Graph Minors

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1 Overview of the Field

Graphs are finite and may have loops and multiple edges. A graph H is a minor of a graph K if H can be obtained from a subgraph of K by contracting edges.

One of the most important work in graph theory is the Graph Minor Theory developed by Robertson and Seymour. It took more than 20 years to publish this seminal work in a series of 20+ long papers. All together there are 23 papers, Graph Minors I–XXIII. The first 20 of these have been written back in the 1980's and have already appeared, the last one being published in 2004.

Graph Minors project resulted in many theoretical advances, (e.g. a proof of Wagner's conjecture), but it also has algorithmic applications, and some of the methods have been successfully used in practical computation.

Two most important and actually best known results concerning graph minor theory are presented below. The first one was conjectured by Wagner and was known as Wagner's Conjecture. It implies that graphs are well-quasi ordered with respect to the graph minor relation.

Theorem 1 *For every infinite sequence $G_1, G_2, \ldots$ of graphs, there exist distinct integers $i < j$ such that G_i is a minor of G_j.*

The second is related to one of the most important problems in theoretical computer Science. Graph minors are intimately related to the k-Disjoint Paths Problem: given a graph G and k pairs (s_1, t_1), ..., (s_k, t_k) of vertices of G, decide whether there are k mutually vertex disjoint paths of G, the ith path linking s_i and t_i for $i = 1, \ldots, k$. If k is part of the input of the problem, then this is one of the well-known NP-complete problems by Karp, and it remains NP-complete even if G is restricted to be planar. However, for any fixed number of pairs, the situation changes.

Theorem 2 *For every fixed integer k, there is a polynomial time algorithm to resolve the k-Disjoint Paths Problem. Actually, the time complexity is $O(n^3)$, where n is the order of the input graph G.*

This result is, in a sense, surprising since the corresponding problem for digraphs is NP-complete even when we consider the fixed value $k = 2$

The Disjoint Paths Problem is easily seen to be polynomially equivalent to the problem of deciding if a fixed graph H with k edges is a topological minor in G. Consequently, it is also polynomially equivalent to the H-Minor Problem of deciding if a fixed graph H is a minor in G.

Theorem 3 *For every fixed graph H, there exists an $O(n^3)$ algorithm for deciding if a given graph of order n contains H as a minor.*

2 Recent Developments and Open Problems

Recently, several researchers are conducting researches to extend Graph Minor project. Let us give a few examples.

Jim Geelen, Bert Gerards and Geoff Whittle are undertaking a program of research aimed at extending the results and techniques of the Graph Minor Project of Robertson and Seymour to matroids. In particular, they are trying to find the structure of minor-closed classes of matroids representable over a fixed finite field. This requires a peculiar synthesis of graphs, topology, connectivity, and algebra. They expect the structure theory will help in proving Rota's conjecture that for every field there are only finitely many excluded minors for representability over that field, and Robertson and Seymour's Well-Quasi-Ordering conjecture that for any finite field any infinite list of matroids representable over that field contains two members such that one is a minor of the other. They also expect the theory will help to find an efficient algorithm for recognising a fixed minor-closed property over a fixed finite field. They have already come up with Graph Minor X for matroids.

The heart of Graph Minor Theory is a decomposition theorem capturing the structure of all graphs excluding a fixed minor. At a high level, the theorem says that every such graph can be decomposed into a collection of graphs each of which can be "almost" embedded into a surface of bounded genus, combined in a tree-like structure.

This decomposition theorem is now well-understood. This leads to several deep results. For example, Bohme, Kawarabayashi, Maharry and Mohar used this to improve the degree of the connectivity that forces a complete graph minor and a complete bipartite minor in large graphs. This improves the connectivity degree given by Thomason and Kostochka, independently, almost 25 years. Another example is a proof of Jorgensen's conjecture for large graphs by Robin Thomas and his team. Robertson, Seymour and Thomas proved Hadwiger's conjecture for K_6-minor-free case. Their result says that any minimal counterexample to Hadwiger's conjecture for K_6-minor-free case is an apex graph, i.e., it has a vertex v such that $G - v$ is planar. This is implied by Jorgensen's conjecture which says that every 6-connected non-apex graph has a K_6-minor. Since a structure theorem for graphs without K_6-minor has been an important problem in graph minor area (and it is even hopeless right now.), this solution for large graphs may give new insight in this direction.

3 Objectives of the workshop

A monumental project in graph theory by Robertson and Seymour was recently completed. This is now called "Graph Minor Theory", and Graph Minors project resulted in many theoretical advances, (e.g. a proof of Wagner's conjecture), but it also has algorithmic applications, and some of the methods have been successfully used in practical computation.

But currently, Graph Minor theory is reasonably understood by many, and several researchers have been working on extensions of Graph minor project. A research program conducted by Jim Geelen, Bert Gerards and Geoff Whittle are extending the results and techniques of the Graph Minor Project of Robertson and Seymour to matroids. They have already published over 10 papers in Journal of Combinatorial Theory Ser. B (All Graph minor theory papers, except for "Graph Minors II", appeared in Journal of Combinatorial Theory Ser. B.). So this project is really "tour de force".

Also, techniques and tools from Graph Minor Theory are reasonably understood by many, and some researchers have been working on exact structural descriptions using them. Let us observe that the decomposition theorem capturing the structure of all graphs excluding a fixed minor is, in a sense, an approximate structure theorem since this structure could contain the minor which we would like to exclude. A proof of Jorgensen's conjecture for large graphs by Robin Thomas and his team may give new insight in this direction since an apex graph, i.e., it has a vertex v such that $G - v$ is planar, clearly does not contain K_6-minors.

As we see the above two examples, there are now research programs which are a far reaching generalizations of Graph Minor Theory and are deep understandings of techniques and tools from Graph Minor Theory.

So we feel that it is now time to gather many researchers who are working on Graph Minor area, and present "state of art" of their current projects. In particular, it seems important to report where these projects stand and where these projects would go.

In particular, we shall focus on the following two points: extensions of Graph Minor Theory, and applications of Graph Minor Theory techniques and tools.

4 Presentation Highlights

Recent research interests: Computing excluded minors; Decompositions of hypergraphs, digraphs and structures; Graph structure and logic.

Isolde Adler, Humboldt University Berlin

Computing excluded minors. Robertson and Seymour showed that every graph class that is closed under taking minors (every *minor ideal*) has a polynomial time decision algorithm. The algorithm uses the excluded minor characterisation of the minor ideal. Nevertheless, for many minor ideals we do not know the excluded minor characterisation. Since the 1980s, much research has been done to overcome this deficiency. But many interesting problems still remain unsolved. Combining graph structure theory and methods from logic, we recently showed how to compute the excluded minor characterisation for the union of two minor ideals, given the characterisations of the respective ideals (Adler, Grohe, Kreutzer, Computing excluded minors, SODA 2008). A small collection of related open problems is available on `http://www2.informatik.hu-berlin.de/~adler/publications/obstructions-open- problems.pdf`.

Decompositions of hypergraphs, digraphs and structures. Extending graph structure theory to hypergraphs, digraphs, matroids and logical structures is an active and interesting field of research. While graph tree-width has a variety of natural characterisations, the picture becomes more heterogeneous in other cases, such as hypertree-width of hypergraphs (Adler, Gottlob, Grohe, Hypertree-Width and Related Hypergraph Invariants, European J. Comb. 2007) and directed tree-width of digraphs (Adler, Directed Tree-Width Examples, JCTB 2007). Interesting questions include finding analogues of the Excluded Grid Theorem for hypergraphs and digraphs.

Rank-width of graphs (equivalent to clique-width) is a very successful notion introduced by Oum and Seymour. While bounded tree-width implies bounded rank-width, the converse is false in general. Many problems that are hard in general become tractable on bounded rank-width, and graphs of bounded rank-width are recognisable in polynomial time. It is an interesting open question to find an extension of this notion to logical structures, with similar favourable properties.

Graph structure and logic. Evaluating conjunctive queries (CQs) on databases is NP-hard in general. The problem is eqivalent to constraint satifaction problems from artificial intelligence and to the homomorphism problem of relational structures. It is well known that CQ evaluation becomes tractable if the underlying graph of the CQ has bounded tree-width. Using tree decompositions with more refined *width* measures, it is possible to obtain larger classes of tractable instances (Adler, Tree-width and functional dependencies in databases, PODS 2008). CQ evaluation is a special case of evaluating first order formulas in logical structures. Can we restrict the structure of first order formulas in order to obtain tractability? Which are the largest classes of tractable instances?

