Reconstruction in Doppler tomography

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1 Introduction

Doppler tomography is applied for
imaging of liquid or gas flows, ultrasound diagnostic, optics, plasma physics etc.
Physical background:

▲ Doppler spectroscopy (projection of ion velocity),
▲ Zeeman effect polarimetry (projection of the poloidal magnetic field),
▲ Doppler effect in moving medium:

1.1 Travel time measurements

c - the sound speed,
v - the local velocity of the medium,
$s = 0$, $s = S$ are the positions of the source and the receiver,
T - the travel time:

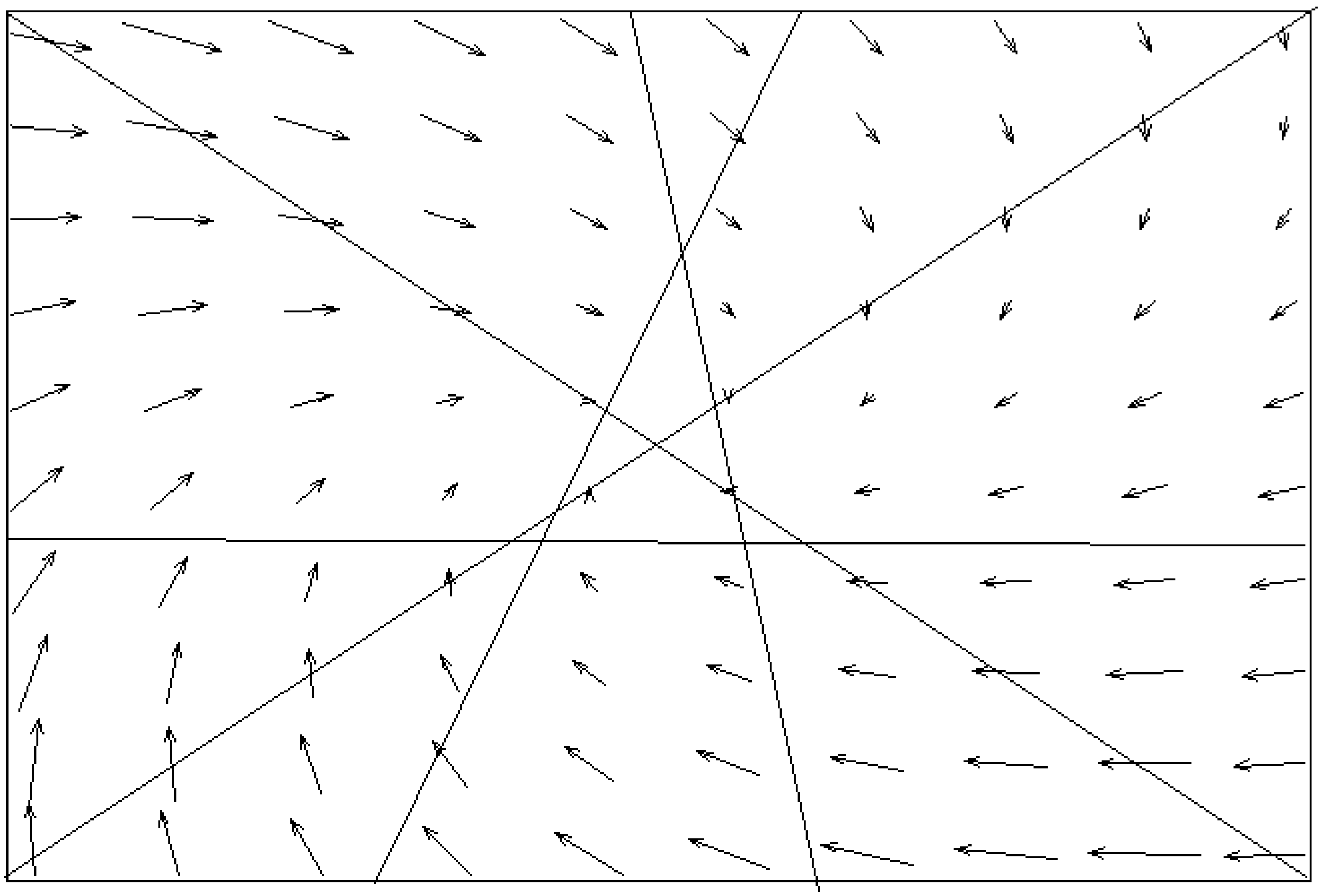
$$T = \int_0^S \frac{\mathrm{d}s}{c(x) + (\theta, v(x))},$$

