a}{V}(\mathbf{v} - \mathbf{u}) - \nabla p \, ,$$

where η is the effective viscosity of the suspension, $\mathbf{v}$ is the velocity of the fluid and p is the pressure.

Model II

Consider a standing time-harmonic acoustic plane wave of frequency ω with pressure

$$p = A\cos(\omega t)\cos(\mathbf{k}\cdot\mathbf{r}+\varphi)\ , \tag{1}$$

where we have assumed that the speed of sound c_s is constant with $k = \omega/c_s$.

Let $\mathbf{R}_1,\ldots,\mathbf{R}_N$ denote the positions of the particles in the fluid. Then, since each particle moves independently

$$\mathbf{R}_i = \mathbf{R}_{0i} - \frac{A}{\rho\omega^2}\cos(\omega t)\sin(\mathbf{k}\cdot\mathbf{r}+\varphi)\mathbf{k}\ , \quad i = 1,\ldots,N\ .$$

The number density of particles is defined by $\varrho = \sum_i \delta(\mathbf{r}-\mathbf{R}_i(t))$. Define the small parameter $\epsilon = A\cos(\omega t)/(\rho c_s^2) \ll 1$. The number density is modulated according to

$$\varrho(\mathbf{r}) = \varrho_0(\mathbf{r})\left[1+\epsilon\cos(\mathbf{k}\cdot\mathbf{r}+\varphi)\right]\ ,$$

where $\varrho_0(\mathbf{r})$ is the equilibrium number density of the particles.

Propagation of diffuse light I

Within the textcolorreddiffusion approximation to the RTE, the energy density u obeys

$$-\nabla \cdot \left[D n^2 \nabla \left(\frac{u}{n^2} \right) \right] + c\mu_a u = 0 \quad \text{in} \quad \Omega \, ,$$
$$u + \ell \frac{\partial u}{\partial n} = \delta(\mathbf{r} - \mathbf{r}_0) \quad \text{on} \quad \partial\Omega \, ,$$

where μ_a is the absorption coefficient and n is the index of refraction. The diffusion coefficient D is defined by $D = c\left[\mu_a + (1-g)\mu_s\right]/3$, where μ_s is the scattering coefficient and g is the anisotropy of scattering. The scattering and absorption coefficients are related to the number density by $\mu_s = \varrho\sigma_s$ and $\mu_a = \varrho\sigma_a$, where σ_s and σ_a denote the scattering and absorption cross sections of the particles. We account for variations in the index of refraction according to $n(\mathbf{r}) = n_0\left[1 + \epsilon\gamma\cos(\mathbf{k}\cdot\mathbf{r} + \varphi)\right]$, where n_0 is the index of refraction in the absence of the acoustic wave and γ is the elasto-optical constant. The optical properties of the medium are thus modulated by the acoustic wave.

Propagation of diffuse light II

Define $\psi_\epsilon = u/n^2$ and $g = n^2\delta(\mathbf{r} - \mathbf{r}_0)$. We then have

$$-\nabla \cdot D_\epsilon \nabla \psi_\epsilon + \alpha_\epsilon \psi_\epsilon = 0 \quad \text{in} \quad \Omega \,,$$
$$\psi_\epsilon + \ell \frac{\partial \psi_\epsilon}{\partial n} = g \quad \text{on} \quad \partial\Omega \,.$$

Here

$$\begin{aligned} \alpha_\epsilon(\mathbf{r}) &= \alpha_0(\mathbf{r}) \left[1 + \epsilon(2\gamma + 1)\cos(\mathbf{k} \cdot \mathbf{r} + \varphi)\right] \,, \\ D_\epsilon(\mathbf{r}) &= D_0(\mathbf{r}) \left[1 + \epsilon(2\gamma - 1)\cos(\mathbf{k} \cdot \mathbf{r} + \varphi)\right] \,, \end{aligned}$$

where α_0 and D_0 are the absorption and diffusion coefficients in the absence of the acoustic wave.