$K_{2,t}$ minors in dense graphs

Maria Chudnovsky/Columbia University

Let H be a graph. If G is an n-vertex simple graph that does not contain H as a minor, what is the maximum number of edges that G can have? This is at most linear in n, but the exact expression is known only for very few graphs H. For instance, when H is a complete graph K_t, the "natural" conjecture, $(t-2)n - \frac{1}{2}(t-1)(t-2)$, is true only for $t \leq 7$ and false for large t.

In this talk we will discuss the maximum number of edges when H is the complete bipartite graph $K_{2,t}$. We show that in this case, the analogous "natural" conjecture, $\frac{1}{2}(t+1)(n-1)$(for all $t \geq 2$), is the truth for infinitely many n.

This is joint work with Bruce Reed and Paul Seymour

Graph Transformations expressed in logic and applications to structural graph theory

Bruno Courcelle, Bordeaux University, France

Most properties interesting for Graph Theory can be formally written in Second-Order logic, by formulas that can use quantifications on relations of fixed arity on vertex sets .

Monadic Second-Order (MSO) logic is the restriction of this language using only quantifications on sets ("monadic" relations). It is still quite powerful, and can express basic properties like k -colorability, minor or vertex-minor inclusion, planarity, k-connectivity.

Expressions of graph properties by MSO formulas are interesting because they yield :

1) Fixed parameter tractable algorithms for parameters tree-width, clique-width or rank-width, of such properties ;

2) Effective computability of obstruction sets for minor-closed graph classes for which we have characterizations by MSO formulas (independent of the obstructions) and upper-bounds on the tree-widths of obstructions.

There are deep links between graph structure and logical decidability for MSO formulas. MSO logic and graph decompositions play a fundamental role in the extension of the Theory of Languages and Automata to the description of sets of finite graphs.

Monadic Second-Order logic can also be used to specify many types of *graph transformations*, in particular, to take various examples :

1) the mapping from a graph to its set of minors or vertex-minors,

2) certain canonical graph decompositions (modular decomposition, split decomposition, decomposition in 3-connected components),

3) encodings of directed graphs as vertex-labelled undirected graphs (this is useful to transfer to directed graphs certain results about rank-decompositions of undirected graphs, and to obtain FPT algorithms for checking MSO properties on directed graphs of bounded clique-width),

4) the mapping from tree-decompositions (or rank-decompositions) of graphs, suitably encoded as labelled trees, to the decomposed graphs,

5) a planar embedding of a linearly ordered planar graph, or a chord diagram representing a linearly ordered circle graph.

We call them *MSO transductions*, this terminology is borrowed from Language Theory.

MSO logic can actually be used to write graph properties in two ways, by means of two logical structures associated with a given graph. A graph G can be given (basic way) as the vertex set $V(G)$ and a binary relation for adjacency, or as its *incidence* (bipartite) *graph* consisting of the set $V(G) \cup E(G)$ of vertices and edges, and a binary incidence relation. This second possibility is more expressive because it allows edge set quantifications. We will distinguish two kinds of MSO transductions : the basic ones that transform graphs into graphs and those that transform incidence graphs into incidence graphs, that we call *MSO*^{Inc}-*transductions*.

MSO transductions are related with graph structure notions by the following results classified in three categories :

I : *Logical characterizations of bounded tree-width and bounded clique-width /rank-width* in terms of trees and MSO transductions :

I.1 A set of graphs has bounded tree-width (resp. path-width) iff it is the image of a set of trees (resp. of paths) under an *MSO*^{Inc}-transduction.

Hence : The image of a set of graphs of bounded tree-width (resp. path-width) under an *MSO*^{Inc}-transduction has bounded tree-width (resp. path-width) (effective proofs, but bad bounds).

I.2 A set of graphs has bounded clique-width (or rank-width) iff it is the image of a set of trees under an *MSO*-transduction.

Hence : The image of a set of graphs of bounded clique-width (or rank-width) under an *MSO*-transduction has bounded clique-width (or rank-width).

(Courcelle and Engelfriet, 1990-1995)

II : *Graph structure and decidability of MSO logic* :

II.1 The set of graphs of tree-width at most k has a decidable *MSO*^{Inc}-theory (one can decide if a given formula is true in all graphs of tree-width at most k).

II.1' If a set of graphs has a decidable *MSO*^{Inc}-theory, it has bounded tree-width (Seese, 1991).

These two results extend *easily* to relational structures (say directed hypergraphs of bounded rank) through their incidence graphs.

II.2 The set of graphs of clique-width at most k has a decidable *CMSO*-theory.

II.2' If a set of graphs has a decidable *CMSO*-theory , it has bounded clique-width (Courcelle and Oum 2007). (CMSO = MSO with an even cardinality set predicate).

The proofs of II.1' and II.2' use deep theorems about minors and vertex-minors, in particular, the Excluded Grid Theorem (excluding a grid implies bounded tree-width).

About extensions :

II.3 A set of relational structures that is the image of the set of trees under an *MSO*-transduction has a decidable *CMSO*-theory (Courcelle, 1992).

II. **Open problem** : Is it true that :

If a set of relational structures has a decidable CMSO-theory , it is the image of a set of trees under an MSO-transduction ?

III : *Encoding powers of graph classes via MSO*^{Inc}*-transductions : a linear hierarchy of graph classes.*

(Recent, joint work with A. Blumensath, Darmstadt, Germany)

For graph classes $\mathcal{C}$ and $\mathcal{D}$, we let $\mathcal{C} \leq \mathcal{D}$ if C is included in the image of $\mathcal{D}$ under an *MSO*^{Inc}-transduction. Classes $\mathcal{C}$ and $\mathcal{D}$ are equivalent, $\mathcal{C} \equiv \mathcal{D}$, if $\mathcal{C} \leq \mathcal{D}$ and $\mathcal{D} \leq \mathcal{C}$, and $\mathcal{C} < \mathcal{D}$ if $\mathcal{C} \leq \mathcal{D}$ and they are not equivalent. Letting $\mathcal{T}_n$ be the class of rooted trees of height at most n :

Linear Hierarchy Theorem :

$$\mathcal{T}_1 < \ldots < \mathcal{T}_n < \ldots < \text{PATHS} < \text{TREES} < \text{GRIDS} \equiv \text{ALL GRAPHS}$$

and every class of graphs is equivalent to one in this hierarchy.

We define the new notion of *n-depth tree-width* of a graph G as the minimal width of a tree-decomposition of G with underlying tree in $\mathcal{T}_n$.

The corresponding graph classes are those of bounded n-depth tree-width for n = 1, ... , bounded path-width, bounded tree-width, unbounded tree-width.

The proof uses logical tools, counting arguments, the Excluded Tree and Grid Theorems (Graph Minors I and V) and the following

Excluded Path Theorem :

A graph class excludes a path iff it has bounded n-depth tree-width for some n.

Open problems :

1) The obstructions to n-depth tree-width $< k$ can be (theoretically) computed. What are they ?

2) Does a similar Linear Hierarchy Theorem hold for *CMSO*-transductions?

Several steps are missing for its proof. In particular, is it true that :

If a set of bipartite undirected graphs has unbounded linear rank-width, it contains all trees as vertex-minors ?

where rank-width is defined like rank-width with paths instead of cubic trees.

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The degenerate value of classes of graphs

Guoli Ding, LSU

For any minor closed class of graphs $\mathcal{C}$, its *degenerate value* is the smallest integer d such that every graph in $\mathcal{C}$ can be vertex-bipartitioned into a part of bounded tree-width (the bound depending only on $\mathcal{C}$), and a part that is d-degenerate (a graph is d-degenerate if it can be reduced to the empty graph by repeatedly deleting vertices of degree $\leq d$). In this talk we present some related results.

Minors-related research interests

Zdeněk Dvořák, SFU

The *grad (Greatest Reduced Average Density) with rank* r of a graph G is equal to the largest average density of a graph G' that can be obtained from G by removing some of the vertices (and possibly edges) and then contracting vertex-disjoint subgraphs of radius at most r to single vertices (arising parallel edges are suppressed). The grad with rank r of G is denoted by $\nabla_r(G)$. In particular, $2\nabla_0(G)$ is the maximum average degree of a subgraph of G. Given a function $f : N \to R^+$, a graph has *expansion bounded by* f if $\nabla_r(G) \leq f(r)$ for every integer r. A class $\mathcal{G}$ of graphs has *expansion bounded by* f if the expansion of every $G \in \mathcal{G}$ is bounded by f. Finally, we say that a class of graphs $\mathcal{G}$ has *bounded expansion* if there exists a function f such that the expansion of $\mathcal{G}$ is bounded by f.