If $|v| << c$, then

$$T \approx \int_0^S \frac{\mathrm{d}s}{c(x)} - \int_0^S \frac{(\theta, v(x))\,\mathrm{d}s}{c^2(x)}.$$

If $c(x) = c$, then

$$\int_0^S (\theta, v(x))\,\mathrm{d}s \approx \frac{S}{c} - T.$$



2 Differential forms and integrals

▼ Let $\mathrm{f} = \sum f_{i_1 \ldots i_k} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$ be a k-differential form in $\mathbf{V} \cong \mathtt{R}^3$, $k = 0, 1, 2, 3$.

0-form $a = a(x)$;

1-form $\mathrm{f} = f_1(x)\,\mathrm{d}x_1 + f_2(x)\,\mathrm{d}x_2 + f_3(x)\,\mathrm{d}x_3$;

2-form $\mathrm{g} = g_{12}(x)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 + g_{23}(x)\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + g_{31}(x)\,\mathrm{d}x_3 \wedge \mathrm{d}x_1$;

3-form $\mathrm{h} = h_{123}(x)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$.

Exterior differential: $\mathrm{f} = \mathrm{d}a,\ \mathrm{g} = \mathrm{df},\ \mathrm{h} = \mathrm{dg};\ \mathrm{dd} = 0$.

Coordinateless notations:

$\mathrm{f}(x;\theta) = f_1(x)\,\theta_1 + f_2(x)\,\theta_2 + f_3(x)\,\theta_3,\ x,\ \theta = (\theta_1, \theta_2, \theta_3) \in \mathbf{V}$,

$\mathrm{g}(x, \theta, \eta) = \frac{1}{2}\left[g_{12}(\theta_1\eta_2 - \theta_2\eta_1) + g_{23}(\theta_2\eta_3 - \theta_3\eta_2) + g_{31}(\theta_3\eta_1 - \theta_1\eta_3)\right]$,

$\mathrm{h}(x, \theta, \eta, \xi) = \frac{1}{6}\ldots$

Doppler transform: 7

▼ A function a defined on $\mathbf{V}$ is fast decreasing, if $a(x) = O\left(|x|^{-q}\right)$, as $|x| \to \infty$ in $\mathbf{V}$ for $q = 0, 1, 2, \ldots$.

▼ S_m is the space of 1-forms f such that the function $\mathrm{f}(x;\theta)$ is fast decreasing as well as all x-derivatives up to the order m for any fixed θ.

For a 1-form $\mathrm{f} \in \mathsf{S}_0$ the integral

$$\mathrm{R}(\rho) = \int_\rho \mathrm{f}$$

is defined for any oriented curve ρ in $\mathbf{V}$.

We have $\mathrm{R}(\mathrm{d}a, \lambda) = 0$ for any fast decreasing function a.

▲ A vector field $v = (v_1, v_2, v_3)$ is replaced by the 1-form $\mathrm{f} = v_1\mathrm{d}x_1 + v_2\mathrm{d}x_2 + v_3\mathrm{d}x_3$, so that

$$\int (\theta, v)\,\mathrm{d}s = \int_\lambda \mathrm{f}$$

Write $\mathrm{R}(x, \theta) = \mathrm{R}(\rho(x, \theta))$, where $\rho(x, \theta) = \{y = x + t\theta,\ t \geq 0\}$, that is

$$\mathrm{R}(x;\theta) = \int_0^\infty \mathrm{f}(x + s\theta; \theta)\,\mathrm{d}s, \quad x, \theta \in \mathbf{V}.$$

We have $\mathrm{R}(x, t\theta) = \mathrm{sgn}t\ \mathrm{R}(x, \theta)$ for any $t > 0$.

The sum $\mathrm{L}(x, \theta) = \mathrm{R}(x, \theta) - \mathrm{R}(x, -\theta)$ is equal to the integral of f over the line $\lambda(x, \theta) = \{y = x + t\theta, t \in \mathtt{R}\}$.