The solution to the above PDE is given by

$$\psi_\epsilon(\mathbf{r}) = \psi_0(\mathbf{r}) - \epsilon \int d^3r' \big[(2\gamma + 1) G(\mathbf{r}, \mathbf{r}')\psi_0(\mathbf{r}')\alpha_0(\mathbf{r}')$$
$$+ (2\gamma - 1)\nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') \cdot \nabla \psi_0(\mathbf{r}') D_0(\mathbf{r}')\big] \cos(\mathbf{k} \cdot \mathbf{r}' + \varphi) + O(\epsilon^2) \,.$$

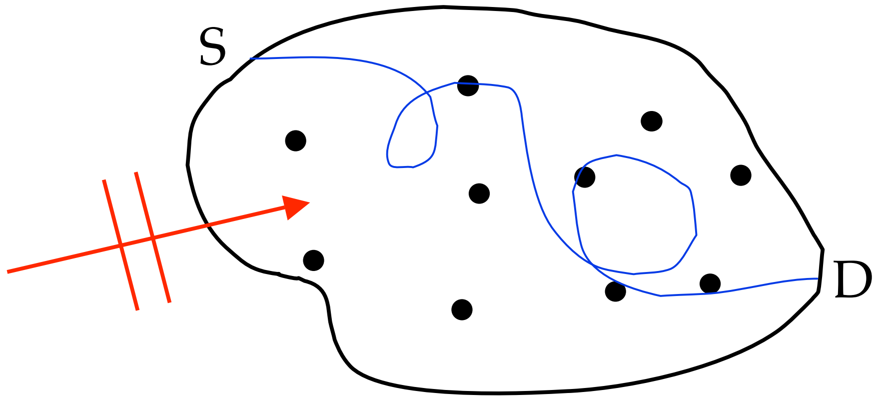
Size of acousto-optic effect

The relative change in intensity due to the presence of an acoustic plane wave is of the order

$$\frac{\Delta I}{I} \approx \epsilon \left[1 + (\kappa L)^2\right] e^{-\kappa L} ,$$

where L is the source-detector separation and $\kappa = \sqrt{\alpha/D}$. Choosing typical values of the above parameters in tissue: $\kappa = 1\ \mathrm{cm}^{-1}$, $L = 1\ \mathrm{cm}$ and $\epsilon = 10^{-3}$. To estimate ϵ, choose $\rho = 1\ \mathrm{g\ cm}^{-3}$, $c_s = 1.5 \times 10^5\ \mathrm{cm\ s}^{-1}$ and $A = 10^6\ \mathrm{Pa}$. We find that $\Delta I/I \approx 10^{-3}$ which is expected to be observable. Note $\Delta I/I$ would be significantly smaller for the case of a focused beam, assuming equivalent incident power.

The experiment



Fix the source and detector and measure the transmitted intensity as the wavevector of the acoustic wave is varied

Inverse problem

The inverse problem of AOI is to reconstruct α_0 and D_0 from ψ_ϵ. Define $\phi = \partial\psi/\partial\epsilon|_{\epsilon=0}$, which can be determined from measurements carried out in the presence and absence of the acoustic wave.

$$\phi(\mathbf{r}) = \int K(\mathbf{r}, \mathbf{r}') \cos(\mathbf{k} \cdot \mathbf{r}' + \varphi) d^3 r' \ ,$$

where

$$K(\mathbf{r}, \mathbf{r}') = [(2\gamma + 1) G(\mathbf{r}, \mathbf{r}') \psi_0(\mathbf{r}') \alpha_0(\mathbf{r}') + (2\gamma - 1) \nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') \cdot \nabla \psi_0(\mathbf{r}') D_0(\mathbf{r}')] \ .$$

Suppose we fix the positions of the optical source and detector and vary the wave vector $\mathbf{k}$ and the phase φ. It is then possible to recover $K(\mathbf{r}, \mathbf{r}')$ by inversion of a Fourier transform.

Inverse problem (iterative method)

Recall that we know K from measurements.

$K(\mathbf{r}, \mathbf{r}') = [(2\gamma + 1)G(\mathbf{r}, \mathbf{r}')\psi_0(\mathbf{r}')\alpha_0(\mathbf{r}') + (2\gamma - 1)\nabla_{\mathbf{r}'}G(\mathbf{r}, \mathbf{r}') \cdot \nabla\psi_0(\mathbf{r}')D_0(\mathbf{r}')]$.