Many classes of graphs have bounded expansion, including all proper minor-closed classes of graphs, and many interesting results previously known for (some) proper minor-closed classes generalize naturally to classes of graphs with bounded expansion (see e.g. the survey *Structural properties of sparse graphs* by J. Nešetřil and P. Ossona de Mendez). Below, I list two problems regarding the classes of graphs with bounded expansion.

- Nešetřil and Ossona de Mendez have shown that every Σ_1-FOL formula can be decided in linear time for graphs in a class of graphs with bounded expansion. In fact, in a joint work with D. Král' and R. Thomas, we have found a semidynamic data structure that enables us (after linear-time preprocessing) answer Σ_1-FOL queries (and locate the witnesses) in constant time per operation. Can all FOL formulas be decided in linear (or at least, polynomial with the exponent independent on the complexity of the formula) time?

  A related result of Grohe shows that FOL formulas can be decided in $O(n^{1+\varepsilon})$ for classes of graphs with bounded local treewidth. While the notion of bounded expansion is independent on bounded local treewidth (there exist graph classes having any combination of these properties), there exists a common generalization of both concepts (classes of nowhere-dense graphs), and it seems likely that a linear-time algorithm for bounded expansion would generalize to $O(n^{1+\varepsilon})$ algorithm for nowhere-dense graphs.

- Do graphs with bounded expansion have some structure? In general, the answer is no, as random graphs (with density $O(1/n)$) have w.h.p. bounded expansion. However, there are indications that graphs with subexponential (or polynomially-bounded) expansion have some structure (for example, they have sublinear separators). Does there exist some characterization of such classes, or a decomposition theorem for them?

Minor-minimal 5-connected graphs

Gašper Fijavž, University of Ljubljana

A graph G is called *minor-minimal k-connected* if G is (vertex) k-connected, yet every proper minor of G has conectivity less than k.

For every integer k the set of minor minimal k-connected graphs is finite. Every 3-connected graph contains a K_4 minor, and Wagner's decomposition theorem easily implies that every 4-connected graph contains either K_5 or $K_{2,2,2}$ as a minor. The sets of minor minimal k-connected graphs for $k \geq 5$ are not known.

Conjecture 1 *The set of minor minimal 5-connected graphs contains exactly six graphs: K_6, $K_{2,2,2,1}$, $C_5 * 3K_1$ (complete join of a 5-cycle with three independent vertices), icosahedron I, and graphs G_1 and G_2 depicted below (where every pair of either green or red vertices should be identified into a single one).*

Dirac has shown that every 5-connected planar graph contains icosahedron as a minor, and we know that every 5-connected projective-planar graph contains one of K_6, G_1, G_2 or I as a minor.

The graphs $K_{2,2,2,1}, C_5 * 3K_1, I, G_1$ and G_2 are all apex. Robertson, Seymour, and Thomas have shown that if an apex graph is embedded in a nonorientable surface, then its face-width is at most two. As planar graphs do not admit embeddings in nonspherical surfaces with face-width ≥ 3, an apex graph cannot be embedded with face-width ≥ 4.

Conjecture 2 *Every 5-connected graph which embeds in a (nonspherical) surface with face-width ≥ 4 contains K_6 as a minor.*

Mohar has constructed examples of apex embeddings with face-width 3, hence 4 in above conjecture is best possible.

On the other hand, the theory of graph minors implies that for every surface Σ there exists a constant c_Σ so that if G embeds in Σ with face-width c_Σ, then G contains K_6 as a minor.

We know that every 5-connected projective-planar graph with face-width ≥ 3 contains a K_6-minor. Using the result of Brunet, Mohar, and Richter it is not difficult to construct a sufficiently dense toroidal-grid minor (which contains K_6 as a minor) whenever G is embedded with large face-width (which does not depend on the surface).

Treewidth computation and extremal combinatorics

Fedor V. Fomin, University of Bergen, Norway

Abstract: For a given graph G and integers $b, f \geq 0$, let S be a subset of vertices of G of size $b+1$ such that the subgraph of G induced by S is connected and S can be separated from other vertices of G by removing f vertices. We prove that every graph on n vertices contains at most $n/\frac{b+f}{b}$ such vertex subsets. This result from extremal combinatorics appears to be very useful in the design of several enumeration and exact algorithms. This significantly improves previous algorithms for these problems.

Binary matroid minors.

Jim Geelen

Abstract: In joint work with Bert Gerards and Geoff whittle, we are trying to generalize the graph minors project, of Neil Robertson and Paul Seymour, to matroids representable over any fixed finite field. Recently we have extended the graph minors structure theorem to the class of binary matroids, and, in the talk we will discuss this theorem and its potential applications.

Our structure theorem is quite technical, but in essence it shows that the members of any proper minor-closed class of binary matroids can be constructed from graphs. We hope to use the structure theorem to solve the following problems and conjectures.

Conjecture 3 (WQO Conjecture) *In any infinite sequence of binary matroids there are two matroids one containing the other as a minor.*

Conjecture 4 (Minor-testing problem) *Testing for a fixed minor can be solved efficiently in the class of binary matroids.*

Conjecture 5 (Shortest circuit problem) *The shortest circuit problem can be solved efficiently in any proper minor-closed class of binary matroids.*

We also hope to use the binary matroid structure theorem to obtain a structure theorem for vertex-minor closed classes of graphs. This would hopefully enable us to solve the WQO Conjecture for vertex-minors and to efficiently solve the vertex-minor testing problem.

Vertex minors play an important role in Measurement Based Quantum Computing. Measurement Based Quantum Computing is a model of Quantum Computing where the states are described by graphs having a quantum bit at each vertex. There are three measurements that can be applied to a bit, resulting in a new graph-state that is an elementary vertex-minor of the initial state.

Conjecture 6 (Classical simulation of Quantum Computing) *For any vertex-minor closed class of graph-states, the associated quantum calculations can be efficiently simulated on a classical computer.*

The d-rank chromatic number

Luis Goddyn, Simon Fraser University

We define here a generalized natural notion of circular chromatic number, and dually, a generalized circular flow number.

Let $\vec{G}$ be a directed graph of order n, and let $\pi = (V_0, V_1, \dots, V_d)$ be an ordered partition of $V(G)$ into $d+1$ nonempty parts. The set of arcs from V_i to V_j where $i < j$ (resp. $j < i$) is denoted $\delta^+(\pi, \vec{G})$ (resp. $\delta^-(\pi, \vec{G})$). For any $d \in \{1, 2, \dots, n-1\}$, the *$d$-rank flow number* of a graph G is defined to be

$$\phi_d(G) = 1 + \min_{\vec{G}} \max_{\pi} \frac{|\delta^+(\pi, \vec{G})|}{|\delta^-(\pi, \vec{G})|}$$

Here $\vec{G}$ varies over the orientations of G and π varies over the ordered partitions of $V(G)$ into $d+1$ nonempty parts. We interpret $\frac{0}{0}$ to equal 1, and $\frac{a}{0} = \infty$ if $a > 0$. We also define the *d-bounded flow number* by $\phi_{\leq d}(G) = \max\{\phi_{d'}(G) \mid 1 \leq d' \leq d\}$.

The Hoffman circulation theorem asserts that $\phi_1(G) = \phi_c(G)$, the circular flow number of G. Questions regarding ϕ_d naturally arise. For example, Seymour's 6-flow theorem asserts that every bridgeless graph G satisfies $\phi_1(G) \leq 6$. Is ϕ_d similarly bounded for a fixed $d > 1$? Particularly interesting is the value of $\phi_{\leq n-1}(G)$, which we conveniently write as $\phi_\infty(G)$. How quickly does $\phi_\infty(G)$ grow for reasonably connected graphs? A recent result of Chávez, Goddyn and Hochstättler provides the following.

Theorem 4 *For any bridgeless graph G of order n we have $\phi_\infty(G) \leq 3n - 2$.*

Using a table of tournaments supplied by Brendan McKay, one can readily compute $\phi_d(G)$ for complete graphs $G = K_n$, $n \leq 7$. The following table indicates that the behaviour of $\phi_\infty(K_n)$ for large n is difficult to guess.

n	3	4	5	6	7
$\phi_\infty(K_n)$	3	6	$\frac{10}{3}$	$\frac{15}{4}$	3

Intuitively, one might suspect that $\phi_d(G) \leq \phi_{d+1}(G)$ for any graph G and $1 \leq d \leq n-2$. However, we find that this is false since $\phi_\infty(K_7) = \phi_4(K_7) > \phi_5(K_7)$.

