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Non-standard analysis within second order arithmetic

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Backgrounds

1. Model theoretic non-standard arguments

Within a countable model of $\mathbf{WKL_0}$ or $\mathbf{ACA_0}$, we can do non-standard analysis by means of weak saturation, standard part principle,. . .

- Non-standard arguments for $\mathbf{WKL_0}$ (Tanaka)
  - existence of Haar measure (Tanaka/Yamazaki)
- Non-standard arguments for $\mathbf{ACA_0}$
  - Riemann mapping theorem (Y)

Backgrounds

1. Model theoretic non-standard arguments

2. Non-standard arithmetic

Big five systems are characterized by non-standard arithmetic (Keisler).

We combine 1 and 2 for the following aims:

- Use non-standard arguments to do (standard) analysis in subsystems of $\mathbf{Z}_2$ easily.
- Do Reverse Mathematics for non-standard analysis.
- Characterize subsystems of $\mathbf{Z}_2$ from non-standard view point.

Backgrounds

For the previous aims, we need systems of non-standard second order arithmetic as the following:

1. Expansions of second order arithmetic and non-standard arithmetic.
2. We can do analysis in both ‘standard structure’ and ‘non-standard structure’.
3. We can use typical non-standard priciples such as ‘standard part priciple’, ‘transfer principle’,. . .
4. If we prove a ‘standard theorem’ within a ns-system, then we can find a ‘standard proof’ in (standard) second order arithmetic.

Contents

Language $\mathcal{L}_2^*$

Language of non-standard second order arithmetic ($\mathcal{L}_2^*$) are the following:

- s number variables: $x^{\mathrm{s}}, y^{\mathrm{s}}, \ldots,$
- $*$ number variables: $x^*, y^*, \ldots,$
- s set variables: $X^{\mathrm{s}}, Y^{\mathrm{s}}, \ldots,$
- $*$ set variables: $X^*, Y^*, \ldots,$
- s symbols: $\mathbf{0}^{\mathrm{s}}, \mathbf{1}^{\mathrm{s}}, =^{\mathrm{s}}, +^{\mathrm{s}}, \cdot^{\mathrm{s}}, <^{\mathrm{s}}, \in^{\mathrm{s}},$
- $*$ symbols: $\mathbf{0}^*, \mathbf{1}^*, =^*, +^*, \cdot^*, <^*, \in^*,$
- function symbol: $\surd$.

s-structure and $*$-structure

M^{s}: range of $x^{\mathrm{s}}, y^{\mathrm{s}}, \ldots$,

M^*: range of $x^*, y^*, \ldots$,

S^{s}: range of $X^{\mathrm{s}}, Y^{\mathrm{s}}, \ldots$,

S^*: range of $X^*, Y^*, \ldots$.

$V^{\mathrm{s}} = (M^{\mathrm{s}}, S^{\mathrm{s}}; 0^{\mathrm{s}}, 1^{\mathrm{s}}, \ldots)$: s-$\mathcal{L}_2$ structure.

$V^* = (M^*, S^*; 0^*, 1^*, \ldots)$: $*$-$\mathcal{L}_2$ structure.

$\surd : M^{\mathrm{s}} \cup S^{\mathrm{s}} \to M^* \cup S^*$: embedding.

We usually regard M^{s} as a subset of M^*.

(Notations)

Let φ be an $\mathcal{L}_2$-formula.

- φ^{s} : $\mathcal{L}_2^*$ formula constructed by adding $^{\mathrm{s}}$ to any $\mathcal{L}_2$ symbols in φ.
- φ^* : $\mathcal{L}_2^*$ formula constructed by adding * to any $\mathcal{L}_2$ symbols in φ.
- $\check{x}^{\mathrm{s}} := \surd(x^{\mathrm{s}})$.
- $\check{X}^{\mathrm{s}} := \surd(X^{\mathrm{s}})$.

We usually omit $^{\mathrm{s}}$ and * of relations $=, \leq, \in$.

We often say "φ holds in V^{s} (in V^*)" when φ^{s} (φ^*) holds.

Typical axioms of non-standard analysis

$\mathbf{emb}$: "$\checkmark$ is an injective homomorphism".