▲ The Doppler transform $\mathrm{R}(x, \theta)$ is invariant with respect to the gauge transformation $\mathrm{f} + \mathrm{d}a$, where a an arbitrary ast decreasing function, since $\mathrm{R}(\mathrm{d}a) = 0$.

▲ The differential df of a 1-form f is gauge invariant.

Inversion problem: *to recover the form* df *from knowledge of integrals* $\mathrm{R}(\mathrm{f}, \rho)$ *on a n-dimensional manifold Λ of rays ρ in $\mathbf{V}^n$.*

2.1 The case $n = 2$

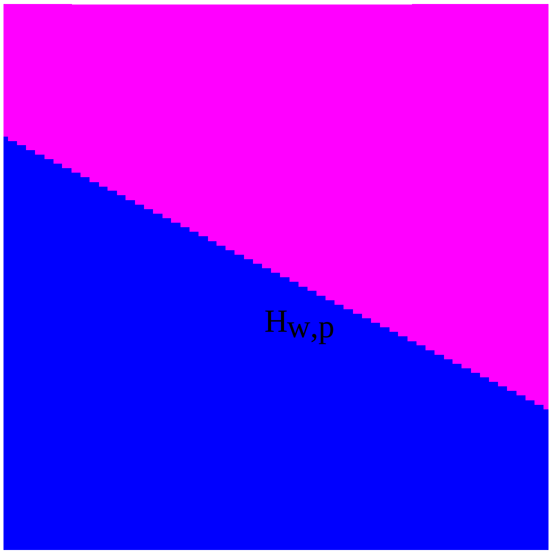
`Norton, Braun-Hauck, Juhlin, Sparr-Stråhlén,...Howard-Wells,...Osman-Prince,...`

Proposition *For an arbitrary 1-form* $\mathrm{f} \in \mathsf{S}_1$ *on a Euclidean plane* $\mathbf{V}$ *and any* $x \in \mathbf{V}$, $\theta \in \mathbf{V} \backslash \{0\}$

$$\mathrm{L}\,(x, \theta) = \int_{\lambda(x,\theta)} \mathrm{F}\mathrm{d}s = \partial_p \int_{H_{\omega,p}} \mathrm{df}, \tag{1}$$

where H *is the half-plane such that* $\partial H = \lambda\,(x, \theta)$.

◀ Apply the Cauchy-Green formula. ▶



▲ Write $\mathrm{df} = F\mathrm{d}S,$ where $\mathrm{d}S$ is the area element and F is a fast decreasing function in $\mathbf{V}$.

$$\partial_{\mathbf{p}} \int_{H_{\omega,\mathbf{p}}} \mathrm{df} = \partial_p \int_{H_{\omega,\mathbf{p}}} F\mathrm{d}S = \int_{\lambda(x,\theta)} F\mathrm{d}s$$

The right-hand side equals to the Radon transform of the function F.

2.2 The case $n = 3$

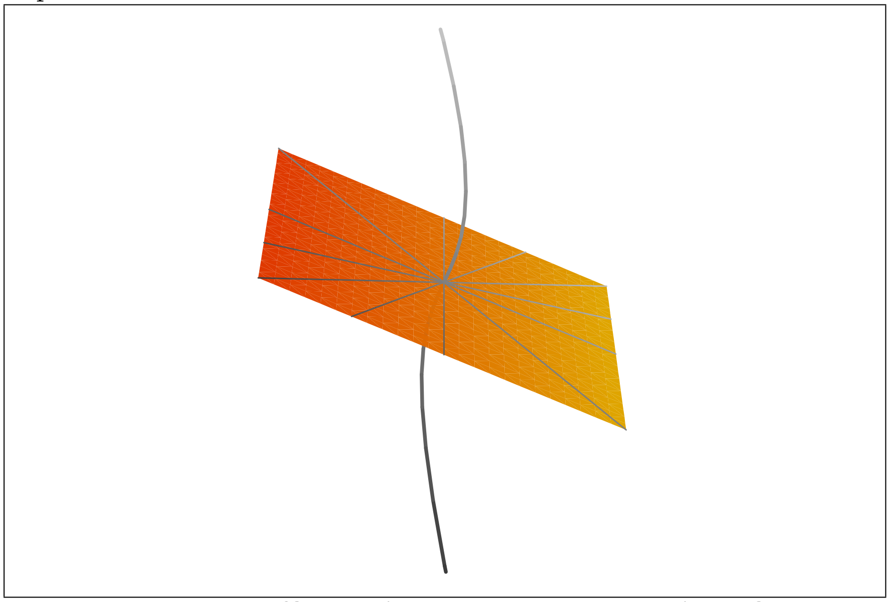
▲ In the 3D case the complete 4D-data of line integrals are redundant.