The notion of ϕ_d is easily dualized to a generalized chromatic number. Let H be a subgraph of G and let $\vec{H}$ and $\vec{G}$ be orientations of H and G. We define $\delta^+(\vec{H}, \vec{G})$ (resp. $\delta^-(\vec{H}, \vec{G})$) to be the number of arcs in $\vec{H}$ which agree (resp. disagree) in their orientation with the corresponding arc in $\vec{G}$. Recall that the *corank* or

Beti number of H is $|E(H)) - |V(H)| + c(H)$ where $c(H)$ is the number of connected components of H. We define the *d-rank chromatic number* of G to be

$$\chi_d(G) = 1 + \min_{\vec{G}} \max_{\vec{H}} \frac{|\delta^+(\vec{H}, \vec{G})|}{|\delta^-(\vec{H}, \vec{G})|}$$

Here $\vec{G}$ varies over the orientations of G, and $\vec{H}$ varies over all the strong orientations of all the bridgeless subgraphs of G which have corank d. For example, when $d = 1$ we have $\vec{H}$ varying over the two strong orientations of every circuit in G. It follows from Minty's formula that $\chi_1(G) = \chi_c(G)$, the circular chromatic number. As usual, for any plane graph G and its dual G^* we have $\phi_d(G) = \chi_d(G^*)$ for any d. Again, there are far more questions than answers regarding the d-rank chromatic number.

One attractive feature is that $\phi_d(A)$ and $\chi_d(A)$ may be well defined for any real matrix A, and indeed for any oriented matroid. We sketch the definition here. Let A be a real matrix whose columns are indexed by a set E. For any vector $x = y'A$ in the rowspace of A we write $x^+ = \{e \in E \mid x_e > 0\}$ and $x^- = \{e \in E \mid x_e < 0\}$, so $x^+ \cup x^-$ is the support of x. The *nullity* of x is defined to be $|x^+| + |x^-| - s$ where s is the rank of the submatrix of A consisting of those columns of A supported by x. We now define

$$\phi_d(A) = 1 + \min_{\vec{A}} \max_{x} \frac{|x^+|}{|x^-|}$$

Here $\vec{A}$ varies over all matrices obtained from A by multiplying a subset of its columns by -1, and x varies over the set of vectors in the rowspace of $\vec{A}$ whose nullity equals d. A similar definition is well defined for any oriented matroid $\mathcal{O}$ where the minimum is taken over the reorientations of $\mathcal{O}$ and the maximum is (in essence) taken over the set of acyclic orientations of every contraction-minor of $\mathcal{O}$ having rank d. Chávez et al in fact prove the following.

Theorem 5 *For any oriented matroid $\mathcal{O}$ of rank r and without coloops we have $\phi_\infty(\mathcal{O}) \leq 3r + 1$.*

As a corollary, the circular flow number $\phi_1(\mathcal{O})$ is also bounded by $3r + 1$. However, we do not know whether this bound is best possible. Chávez et al have proposed the following.

Conjecture 7 *For any oriented matroid of rank r without coloops we have $\phi_1(\mathcal{O}) \leq 3/2 + \sqrt{2r}$.*

This conjecture holds true, and is best possible, for the class of regular matroids. In particular, the dual of a complete graph of order n has rank $r = \binom{n}{2} - n + 1$ and circular flow number $n \approx 3/2 + \sqrt{2r}$.

References:

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Testing for Group-labelled Minors

Tony Huynh, University of Waterloo

A Γ-labelled graph is an oriented graph with edge labels from a group Γ. Given k pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$ and k group values $\gamma_1, \ldots, \gamma_k$, when are there k disjoint paths $P_1, \ldots, P_k$, where P_i connects s_i and t_i with value γ_i? We can solve this problem in polynomial-time when k is fixed and Γ is finite abelian. This yields a polynomial-time algorithm for testing a fixed group-labelled minor. A complementary result we would also like to prove is that Γ-labelled graphs are well-quasi-ordered, again when Γ is finite abelian.

This is joint work with Jim Geelen, University of Waterloo.

Hadwiger's Conjecture is "decidable"

Ken-ichi Kawarabayashi, National Institute of Informatics

The famous Hadwiger's conjecture asserts that every graph with no K_{t+1}-minor is t-colorable. When $t = 4$, this case is known to be equivalent to the Four Color Theorem by Wagner, and the case $t = 5$ was settled by Robertson, Seymour and Thomas. So far the cases $t \geq 6$ are wide open.

In this paper, we prove the following two theorems:

1. Every minimal counterexample to Hadwiger's conjecture has at most $f(t)$ vertices for some computable function f of t.

2. There is an $O(n^2)$ algorithm to decide Hadwiger's conjecture for fixed t.

In fact, concerning the second result, we prove the following stronger theorem:

For a given graph G and any fixed t, there is an $O(n^2)$ algorithm to output one of the following:

1. a t-coloring of G, or

2. a K_{t+1}-minor of G, or

3. a minor H of G of order at most $f(t)$ such that H does not have a K_{t+1}-minor nor is t-colorable.

The last conclusion implies that H is a counterexample to Hadwiger's conjecture.

The time complexity of the algorithm matches the best known algorithm for 4-coloring planar graphs, due to Appel and Hakken, and Robertson, Sanders, Seymour and Thomas. In fact, we have used the algorithm of 4-coloring planar graphs as a subroutine.

Joint work with Bruce Reed.

The structure of claw-free graphs

Andrew King, McGill University

A lemma of Fouquet implies that any claw-free graph with stability number at least three either is quasi-line or contains an induced C_5 in the neighbourhood of some vertex. By focusing on this distinction between quasi-line and claw-free graphs and using Chudnovsky and Seymour's structure theorem for quasi-line graphs, we find a new proof of their structure theorem for claw-free graphs with stability number at least four and containing no clique cutset.

This is joint work with Bruce Reed.

Algorithms for Finding an Induced Cycle in Planar Graphs and Bounded Genus Graphs

Yusuke KOBAYASHI, University of Tokyo, Japan

This is joint work with Ken-ichi Kawarabayashi.

We consider the problem for finding an induced cycle passing through k given vertices, which we call the *induced cycle problem*. Unfortunately, the problem of finding an induced cycle passing through two given vertices is NP-complete in a general graph. However, if the input graph is constrained to be planar and k is fixed, then the induced cycle problem can be solved in polynomial time.

In particular, an $\mathrm{O}(n^2)$ time algorithm is given for the case $k = 2$ by McDiarmid, Reed, Schrijver and Shepherd, where n is the number of vertices of the input graph.

Our main results in this paper are to improve their result in the following sense.

1. The number of vertices k is allowed to be non-trivially super constant number, up to $k = \mathrm{o}((\frac{\log n}{\log \log n})^{\frac{2}{3}})$. More precisely, when $k = \mathrm{o}((\frac{\log n}{\log \log n})^{\frac{2}{3}})$, then the ICP in planar graphs can be solved in $\mathrm{O}(n^{2+\varepsilon})$ time for any $\varepsilon > 0$.

2. The time complexity is linear if the given graph is planar and k is fixed.

3. The above results are extended to graphs embedded in a fixed surface.

We note that the linear time algorithm (the second result) is independent from the first result.

Let us point out that we give the first polynomial time algorithm for the problem for the bounded genus case. We observe that if k is as a part of the input, then the problem is still NP-complete, and so we need to impose some condition on k.

Matroids of Bounded Local Branch-width

Daniel Král', ITI, Charles University, Prague

A classical result of Courcelle from early 1990's asserts that for every MSOL formula φ and every k, there exists a linear-time algorithm deciding φ for graphs with tree-width at most k. The counterpart of this result for matroids has been established by Hliněný [J. Comb. Theory Ser. B 96 (2006), 325–351] who showed that for every MSOL formula φ, every k and every finite field F, there exists a cubic-time algorithm deciding φ for matroids represented over F whose branch-width is at most k.

Our work is inspired by a result of Grohe [J. ACM 48 (2001), 1184–1206] who showed that for every FOL formula φ and every class of graphs of bounded local tree-width, there exists an almost linear-time algorithm deciding φ for graphs from the class. We show that for every FOL formula φ, every class of matroids of bounded local branch-width and every finite field F, there exists an almost cubic-time algorithm deciding φ for matroids from the class that are represented over F.

A class of matroids has bounded local branch-width if there exists a function $f : N \to N$ such that for every matroid M from the class, the k-neighborhood of each of its elements has branch-width at most $f(k)$ for every $k \geq 1$. An element e' is contained in the k-neighborhood of e if there exist elements $e_0, \ldots, e_\ell$, $0 \leq \ell \leq k$ such that $e_0 = e$, $e_\ell = e'$ and M has a cycle of size at most k containing both e_{i-1} and e_i for every $i = 1, \ldots, \ell$. If a class of graphs has bounded local tree-width, then the class of graphic matroids of graphs from this class has bounded local branch-width and thus our result can be understood as an extension of Grohe's result to representable matroids.