$\mathbf{e}$: $\forall x^* \forall y^{\mathrm{s}} (x^* < \check{y}^{\mathrm{s}} \to \exists z^{\mathrm{s}} (x^* = \check{z}^{\mathrm{s}}))$.

$\mathbf{fst}$: $\forall X^* (\mathrm{card}(X^*) \in M^{\mathrm{s}}$

$\to \exists Y^{\mathrm{s}} \forall x^{\mathrm{s}} (x^{\mathrm{s}} \in Y^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in X^*)$.

$\mathbf{st}$: $\forall X^* \exists Y^{\mathrm{s}} \forall x^{\mathrm{s}} (x^{\mathrm{s}} \in Y^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in X^*)$.

$\mathbf{\Sigma}^i_j$**overspill(saturation) :**

$\forall x^* \forall X^* (\forall y^{\mathrm{s}} \exists z^{\mathrm{s}} (z^{\mathrm{s}} \geq y^{\mathrm{s}} \wedge \varphi(\check{z}^{\mathrm{s}}, x^*, X^*)^*)$

$\to \exists y^* (\forall w^{\mathrm{s}} (y^* > \check{w}^{\mathrm{s}}) \wedge \varphi(y^*, x^*, X^*)^*))$

for any $\Sigma^i_j(\mathcal{L}_2)$-formula $\varphi(z, x, X)$.

Typical axioms of non-standard analysis

$\boldsymbol{\Sigma^i_j}\mathbf{equiv}$: $(\varphi^{\mathrm{s}} \leftrightarrow \varphi^*)$

for any $\boldsymbol{\Sigma^i_j(\mathcal{L}_2)}$-sentence $\boldsymbol{\varphi}$.

$\boldsymbol{\Sigma^i_j}\mathbf{TP}$: $\forall \boldsymbol{x}^{\mathrm{s}} \forall \boldsymbol{X}^{\mathrm{s}} (\boldsymbol{\varphi}(\boldsymbol{x}^{\mathrm{s}}, \boldsymbol{X}^{\mathrm{s}})^{\mathrm{s}} \leftrightarrow \boldsymbol{\varphi}(\check{\boldsymbol{x}}^{\mathrm{s}}, \check{\boldsymbol{X}}^{\mathrm{s}})^*)$

for any $\boldsymbol{\Sigma^i_j(\mathcal{L}_2)}$-formula $\boldsymbol{\varphi(x, X)}$.

ns-systems

$$\textbf{ns-BASIC} = (\mathbf{RCA_0})^{\mathrm{s}} + \mathrm{emb} + \mathrm{e} + \mathrm{fst} + \Sigma^0_1\mathrm{overspill} + \Sigma^1_2\mathrm{equiv} + \Sigma^0_0\mathrm{TP}.$$

$$\textbf{ns-WKL}_\mathbf{0} = (\mathbf{WKL_0})^{\mathrm{s}} + \mathrm{emb} + \mathrm{e} + \mathrm{st} + \Sigma^0_1\mathrm{overspill} + \Sigma^1_2\mathrm{equiv} + \Sigma^0_0\mathrm{TP}.$$

$$\textbf{ns-ACA}_\mathbf{0} = (\mathbf{ACA_0})^{\mathrm{s}} + \mathrm{emb} + \mathrm{e} + \mathrm{st} + \Sigma^1_0\mathrm{overspill} + \Sigma^1_2\mathrm{equiv} + \Sigma^1_1\mathrm{TP}.$$

- **ns-WKL$_\mathbf{0}$** is an extension of **WKL$_\mathbf{0}$*** introduced by Keisler.
- **ns-ACA$_\mathbf{0}$** is an extension of **ACA$_\mathbf{0}$***.

ns-systems

We can show the following.

Proposition 1.

1. **ns-BASIC** is a conservative extension of $\mathbf{RCA_0}$.
2. $\mathbf{ns\text{-}WKL_0} = \mathbf{ns\text{-}BASIC} + \mathbf{st}$.
3. $\mathbf{ns\text{-}ACA_0} = \mathbf{ns\text{-}BASIC} + \mathbf{st} + \mathbf{\Sigma^1_1 TP}$.