The variety of lines that are parallel to either of two given planes has dimension 3; a reconstruction can be done by reduction to 2D case: `Schuster, Vertgeim (2000)` .

3D case: `Vertgeim, Denisjuk`.

Let $\Gamma \subset \mathbf{V}$ - the set of sources.

Stability condition: *for any point* $q \in \operatorname{supp} \mathrm{f}$ *and any plane* H *through* q *there is at least one point* $p \in H \cap \Gamma$.



This condition is sufficient for a reconstruction, if the first derivatives of $R(\rho)$ *are known for all rays* ρ *with sources on* Γ. In particular, the reconstruction is possible on any chord of a curve Γ.

Notations: Fix a Euclidean structure in $\mathbf{V}$, denote $H_{p,\omega} \doteq \{y \in \mathbf{V}; \langle \omega, y \rangle = p\}$ for any $\omega,\ |\omega| = 1$ and $p \in \mathtt{R}$.

For any vector $\xi \neq 0$ the directional derivatives are

$$a_{\xi}(x) = (\xi, \mathrm{d}a(x)),\ \mathrm{R}_{\xi}(x;\theta) = (\xi, \mathrm{d}_x \mathrm{R}(x;\theta)),\ \partial_{;\xi}\mathrm{R}(x;\theta) = (\xi, \mathrm{d}_{\theta}\mathrm{R}(x;\theta)).$$

Proposition. *Let* f *be a 1-form of the class* S_3. *For an arbitrary plane* H *an arbitrary point* $y \in H$ *and any vector* ξ *parallel to* H *we have*

$$\partial_{\mathbf{p}} \int_H \mathrm{df}(x;\xi,\omega)\,\mathrm{d}H(x) = \int_S \partial_{\xi;\omega\omega}\mathrm{R}(y;\theta)\,\mathrm{d}\varphi(\theta), \tag{2}$$

where $\mathrm{d}H$ *is the Euclidean area element on* H,
$\mathrm{d}\varphi$ *is the angular measure on the unit circle* $S \subset H$.

Theorem. *Let* f *be a 1-form of the class* S_2 and $\Gamma \subset \mathbf{V}$ *be a set such that any hyperplane* H *that meets the support of* f *meets also* Γ.
The form df *can be reconstructed from data of first derivatives of the integral* $\mathrm{R}(x,\theta)$ *for rays* $\rho(x,\theta)$, $x \in \Gamma, |\theta| = 1$.

◀ For arbitrary vectors $\eta, \xi \in \mathbf{V}$ and a plane H we set

$$\mathrm{I}_H(\eta,\xi) = \partial_{\mathbf{p}} \int_H \mathrm{df}(x;\eta,\xi)\,\mathrm{d}H$$

The function I can be determined from the given integral data. If both vectors η, ξ are parallel to H, the equation $\mathrm{I}_H(\eta, \xi) = 0$ follows from partial integration. If η parallel to H and $\xi = \omega$ it is known by the formula (2) applied to a point $y \in H \cap \Gamma$.

For arbitrary vectors (η, ξ), we can write $\xi = a\omega + \xi'$, and $\eta = b\xi + \eta'$ for some numbers a and b, where ξ', η' are parallel to H.

If $a = b = 0$, then $\mathrm{I}_H(\eta, \xi) = 0$.

Suppose that $a \neq 0$. We have the equation

$$\mathrm{I}_H(\eta, \xi) = \mathrm{I}_H(\eta', \xi) = \mathrm{I}_H(\eta', a\omega) = a\mathrm{I}_H(\eta', \omega),$$

where the right-hand side is known.