This is a report on work in progress and contains results of my joint work with Tomáš Gavenčiak.

Recent work: Meta-Theorems, Computing Minor Obstructions, Digraph Decompositions.

Stephan Kreutzer. University of Oxford

1. *Algorithmic Meta-Theorems.* I am generally interested in algorithmic meta-theorems, or put differently, the parameterized complexity of logical systems. Here we aim for results of the form: *every graph property definable in first-order logic can be decided by fixed-parameter algorithms on graph classes excluding a minor* (Flum, Grohe LICS 2001). We have recently generalised this to a new property of graph classes called "locally excluded minors", which properly generalises minor ideals and graph classes with bounded local tree-width (Dawar, Grohe, Kreutzer LICS 07).

   Besides interest from a logic perspective, such results establish fixed-parameter algorithms for a broad range of problems on the graphs in question and hence are of interest from an algorithmic perspective also.

   On the positive side, we aim for finding graph classes as general as possible on which first-order definable properties can be verified efficiently. Candidates for next steps would be *bounded expansion classes* and *nowhere dense structures*. On the negative side, we aim for conditions on graph classes that ensure that first-order model-checking is not fixed-parameter tractable. One example is that first-order model-checking is not fixed-parameter tractable on the class of k-degenerate graphs, for $k > 4$ (Kreutzer 08), but tighter and more interesting bounds remain to be established.

2. *Computing Minor Obstructions.* While the Graph Minor Theorem shows that every minor ideal, i.e. minor closed class of graphs, excludes a finite class of excluded minors, the actual excluded minors remain unknown for many natural minor ideals. However, for some minor ideals the excluded minors have been shown to be computable or have actually been computed, but for other natural minor ideals this remains an open problem.

   It would be interesting to establish further decidability results for computing excluded minors and to use these results to actually compute excluded minor characterisations.

3. *Directed graphs.* While the theory of graph minors has initiated a rich structural and algorithmic theory of undirected graphs, a similar theory for directed graphs remains open. Analogues of tree-width for directed graphs have been proposed by various authors, however, it is fair to say that we are very far from a convincing theory. Further advances in this direction would be very interesting.

Graph Minor Recognition and Disjoint Routed Paths

Zhentao Li, McGill University

We have recently shown a linear time algorithm which determines whether an input graph contains K_5 as a minor and outputs a K_5-model if the input graph contains one.

We then modify this algorithm to solve the k disjoint routed paths (k-DRP) problem when $k = 2$ in linear time.

Problem 1 (k-DRP) *Given a graph G and k pairs of vertices $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$, does there exists vertex disjoint paths in G connecting each s_i to t_i.*

This is joint work with Bruce Reed and Rohan Kapadia.

List-Color-Critical Graphs on a Fixed Surface

Bojan Mohar, Simon Fraser Univsersity

A *k-list-assignment* for a graph G assigns to each vertex v of G a *list* $L(v)$ of *admissible colors*, where $|L(v)| \geq k$. A graph is *k-list-colorable* (or *k-choosable*) if it can be properly colored from the lists for every k-list-assignment.

We prove the following conjecture posed by Thomassen in 1994: "There are only finitely many list-color-critical graphs with all lists of cardinality at least 5 on any fixed surface."

This generalizes the well-known result of Thomassen on the usual graph coloring case. We use this theorem and specific parts of its proof to resolve the complexity status of the following problem about k-list-coloring graphs on a fixed surface S, where k is a fixed positive integer.

Input: A graph G embedded in the surface S.
Question: Is G k-list-colorable? If not, provide a certificate (a list-color-critical subgraph).

The cases $k = 3, 4$ are known to be NP-hard (actually even Π_2^p-complete), and the cases $k = 1, 2$ are easy. Our main results imply that the problem is tractable for every $k \geq 5$. In fact, together with our recent algorithmic result, we are able to solve it in linear time when $k \geq 5$. Our proof yields even more: if the input graph is k-list-colorable, then for any k-list-assignment L, we can construct an L-coloring of G in linear time. This generalizes the well-known linear time algorithms for planar graphs by Nishizeki and Chiba (for 5-coloring), and Thomassen (for 5-list-coloring).

We also give a polynomial time algorithm to resolve the following question:

Input: A graph G in the surface S, and a k-list-assignment L, where $k \geq 5$.
Question: Does G admit an L-coloring? If not, provide a certificate for this. If yes, then return an L-coloring.

If the graph G is k-list-colorable, then our first result gives a linear time solution. However, the second problem is more general, since it provides a coloring (or a small obstruction) for an arbitrary graph in S.

We also use our main theorem to prove another conjecture that was proposed recently by Thomassen: "For every fixed surface S, there exists a positive constant c such that every 5-list-colorable graph with n vertices embedded on S, has at least $c \cdot 2^n$ distinct 5-list-colorings for every 5-list-assignment for G." Thomassen himself proved that this conjecture holds for the usual 5-coloring.

In addition to all these results, we also made partial progress towards a conjecture of Albertson concerning coloring extensions and a progress on similar questions for triangle-free graphs and graphs of larger girth.

Joint work with Ken-ichi Kawarabayashi.

Research projects related to graph minors

Sergey Norin, Princeton University

K_t-minors in large t-connected graphs

Joint work with Robin Thomas.

Thomas conjectured the following exact structural description of large K_t-minor-free t-connected graphs.

Conjecture 8 *For every positive integer t there exists a positive integer $N = N(t)$ such that every t-connected graph G with $|V(G)| \geq N$ and no K_t minor contains a set X of $t-5$ vertices such that $G - X$ is planar.*

It appears that for $t \geq 6$ the structure of large $(t-1)$-connected graphs with no K_t minor is prohibitively complicated. Moreover, for $t \geq 8$ the assumption in the above conjecture that $|V(G)|$ is large is also necessary. In fact, Thomason has shown that there exist $\Theta(t\sqrt{\log t})$-connected graphs with no K_t-minor, but all known examples are bounded in size by a function of t. Thus, the conditions imposed on the connectivity and the size of the graph can not be relaxed.

In 2005 DeVos, Kawarabayashi, Thomas, Wollan and I verified this conjecture for $t \leq 6$. Recently, Thomas and I verified it for $t \leq 8$. We continue working on the general case.

Markov bases for graphs

Joint work with Daniel Král' and Ondrej Pangrác.

With a graph G one can associate a binomial ideal I_G as follows. Given a field $\mathbb{K}$ and a graph G consider two polynomial rings

$$\mathbb{K}[x] := \mathbb{K}[x_{(A,B)} \mid A \cap B = \emptyset, A \cup B = V(G)]$$

and

$$\mathbb{K}[y] := \mathbb{K}[y^e_{aa}, y^e_{ab}, y^e_{aa}, y^e_{bb} \mid e \in E(G)].$$

In the first ring one variable is associated with every (not necessarily proper) two-coloring of vertices of G. In the second ring there are four variables associated with every edge, where the indices aa, ab, ba and bb encode the four possible ways of coloring the ends of e. A map $\phi_G : \mathbb{K}[x] \to \mathbb{K}[y]$ maps every variable of $\mathbb{K}[x]$ to the product of the corresponding $|E(G)|$ variables of $\mathbb{K}[y]$, one per edge. The ideal I_G is defined as the kernel of $\phi(G)$. The study of this rather technical object is motivated by a statistical problem of efficient sampling from discrete conditional distributions.

In particular, the parameter $\mu(G)$, defined as the minimum over all sets of generators of I_G of the maximum degree of a polynomial in the set, is of interest. Such a set of generators, if it consists of binomials, is referred to as a *Markov basis for G*. In recent work with Daniel Král' and Ondrej Pangrác we answered a question of Develin and Sullivant characterizing graphs with $\mu(G) \leq 4$.

Theorem 6 *A graph G has $\mu(G) \leq 4$ if and only if G is series-parallel.*

Many basic questions remain open: If H is a minor G is it necessarily true that $\mu(H) \leq \mu(G)$? Is $\mu(G) \leq 6$ for every planar graph G? Is $\mu(G)$ bounded for graphs G in every proper minor-closed family? What is the order of growth of $\mu(K_t)$ as $t \to \infty$?

Tree-width and Rank-width of H-minor-free Graphs

Sang-il Oum, KAIST

This is a joint work with Fedor Fomin and Dimitrios Thilikos.

We prove that, for a fixed graph H, there is a constant c such that the tree-width and the rank-width of a graph with no H-minor are within a factor of c. To prove this, we prove that, for a fixed integer r, there is a constant c such that for every hypergraph H with $|E(H)|/|V(H)| > c$, the incidence graph of H has a K_r-minor. This can be seen as a generalization of Mader's theorem on graphs.

