# An inexact primal-dual deflected subgradient algorithm with augmented Lagrangians

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#### Outline



2 Minimizing the Lagrangian

3 Deflected Subgradient Method

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The Primal Problem The Dual Problem Duality Properties

### The Primal Problem

#### X reflexive Banach space, H a Hilbert space

#### minimize $\varphi(x)$ s.t. x in X (1)

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 $\varphi: X \to \mathbb{R}_{+\infty}$  proper, weakly-lsc with weakly compact level sets

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## Building up the Dual

#### Take a duality parameterization for (1), i.e.,

 $g: X \times H \to \mathbb{R}_{\pm \infty}$  such that  $g(x, 0) = \varphi(x) \quad \forall x \in X$ .

and an augmenting function  $\sigma: H \to \mathbb{R}$ 

proper, w-lsc, level-bounded, and:

 $\sigma(0) = 0, \quad \sigma(y) \ge \|y\| \, \forall y, \quad \text{and } \operatorname*{Argmin}_{y} \sigma(y) = \{0\}$ 

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## The "augmented" Dual

#### The augmented Lagrangian

 $L: X \times H \times \mathbb{R}_+ \to \mathbb{R}_{\pm \infty}$ 

$$L(x, y, r) := \inf_{z \in H} \{ g(x, z) - \langle z, y \rangle + r\sigma(z) \}$$

The dual function:  $q: H \times \mathbb{R}_+ \to \mathbb{R}_{-\infty}$ 

$$q(y,r) = \inf_{x \in X} L(x,y,r)$$

with dual problem:

maximize q(y,r) s.t. (y,r) in  $H imes \mathbb{R}_+$ 

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# **Duality Properties**

Augmented Lagrangians proposed by Rockafellar and Wets, 1997:

- Strong duality: dual optimal value = primal optimal value
- Saddle point properties: get primal solution using dual one
- Dual problem is convex: use known solution techniques

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The subproblem Approximate solutions  $\epsilon$ -subgradients

### The subproblem

#### Fix (y, r) a dual variable, and $\varepsilon \ge 0$ , find:

 $(\tilde{x},\tilde{z})\in X_{\varepsilon}(y,r)$ 

#### where

 $X_{\varepsilon}(\mathbf{y},\mathbf{r}) := \{(x,z) \in X \times H : g(x,z) - \langle z, y \rangle + r\sigma(z) \le q(y,r) + \varepsilon\}$ 

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## $\epsilon$ -solutions

Let 
$$M_P := \inf_{x \in X} \varphi(x)$$
  $\rightarrow$  optimal primal value  
and  $M_D := \sup_{(y,r) \in H \times \mathbb{R}_+} q(y,r)$   $\rightarrow$  optimal dual value  
Dual solutions= $D_*$   
Fix  $\epsilon_* \ge 0$ :  
 $x_* \in X$  is  $\epsilon_*$ -primal solution if  $\varphi(x_*) \le M_P + \epsilon_*$ 

 $(y_*, c_*) \in H imes \mathbb{R}_+$  is  $\epsilon_*$ -dual solution if

$$q(y_*, c_*) \geq M_D - \epsilon_*$$

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### An *e*-subgradient method

**Update rule:** Given current  $w_k := (u_k, c_k)$ , search along  $\epsilon$ -subgradient direction  $g_k \in \partial_{\epsilon}q(w_k)$ :

 $W_{k+1} = W_k + S_k g_k$ 

where step-size  $s_k > 0$ . An  $\epsilon$ -subgradient of q at  $w_k$  is

 $g_k = (-z_k, \sigma(z_k))$ 

where

 $(x_k, z_k) \in X_{\epsilon}(u_k, c_k)$ 

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The subproblem Approximate solutions  $\epsilon$ -subgradients

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### An *e*-subgradient method

**Update rule:** Given current  $w_k := (u_k, c_k)$ , search along  $\epsilon$ -subgradient direction  $g_k \in \partial_{\epsilon}q(w_k)$ :

 $W_{k+1} = W_k + S_k g_k$ 

where step-size  $s_k > 0$ . An  $\epsilon$ -subgradient of q at  $w_k$  is

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Convergence Result: Results - I Results II

## **Deflected Subgradient Method**

```
Let \alpha > 0, \delta \in (0, 1), \epsilon_* > 0
```

Step 0 Choose  $(u_0, c_0)$  with  $c_0 \ge 0$  and choose  $\epsilon_k \downarrow 0$ . Step k Given  $(u_k, c_k)$ :

Since  $\mathcal{O}_{i_{1}}$  and  $(\alpha_{i_{1}}, \alpha_{i_{2}}) \in \mathcal{X}_{i_{1}}(\alpha_{i_{1}}, \alpha_{i_{2}}) \cup [1, \alpha_{i_{2}} = 0 \text{ and } \alpha_{i_{1}} \leq \alpha_{i_{1}} \text{ STOP}$ Since  $\mathcal{O}_{i_{2}}$  and  $(\alpha_{i_{2}}, \alpha_{i_{2}}, \alpha_{i_{2}}, \alpha_{i_{2}}) \in \mathcal{O}_{i_{2}}$  (SOTO Step  $\mathcal{I}_{i_{2}}$ ) Since  $\mathcal{O}_{i_{2}}$ 

 $\operatorname{Set} = \left\{ \begin{array}{ccc} a_{k+1} & \cdots & a_k = 3 a_{k+1} \\ a_{k+1} & \cdots & a_{k+1} \cdot a_k (1+a_k) \sigma(a_k), \\ a_{k+1} & \cdots & a_{k+1} \cdot a_k (1+a_k) \sigma(a_k), \end{array} \right.$ 

Convergence Results Results - I Results II

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where  $s_k > 0 \alpha_k \in (0, \alpha)$ 

(Gasimov 2002; Gasimov & Ismayilova 2004) (Burachik & Gasimov & Ismayilova & Kaya, 2006, Burachik & Kaya 2007, Burachik & Kaya & Mammadov, 2009)

# **Basic Result**

Convergence Results Results - I Results II

#### For every choice of $s_k, \alpha_k$

If stops at iteration k, then

 $x_k$  is  $\epsilon_*$ -primal optimal, and

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# **Bounded Stepsize**

Take  $\beta > \eta > 0$  and  $\eta_k := \min\{\eta, \|z_k\|\}, \ \beta_k := \min\{\beta, \sigma(z_k)\}$ 

Choose  $s_k \in [\eta_k, \beta_k]$ 

If  $\epsilon_k \leq M_D - q(u_k, c_k) + R\sigma(z_k)$  and  $D_* \neq \emptyset$ , then

- $\{q_k\}$  converges to  $M_D$
- $\{(u_k, c_k)\}$  converges weakly to a dual solution
- If 0 < α<sub>k</sub> < α
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Convergence Results Results - I Results II

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Convergence Results Results - I Results II

# **Unbounded Stepsize**

Take  $\beta > 0$  and a sequence  $\{\theta_k\} \subset [0, \beta), \sum \theta_k = \infty$ 

Define 
$$\eta_k := \frac{\theta_k}{\sigma(z_k)}, \ \beta_k := \frac{\beta}{\sigma(z_k)}$$

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$$\epsilon_k \leq M_D - q(y_k, c_k) + R\sigma(z_k)$$

#### Dual sequence bded iff dual solutions exist

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