The form df can be reconstructed from data of integrals $I_H(\eta, \xi)$ by means of the classical formula of Lorentz:

$$\mathrm{df}(x) = -\frac{1}{8\pi^2}\int_{|\omega|=\mathbf{1}} \partial_p^2 \int_{H_{\omega,\mathbf{p}}} \mathrm{df}(y)\,\mathrm{d}H\Bigg|_{\mathbf{p}=\langle\omega,\mathbf{x}\rangle} \mathrm{d}\omega.$$

We only need to know these integrals for hyperplanes H that meet the support of df. Otherwise the integral vanishes.

3 Range conditions

3.1 Line integrals of functions

The function

$$\mathrm{J}(x, \theta) = \int_{-\infty}^{\infty} \phi(x + r\theta)\,\mathrm{d}r$$

is called X-ray (or the John) transform of $\phi \in \mathsf{S}_0$, where $x, \theta \in \mathbf{V}$. It fulfils $\mathrm{J}(x, t\theta) = t^{-1}\mathrm{J}(x, \theta), t \neq 0$ and the John equations

$$\left(\frac{\partial^2}{\partial\theta_i\partial x_j} - \frac{\partial^2}{\partial\theta_j\partial x_i}\right)\mathrm{J}(x, \theta) = 0,\ i, j = 1, 2, 3. \tag{3}$$

The inverse statement `John(1938):`

Theorem *Any smooth fast decreasing function* $\mathrm{J}(x, \theta)$ *that satisfies these conditions is equal to X-ray transform of a function* $\phi \in \mathsf{S}_\infty$.

Remark: *Given a curve* Γ *in* $\mathbf{V}^3$, *the variety* Λ *of lines* λ *that meet* Γ *is characteristic for the John equation.* In the chart $x_3 = \theta_3 = 1$ the system is reduced to the only equation

$$\left(\frac{\partial^2}{\partial\theta_1\partial x_2} - \frac{\partial^2}{\partial\theta_2\partial x_1}\right)\mathrm{J}(x, \theta) = 0$$

A 3-variety Λ the equation $\Phi(x_1, x_2, \theta_1, \theta_2) = 0$ is characteristic for the John equation if

$$\frac{\partial \Phi}{\partial x_1}\frac{\partial \Phi}{\partial \theta_2} - \frac{\partial \Phi}{\partial x_2}\frac{\partial \Phi}{\partial \theta_1} = 0.$$

3.2 Integrals of forms

The line integrals $\mathrm{L} = \mathrm{L}(x, \theta)$ of a 1-form f fulfil the homogeneity condition $\mathrm{L}(x, t\theta) = \pm\mathrm{L}(x, \theta)$ for $\pm t > 0$ and the system of equations

$$\left(\frac{\partial^2}{\partial \theta_i \partial x_j} - \frac{\partial^2}{\partial \theta_j \partial x_i}\right)^2 \mathrm{L}(x, \theta) = 0, \ \ i, j = 1, 2, 3, \tag{4}$$

The same equations hold at a point x for the ray integrals $\mathrm{R}(x, \theta)$ provided the form f vanishes in a neighborhood of the point x.

▼ The inverse statement is due to `Gelfand-Gindikin-Graev(1980,2000)`:

Theorem *An arbitrary smooth function* $\mathrm{L}(x, \theta)$ *that decreases fast as* $|x \times \theta| \to \infty$ *with all derivatives that fulfils* (4), *is equal to the line transform of a 1-form* f *with coefficients in the Schwartz space* (and vice versa).

The variety Λ of lines λ that touch a curve Γ is a "double" characteristic for (4). The "initial" data on Λ are the functions and its first derivatives.