Some recent work

Dieter Rautenbach, TU Ilmenau, Germany

On packing shortest cycles in graphs

We study the problems to find a maximum packing of shortest edge-disjoint cycles in a graph of given girth g (g-ESCP) and its vertex-disjoint analogue g-VSCP. In the case $g = 3$, Caprara and Rizzi (2001) have shown that g-ESCP can be solved in polynomial time for graphs with maximum degree 4, but it is APX-hard for graphs with maximum degree 5, while g-VSCP can be solved in polynomial time for graphs with maximum degree 3, but it is APX-hard for graphs with maximum degree 4. For $g \in \{4, 5\}$, we show that both problems allow polynomial time algorithms for instances with maximum degree 3, but are APX-hard for instances with maximum degree 4. For each $g \geq 6$, both problems are APX-hard already for graphs with maximum degree 3. (Joint with Friedrich Regen.)

Cycle Length Parities and the Chromatic Number

In 1966 Erdős and Hajnal proved that the chromatic number of graphs whose odd cycles have lengths at most l is at most $l + 1$. Similarly, in 1992 Gyárfás proved that the chromatic number of graphs which have at most k odd cycle lengths is at most $2k + 2$ which was originally conjectured by Bollobás and Erdős.

Here we consider the influence of the parities of the cycle lengths modulo some odd prime on the chromatic number of graphs. As our main result we prove the following: Let p be an odd prime, $k \in \mathbb{N}$ and $I \subseteq \{0, 1, \ldots, p-1\}$ with $|I| \leq p-1$. If G is a graph such that the set of cycle lengths of G contains at most k elements which are not in I modulo p, then $\chi(G) \leq \left(1 + \frac{|I|}{p-|I|}\right) k + p(p-1)(r(2p, 2p) + 1) + 1$ where $r(p, q)$ denotes the ordinary Ramsey number. (Joint with Christian Löwenstein and Ingo Schiermeyer.)

Edge-Injective and Edge-Surjective Vertex Labellings

For a graph $G = (V, E)$ we consider vertex-k-labellings $f : V \to \{1, 2, \ldots, k\}$ for which the induced edge weighting $w : E \to \{2, 3, \ldots, 2k\}$ with $w(uv) = f(u) + f(v)$ is injective or surjective or both.

We study the relation between these labellings and the number theoretic notions of an additive basis and a Sidon set, present a new construction for a so-called restricted additive basis and derive the corresponding consequences for the labellings.

We prove that a tree of order n and maximum degree Δ has a vertex-k-labelling f for which w is bijective if and only if $\Delta \leq k = n/2$. Using this result we prove a recent conjecture of Ivančo and Jendrol' concerning edge-irregular total labellings for graphs that are sparse enough. (Joint with Stephan Brandt, Jozef Miškuf and Friedrich Regen.)

On Spanning Tree Congestion

We prove that every connected graph G of order n has a spanning tree T such that for every edge e of T the edge-cut defined in G by the vertex sets of the two components of $T - e$ contains at most $n^{\frac{3}{2}}$ many edges which solves a problem posed by Ostrovskii (Minimal congestion trees, *Discrete Math.* **285** (2004), 219-226.). (Joint with Christian Löwenstein and Friedrich Regen.)

Locally Dense Independent Sets in Regular Graphs of Large Girth

We present an example for a new approach which seems applicable to every graph theoretical concept defined by local conditions and regular graphs of large girth. It combines a random outer procedure processing the graph in rounds with a virtually arbitrary algorithm solving local instances within each round and combines the local solutions to a global one. The local uniformity of the considered instances and the randomness of the outer procedure make the asymptotic analysis possible. Here we apply this approach to the simplest yet fundamental example of a locally defined graph theoretical concept: independent sets in graphs.

For an integer $d \geq 3$ let $\alpha(d)$ be the supremum over all α with the property that for every $\epsilon > 0$ there exists some $g(\epsilon)$ such that every d-regular graph of order n and girth at least $g(\epsilon)$ has an independent set of cardinality at least $(\alpha - \epsilon)n$.

Considerably extending the work of Lauer and Wormald (Large independent sets in regular graphs of large girth, *J. Comb. Theory, Ser. B* **97** (2007), 999-1009) and improving results due to Shearer (A note on the independence number of triangle-free graphs, II, *J. Comb. Theory, Ser. B* **53** (1991), 300-307) and Lauer and Wormald, we present the best known lower bounds for $\alpha(d)$ for all $d \geq 3$. (Joint with Frank Göring, Jochen Harant, and Ingo Schiermeyer.)

Current Research Interests of Bruce Richter

Bruce Richter

Extending topological theorems of finite graphs to infinite cases

There is a growing literature on the subject of extending topological theorems about finite graphs to infinite cases. Diestel and his students have produced many examples (cycle spaces, MacLane's Theorem, Whitney's Theorem, Tutte's theorems on peripheral cycles, etc.) related to either the Freudenthal compactification of a locally finite graph or a slight generalization of that. I have been more interested in coming at these questions from the other side, for example: what properties of the topological space are required in order for it to make sense to define the cycle space of the topological space to have a theory that is recognizable? Thomassen and Vella have recently introduced the notion of a graph-like space, which is a topological generalization to the infinite situation. These properly include the compact spaces employed by Diestel et al. Christian, Rooney, and I have recently proved the theorems of MacLane and Whitney in graph-like spaces; Thomassen and Vella proved Menger's Theorem.

One can sometimes hope for much more. A few years ago, Thomassen and I proved that 3-connected, compact, locally connected subsets of the sphere uniquely embed in the sphere. Thomassen proved that a 2-connected, compact, locally connected space is homeomorphic to a subset of the sphere if and only if it has no $K_{3,3}$ or K_5; Rooney, Thomassen, and I have recently found the other obstructions (there are 6) to embedding a compact, locally connected space in the sphere.

There is obviously a lot of work left to be done. I am particularly interested at the moment in beginning the study of embeddings of graph-like spaces into surfaces.

2-crossing-critical graphs

Bogdan Oporowski has made substantial progress on trying to determine all 2-crossing-critical graphs. I have gotten sucked into this project. But I am also interested in determining all "large" 3-crossing-critical graphs. (A graph is *k-crossing-critical* if its crossing number is $\geq k$ and all proper subgraphs have crossing number $< k$. The graph $C_3 \times C_3$ has crossing number 3, but all its proper subgraphs have crossing number ≤ 1, so it is both 2-crossing-critical and 3-crossing-critical.)

Approximating Acyclicity Parameters of Sparse Hypergraphs

Dimitrios M. Thilikos, Department of Mathematics National and Kapodistrian University of Athens

The notions of hypertree width and generalized hypertree width were introduced by Gottlob, Leone, and Scarcello (PODS'99, PODS'01) in order to extend the concept of hypergraph acyclicity. These notions were further generalized by Grohe and Marx in SODA'06, who introduced the fractional hypertree width of a hypergraph. All these width parameters on hypergraphs are useful for extending tractability of many problems in database theory and artificial intelligence. Computing each of these width parameters is known to be an NP-hard problem. Moreover, the (generalized) hypertree width of an n-vertex hypergraph cannot be approximated within a factor $c \log n$ for some constant $c > 0$ unless $\mathsf{P} \neq \mathsf{NP}$. In this paper, we study the approximability of (generalized, fractional) hyper treewidth of sparse hypergraphs where the criterion of sparsity reflects the sparsity of their incidence graphs. Our first step is to prove that the (generalized, fractional) hypertree width of a hypergraph $\mathcal{H}$ is constant-factor sandwiched by the treewidth of its incidence graph, when the incidence graph belongs to some apex-minor-free graph class (the family of apex-minor-free graph classes includes planar graphs and graphs of bounded genus). This determines the combinatorial borderline above which the notion of (generalized, fractional) hypertree width becomes essentially more general than treewidth, justifying that way its functionality as a hypergraph acyclicity measure. While for more general sparse families of hypergraphs treewidth of incidence graphs and all hypertree width parameters may differ arbitrarily, there are sparse families where a constant factor approximation algorithm is possible. In particular, we give a constant factor approximation polynomial time algorithm for (generalized, fractional) hypertree width on hypergraphs whose incidence graphs belong to some H-minor-free graph class. This extends the results of Feige, Hajiaghayi, and Lee from STOC'05 on approximating treewidth of H-minor-free graphs.