Suppose that $\alpha_0 = 0$. Then $D_0 = \mathcal{A}[D_0]$, where the nonlinear operator $\mathcal{A}$ is defined by

$$\mathcal{A}[D_0](\mathbf{r}') = \frac{K(\mathbf{r}, \mathbf{r}')}{(2\gamma - 1)\nabla_{\mathbf{r}'}G(\mathbf{r}, \mathbf{r}') \cdot \nabla\psi_0(\mathbf{r}')} \; .$$

Applying fixed-point iteration we have

$$D_0^{(n+1)} = \mathcal{A}[D_0^{(n)}] \, , \quad n = 1, 2, \ldots \; .$$

At each step it is necessary to compute the Green's function G, which depends upon the current estimate of D_0. Measurements from two independent sources are required to reconstruct both α_0 and D_0.

Inverse problem and the 0-Laplacian

For a point source, we have seen that

$$f(\mathbf{r}) = (2\gamma - 1)D_0(\mathbf{r})(\nabla\psi_0(\mathbf{r}))^2 + (2\gamma + 1)\alpha_0(\mathbf{r})\psi_0^2(\mathbf{r})$$

is known from measurements.

Suppose that $\alpha_0 = 0$. Then

$$D_0(\mathbf{r}) = \frac{f(\mathbf{r})}{(2\gamma - 1)(\nabla\psi_0(\mathbf{r}))^2} \; .$$

We find that ψ_0 obeys the nonlinear equation

$$\begin{aligned} \nabla\cdot\left[\frac{f}{(\nabla\psi_0)^2}\nabla\psi_0\right] &= 0 \quad \text{in} \quad \Omega \, , \\ \psi_0 + \ell\frac{\partial\psi_0}{\partial n} &= g \quad \text{on} \quad \partial\Omega \, . \end{aligned}$$

Once ψ_0 is found by solving the above PDE, we can recover the diffusion coefficient D_0.

Reconstruction of absorption and scattering

Consider the problem of recovering both α_0 and D_0. We require data from two sources g_1 and g_2.

$$f_k(\mathbf{r}) = (2\gamma - 1)D_0(\mathbf{r})(\nabla\psi_k(\mathbf{r}))^2 + (2\gamma + 1)\alpha_0(\mathbf{r})\psi_k^2(\mathbf{r}) \ , \quad k = 1, 2 \ .$$

Solving for α_0 and D_0 we find

$$\begin{aligned} \alpha_0(\mathbf{r}) &= \frac{f_1(\mathbf{r})(\nabla\psi_2(\mathbf{r}))^2 - f_2(\mathbf{r})(\nabla\psi_1(\mathbf{r}))^2}{(2\gamma+1)\left[\psi_1^2(\mathbf{r})(\nabla\psi_2(\mathbf{r}))^2 - \psi_2^2(\mathbf{r})(\nabla\psi_1(\mathbf{r}))^2\right]} \ , \\ D_0(\mathbf{r}) &= \frac{f_2(\mathbf{r})\psi_1^2(\mathbf{r}) - f_1(\mathbf{r})\psi_2^2(\mathbf{r})}{(2\gamma-1)\left[\psi_1^2(\mathbf{r})(\nabla\psi_2(\mathbf{r}))^2 - \psi_2^2(\mathbf{r})(\nabla\psi_1(\mathbf{r}))^2\right]} \ . \end{aligned}$$

The ψ_k are then obtained by solving the system of nonlinear equations

$$\begin{aligned} -\nabla \cdot D_0 \nabla\psi_k + \alpha_0\psi_k &= 0 \quad \text{in} \quad \Omega \ , \\ \psi_k + \ell\frac{\partial\psi_k}{\partial n} &= g_k \quad \text{on} \quad \partial\Omega \ , \end{aligned}$$

Once the ψ_k are found, we can then recover α_0 and D_0.

Remarks

The inverse problem of AOI is expected to be well-posed, requiring only a Fourier inversion followed by fixed-point iteration. In contrast, the inverse problem of optical tomography is severely ill-posed with logarithmic stability. This ill-posedness is responsible for the relatively low resolution of the method.

The 0-Laplacian was studied by Ammari and collaborators in a different physical context.

The diffusion approximation is valid when the energy density varies slowly on the scale of the transport mean free path. This condition breaks down when the acoustic wavelength is sufficiently small. It would thus be of interest to extend the theory we have developed to the transport regime.

Open problems:

convergence, stability, error estimates for iterative algorithm

existence, uniqueness for 0-Laplacian and its generalization

Conclusions

We have developed a tomographic method for acousto-optic imaging.

Neither interferometric measurements of tagged photons nor the use of a focused ultrasound beam is required.

Our approach is based on the solution to an inverse problem for the diffusion equation with interior control of boundary measurements.