4 Rays tangent to a surface

The variety Λ of rays tangent to a surface S is characteristic for the John equation and double characteristic for (4). A simple reconstruction formula for the Doppler transform is as follows:

Theorem *Let* S *be a smooth surface in an oriented Euclidean space* $\mathbf{V}$,

H be a plane nowhere tangent to S. For an arbitrary $f \in \mathsf{S}_3$ we have

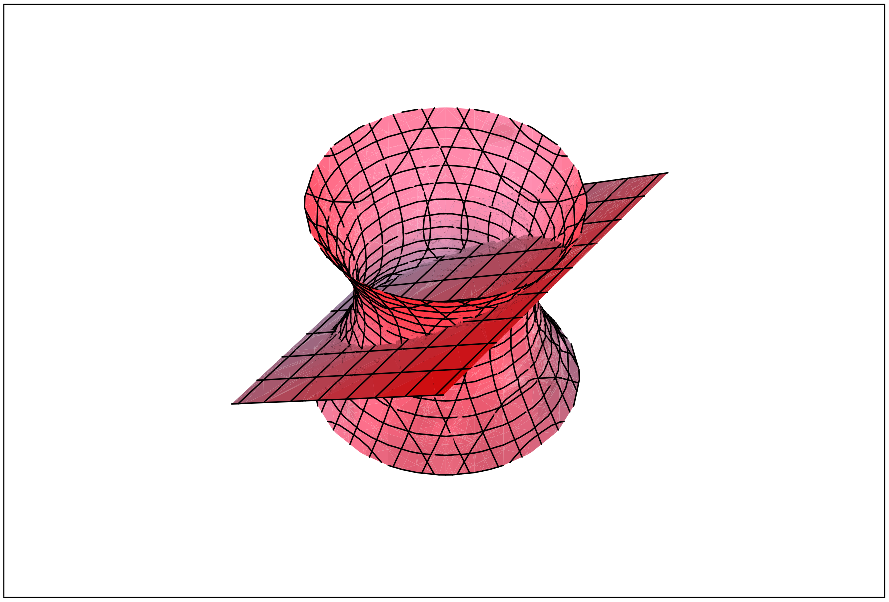
$$\partial_{\mathbf{p}} \int_H \mathrm{df}(x; \theta, \omega)\, \mathrm{d}H = \int_{\mathbf{C}} \left[\kappa \partial_{\theta;\omega\omega} \mathrm{R}(y; y') - [\theta, \omega, y'] \, \mathrm{R}_{\omega\omega}(y; y')\right] \mathrm{d}s,$$

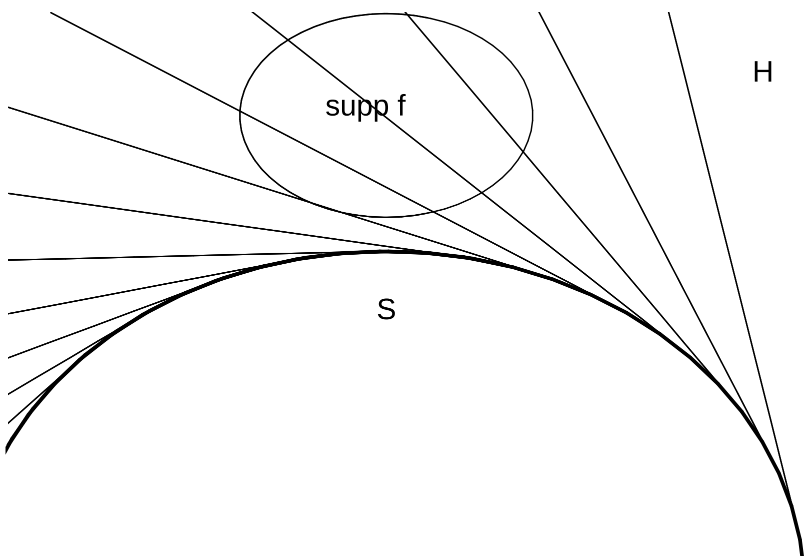
where

(i) $y = y(s)$, $0 \le s \le s.$ *is the equation of the curve* $\mathbf{C} \doteq S \cap H$ *such that* $|y'| = 1,\ y' = \partial y / \partial s$,

(ii) $\kappa = [y', y'', \omega] > 0$ *is the curvature of* $\mathbf{C}$,

(iii) $\mathrm{supp}\, f \cap H$ *is contained in the image of the map* $Y : (0, s.) \times (0, \infty) \to H$, $(s, r) \mapsto y(s) + ry'(s)$.





Rays tangent to the curve $S \cap H$

5 Some references

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