(Joint work with Fedor V. Fomin and Petr A. Golovach)

Recent work

Robin Thomas

I have been working with Sergey Norin on the following conjecture of mine:

Conjecture 9 *For every integer $t \geq 1$ there exists an integer N_t such that every t-connected graph G on at least N_t vertices with no K_t minor has a set X of at most $t-5$ vertices such that $G\backslash X$ is planar.*

In earlier joint work with Kawarabayashi, Norin and Wollan we were able to prove Conjecture 9 for $t = 6$. That provided partial confirmation for a problem of Jørgensen, who conjectured that Conjecture 9 holds for all 6-connected graphs when $t = 6$; that is, that $N_6 = 0$. Our N_6 is not small, probably around $10^{10^{10}}$.

More recently, Norin and I were able to find two different proofs of the result from the previous paragraph, and have extended one of them to $t = 8$. That is of interest, because $t = 8$ is the first value where it is known that the integer N_t is necessary, and it is the first value for which the excluded K_t theorem of Robertson and Seymour includes the so-called vortices, a structural ingredient that makes application of the theorem much harder.

In our most recent work with Norin we have devised a strategy for attacking Conjecture 9 for general t. It seems that we might be able to prove the conjecture under the stronger hypothesis that the graph G is $(t + 1)$-connected. I hope to be able to confirm or deny this by the time of the conference. Pushing the argument to apply to t-connected graphs might be possible, but there are additional technical difficulties that need to be resolved. It is too early to tell whether that effort will be successful.

In another line of work I have been working with Zdeněk Dvořák and Daniel Král' on 3-coloring triangle-free graphs on surfaces. Our main result is that for every fixed surface Σ we have a linear-time algorithm that correctly decides whether an input triangle-free graph G drawn in Σ is 3-colorable, and if it is, then it actually finds a 3-coloring. This is not entirely new—we have announced a polynomial-time version of this earlier, but now we have been able to speed up the running time to linear time. There are two main ingredients. The first one is a special case when the graph quadrangulates Σ, except for a bounded number of precolored faces, each of bounded size. Here we need the notion of a winding number of a 3-coloring, and prove a 3-coloring result analogous to the Graph Minors VII paper of Robertson and Seymour. For the second ingredient we have an improved theorem, which in this formulation is new:

Theorem 7 *There exists an absolute constant c such that for every 4-critical triangle-free graph G drawn in a surface of Euler genus g the sum of the face-sizes of all faces of G of size at least five is at most cg.*

This generalizes, improves the bound and simplifies the proof of a theorem of Thomassen stating that there are only finitely many 4-critical graphs of girth at least five on any given surface. To speed up the algorithm to linear time we use techniques from the theory of graph minors and a result about "low tree-depth partitions" of Nešetřil and Ossona de Mendez.

Graph Minors and the Erdős-Pósa Property

Paul Wollan, University of Hamburg

A family $\mathcal{F}$ of graphs has the *Erdős-Pósa Property* (EP-property) if there exists a function $f(k)$ such that for all positive integers k, a graph G either contains k vertex disjoint subgraphs each isomorphic to a graph in $\mathcal{F}$, or there exists a set $X \subseteq V(G)$ such that $G - X$ contains no subgraph in $\mathcal{F}$. The name arises from a classic paper of Erdős and Pósa where they proved that such a function $f(k)$ exists for the family $\mathcal{C}$ of cycles. My recent work has considered the EP-property for two separate families $\mathcal{F}$ of graphs.

First, we consider cycles with modularity constraints. A family of graphs for which the EP-property does not hold is the family $\mathcal{C}_{odd}$, the set of odd cycles. This was first observed by Thomassen who gave a family of projective planar graphs forming a counter-example. Reed gave a related construction called *Escher walls*. Furthermore, Reed gives a partial characterization of when a graph has neither many disjoint odd cycles nor a small hitting set for the set of odd cycles by showing that such graphs must contain in a specific sense a large Escher wall.

Djeter and Neumann-lara give infinitely many pairs l and m such that the Erdős-Pósa property does not hold for the set of cycles of length l mod m. They propose the following problem.

Problem 1 (Djeter and Neumann-lara) *Classify for which values l and m does the family of cycles of length l mod m have the Erdős-Pósa property.*

Thomassen showed that for all positive integers m, the class of cycles of length 0 (mod m) does have the EP-property. We give the following theorem.

Theorem 8 *There exists a function $f(k)$ such that the following holds. For all positive integers k and all positive odd integers m, a graph G either has k disjoint cycles of non-zero length mod m, or there exists a set $X \subseteq V(G)$ of size at most $f(k)$ intersecting all such cycles.*

The examples of Thomassen and Reed can be modified to show that Theorem 8 is no longer true whenever the integer m is allowed to be even.

Theorem 8 is consequence of a more general result on group labeled graphs. Let Γ be an abelian group. A Γ-*labeling* of a graph G is a function $\gamma : E(G) \to \Gamma$. For any subgraph H of G, we let $\gamma(H) = \sum_{e \in E(H)} \gamma(e)$. A *non-zero cycle* in a Γ-labeled graph is simply a cycle C such that $\gamma(C) \neq 0$, where 0 denotes the identity of Γ. Recall that the *order* of any element $\alpha \neq 0$ of a group Γ is the minimum n such that $n\alpha = 0$. We prove the following.

Theorem 9 *There exist constants c and c' such that the following holds. Let G be a Γ-labeled graph where Γ does not have any elements of order two. Then for all positive integers k, either G contains k disjoint non-zero cycles, or there exists a set $X \subseteq V(G)$ with $|X| \leq c^{k^{c'}}$ such that $G - X$ does not contain any non-zero cycles.*

Theorem 8 is an immediate consequence of Theorem 9 as we can consider cycles of non-zero length mod m to be non-zero cycles in the graph labeled by the group $\mathbb{Z}_m$ with every edge having weight one.

Proof of Theorem 9 proceeds by showing that a suitably chosen minimal counter-example G contains a large grid-like subdivision W. Given a graph and a subset A of the vertices, an A-path is a path intersecting A exactly in its endpoints. We let X be the set of vertices of W of degree three. Using our choice of G as a minimal counter-example and the following theorem, we find many disjoint X-paths, each with non-zero weight.

Theorem 10 *There exists a constant c such that the following holds. Let Γ be an abelian group and let γ be a Γ-labeling of a graph G. Let $A \subseteq V(G)$ be a fixed set of vertices of G. Then either there exist disjoint A-paths $P_1, P_2, \ldots, P_k$ with $\gamma(P_i) \neq 0$, or there exists a set $X \subseteq V(G)$ with $|X| \leq ck^4$ such that $G - X$ has no non-zero A path.*

The remainder of the proof consists of using the X-paths along with the subdivision W to find many disjoint non-zero cycles, contradicting our choice of G to be a counter-example.

Several interesting questions arise from the proof methods.

Problem 2 *Does the EP-property hold for $\mathcal{C}_{1,p}$, the family of cycles of length 1 mod p where p is a prime?*

This would suffice to show that the EP-property holds for cycles of length l mod p for all $0 \leq l < p$. The restriction to p prime is indicated by the proof of Theorem 9 and the fact that any element of $\mathbb{Z}_p$ is a generator for the group. The EP-property is often closely related to the existence of efficient algorithms. This offers some evidence that the following question may have an affirmative answer.

Problem 3 *Does there exist a polynomial time algorithm for determining whether a given graph has a cycle of length l mod m for given integers l and m?*

The second family of graphs for which we consider the EP-property consists of clique minors. For all integers t, let $\mathcal{F}_t$ be the set of all graphs that contain K_t as a minor. For all $t \geq 5$, the EP-property does not hold for F_t. For example, a large projective planar grid does not contain even two disjoint K_5 minors, and yet one must delete a large number of vertices to leave the graph K_5-free. However, when the graph is assumed to be highly connected, the EP-property does hold. The following theorem is joint work with Reinhard Diestel and Ken-ichi Kawarabayashi.

Theorem 11 *[Diestel, Kawarabayashi, W.] There exists a function $f = f(p, k)$ such that the following holds. For positive integers k and p, every $((p-3)k+18p)$-connected graph G either contains k disjoint subgraphs in $\mathcal{F}_p$, or there exists a set $X \subseteq V(G)$ with $|X| \leq f(k,p)$ such that $G - X$ does not contain K_p as a minor.*

For any fixed integer p, the function $((p-3)k + 18p)$ obtained for the connectivity is the best possible up to an additive constant. The proof of Theorem 11 relies heavily on the structure theorem of Robertson and Seymour for graphs with no large clique minor. We conclude with one final question.

Problem 4 *Let $\mathcal{M}$ be a fixed minor closed class of graphs. Does there exist a polynomial time algorithm that takes as its input a graph $G \in \mathcal{M}$ and integer t and outputs a minimal size set of vertices (edges) intersecting every K_t minor?*

Problem 4 is not true when the input graph is not restricted to a fixed minor closed class.

The Structure of Cartesian Products

David Wood, The University of Melbourne

I recently studied clique minors in Cartesian products (see `arXiv:0711.1189`). The main result was a rough structural characterisation of Cartesian products with bounded *Hadwiger number* (i.e., the order of the largest clique minor). It says that for connected graphs G and H, each with at least one edge, $G\square H$ has bounded Hadwiger number if and only if at least one of the following conditions are satisfied:

- G has bounded treewidth and H has bounded order,
- H has bounded treewidth and G has bounded order, or
- G has bounded hangover and H has bounded hangover,

where a graph with bounded *hangover* is either a cycle or consists of a path of degree-2 vertices joining two connected subgraphs of bounded order with no edge between the subgraphs.

This result implies that if the product of two sufficiently large graphs has bounded Hadwiger number then it is one of the following graphs:

- a planar grid with a vortex of bounded width in the outerface,
- a cylindrical grid with a vortex of bounded width in each of the two 'big' faces, or
- a toroidal grid (with no vortex).

I found it surprising that every other Cartesian product (of large graphs) has a large clique minor. The proof of the above results reduces to the following example. Let B_n be the tree obtained by adding one leaf adjacent to the middle vertex of P_n. I call $P_n\square B_n$ the *tennis court graph*; the underlying grid is the surface of the court, and the extra edge forms the net. Despite $P_n\square B_n$ looking very much like a grid, Seese and Wessel (1986) observed that $P_n\square B_n$ has unbounded clique minors (since K_n can be drawn in the plane with all crossings on a straight line). This observation was a key in the development of vortices in Robertson and Seymour's work. Pictured here is a more efficient construction of a K_n-minor in $P_{n^2}\square B_n$.

Another key example is $G\square K_2$. This graph consists of two copies of G with an edge between corresponding vertices in the two copies. I proved that Hadwiger's Conjecture holds for all $G\square H$ (where G is fixed and $\chi(G) \geq \chi(H)$) if and only if it holds for $G\square K_2$. So studying clique minors in $G\square K_2$ is interesting. I proved that the Hadwiger number of $G\square K_2$ is tied to the treewidth of G. In particular, if G has treewidth at most k, then $G\square K_2$ has treewidth at most $2k+1$ and thus contains no K_{2k+3}-minor. Conversely, if G has treewidth at least 2^{4k^4}, then $G\square K_2$ contains a K_k-minor.

Trying to improve upon this exponential bound, lead Bruce Reed and I to the following definition (see `arXiv:0809.0724`). A *grid-like-minor of order k* in a graph G is a set $\mathcal{P}$ of paths in G, such that the intersection graph of $\mathcal{P}$ is bipartite and contains a K_k-minor. Observe that the intersection graph of the rows and columns of the grid $P_k \square P_k$ is the complete bipartite graph $K_{k,k}$, which contains a K_{k+1}-minor. Hence, $P_k \square P_k$ contains a grid-like-minor of order $k+1$. Bruce Reed and I proved that every graph with treewidth at least $ck^4\sqrt{\log k}$ contains a grid-like-minor of order k. Conversely, every graph that contains a grid-like-minor of order k has treewidth at least $\frac{k}{2}-1$. So grid-like-minors serve as a canonical witness for a graph to have large treewidth (with polynomial bounding functions, whereas only exponential bounds are known for grid minors). It is easily seen that the intersection graph of a grid-like-minor in G is a minor of $G \square K_2$. It follows that if G has treewidth at least $ck^4\sqrt{\log k}$, then $G \square K_2$ contains a K_k-minor.

I finish with a speculative conjecture that aims to bridge the dichotomy between planar graphs with bounded treewidth and planar graphs with a large grid minor. Robertson, Seymour and Thomas (1994) proved that every n-vertex planar graph G is a minor of the grid $P_{14n} \square P_{14n}$. For certain applications in graph layout, it would be helpful if for each vertex v of G, the preimage of v in the grid has bounded radius. However, this is impossible for certain planar graphs G.

Global Structure of Planar Graphs Conjecture: For every planar graph G there are graphs X and Y, such that both X and Y have bounded treewidth, Y has bounded maximum degree, and G is a minor of $X \boxtimes Y$, such that the preimage of each vertex of G has bounded radius in $X \boxtimes Y$.

This conjecture might be true with Y a path and X having treewidth 2, and it might be true with $\square$ instead of $\boxtimes$.

Judicious partitions of graphs

Xingxing Yu, School of Mathematics, Georgia Institute of Technology.

Judicious partition problems ask for partitions of the vertex set of a graph so that several quantities are optimized simultaneously. I will discuss several judicious partition problems of Bollobas and Scott, and present our recent results on these problems. This is joint work with Baogang Xu.

The circular chromatic number of K_4-free graphs with no odd hole

Yori Zwols, Columbia University

An *odd hole* in a graph is an induced cycle of odd length at least five. In this paper we show that every imperfect K_4-free graph with no odd hole either is one of two basic graphs, or has an even pair or a clique cutset. We use this result to show that every K_4-free graph with no odd hole has circular chromatic number strictly smaller than 4. We also exhibit a sequence $\{G_n\}$ of such graphs with $\lim_{n\to\infty} \chi_c(G_n) = 4$.

5 Scientific Progress Made

So far, we heard three results. The first result, which was done during the workshop, is due to Kawarabayashi, Kruetzer, and Mohar, who proposed the following algorithmic result. The second result was due to Chudnovsky and Seymour, who completed their proof of Rao's well-known conjecture concerning degree-sequence, just after the workshop. The third result was due to Paul Wollan and Bertrand Guenin, who also completed the proof of Gerards' conjecture concerning the even cycle space. In the next three subsections, we shall describe the results.

5.1 Constructing a linkless embedding by Kawarabayashi, Kreutzer and Mohar

We consider embeddings of graphs in the 3-space $\mathbb{R}^3$ (all embeddings in this paper are assumed to be piecewise linear). An embbeding of a graph in $\mathbb{R}^3$ is *linkless* if every pair of disjoint cycles forms a trivial link (in the sense of knot theory), i.e., each of the two cycles (in $\mathbb{R}^3$) can be embedded in a closed topological 2-disk disjoint from the other cycle. Robertson, Seymour and Thomas showed that a graph has a linkless embedding in $\mathbb{R}^3$ if and only if it does not contain as a minor any of seven graphs in Petersen's family (graphs obtained from K_6 by a series of $Y\Delta$ and ΔY operations). They also showed that a graph is linklessly embeddable in $\mathbb{R}^3$ if and only if it admits a *flat* embedding into $\mathbb{R}^3$, i.e. an embedding such that for every cycle C of G, there exists a closed disk $D \subseteq \mathbb{R}^3$ with $D \cap G = \partial D = C$. Clearly, every flat embeddings is linkless, but the converse is not true.

We consider the following algorithmic problem associated with embeddings of graphs in $\mathbb{R}^3$:

Flat and Linkless Embedding

Input: A graph G.

Output: Either detect one of Petersen's family graphs as a minor in G or return a flat (and linkless) embedding in the 3-space.

The first conclusion is a certificate that the given graph has no linkless and no flat embeddings. In this paper we give an $O(n^2)$ algorithm for this problem. Our algorithm does not depend on minor testing algorithms.

5.2 Well-quasi-ordering tournaments and Rao's degree-sequence conjecture by Chudnovsky and Seymour

Rao conjectured about 1980 that in every infinite set of degree sequences (of graphs), there are two degree sequences with graphs one of which is an induced subgraph of the other. In the last month or so we seem to have found a proof.

The problem turns out to be related to ordering digraphs by immersion (vertices are mapped to vertices, and edges to edge-disjoint directed paths). Immersion is not a well-quasi-order for the set of all digraphs, but for certain restricted sets (for instance, the set of tournaments) we prove it is a well-quasi-order. The connection between Rao's conjecture and tournament immersion is as follows. One key lemma reduces Rao's conjecture to proving the same assertion for degree sequences of split graphs (a split graph is a graph whose vertex set is the union of a clique and a stable set); and to handle split graphs it helps to encode the split graph as a directed complete bipartite graph, and to replace Rao's containment relation with immersion.

5.3 Isomorphism theorems for even cycles and even cuts in graphs by Guenin and Wollan

A seminal result of Whitney characterizes when two graphs have the same cycle space (or equivalently the same cut space): namely, this occurs exactly when one graph can be obtained from the other by repeatedly rearranging the graph along one and two vertex cut sets in a special way. We generalize this result to even cycles in signed graphs as well as to even cuts in grafts.