Interpretation of ns-$\mathbf{ACA_0}$ in $\mathbf{ACA_0}$

We interpret $\mathcal{L}_2^*$-formulas by forcing relation $\Vdash$.

(in $\mathbf{ACA_0}$)
For unbounded X and $\mathcal{L}_2^*$-formula ψ, we define

$$X \Vdash \psi \leftrightarrow \text{"for any generic ultrafilter } G \ni X \text{ on } S, \\ (M, S, M^M/G, S^M/G) \models \psi\text{"}.$$

Theorem 2.

1. ns-$\mathbf{ACA_0} \vdash \psi \Rightarrow \mathbf{ACA_0} \vdash (\Vdash \psi)$ for any $\psi \in \mathcal{L}_2^*$.
2. $\mathbf{ACA_0} \vdash (\Vdash \varphi^{\mathrm{s}}) \leftrightarrow \varphi$ for any $\varphi \in \mathcal{L}_2$.

Interpretation of ns-$\mathbf{ACA_0}$ in $\mathbf{ACA_0}$

Corollary 3 (conservativity)**.**
ns-$\mathbf{ACA_0} \vdash \varphi^{\mathbf{s}} \Rightarrow \mathbf{ACA_0} \vdash \varphi$ for any $\varphi \in \mathcal{L}_2$.

Proof.

By Theorem 2.1, a proof **ns**-$\mathbf{ACA_0} \vdash \varphi^{\mathbf{s}}$ can be transformed to a proof $\mathbf{ACA_0} \vdash\Vdash \varphi^{\mathbf{s}}$.
Then, by Theorem 2.2, $\mathbf{ACA_0} \vdash \varphi$. □

Interpretation of ns-$\mathbf{WKL_0}$ in $\mathbf{WKL_0}$

Let $\mathbf{WKL_0}' := \mathbf{WKL_0} +$ "I is a proper cut" $+\{c > I\}$.

(I: a new relation symbol, c: a new constant symbol)

Within $\mathbf{WKL_0}'$, we can define another forcing notion $\Vdash_w$ for self-embeddings.

Theorem 4.

1. ns-$\mathbf{WKL_0} \vdash \psi \Rightarrow \mathbf{WKL_0}' \vdash (\Vdash_w \psi)$ for any $\psi \in \mathcal{L}_2^*$.
2. $\mathbf{WKL_0}' \vdash (\Vdash_w \varphi^{\mathrm{s}}) \leftrightarrow \varphi$ for any $\varphi \in \mathcal{L}_2$.

Corollary 5 (conservativity)**.**

ns-$\mathbf{WKL_0} \vdash \varphi^{\mathrm{s}} \Rightarrow \mathbf{WKL_0} \vdash \varphi$ for any $\varphi \in \mathcal{L}_2$.

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Non-standard analysis in ns-systems

In this section, we show some examples of non-standard analysis in ns-systems. Our aim is to find non-standard counterparts of famous theorems which are equivalent to some ns-systems.

(We often omit $\breve{\cdot}$ for number variables.)

Within **ns-BASIC**, we can define real numbers, open sets, continuous functions, complete separable metric spaces,. . . in both $\mathbf{s}$-structure and $\boldsymbol{*}$-structure.

Definition 1 (**ns-BASIC**: standard part)**.** Let $\boldsymbol{\alpha}^* = \langle \boldsymbol{a}_{\boldsymbol{i}}^* \mid \boldsymbol{i} \in \mathbb{N}^* \rangle \in \mathbb{R}^*$ and $\boldsymbol{\beta}^{\mathrm{s}} = \langle \boldsymbol{b}_{\boldsymbol{i}}^{\mathrm{s}} \mid \boldsymbol{i} \in \mathbb{N}^{\mathrm{s}} \rangle \in \mathbb{R}^{\mathrm{s}}$. $\boldsymbol{\beta}^{\mathrm{s}}$ is said to be a standard part of $\boldsymbol{\alpha}^*$ $(\mathbf{st}(\boldsymbol{\alpha}) = \boldsymbol{\beta})$ if

$$\forall \boldsymbol{i} \in \mathbb{N}^{\mathrm{s}}\ |\boldsymbol{a}_{\boldsymbol{i}}^* - \boldsymbol{b}^{\mathrm{s}}| < \mathbf{2}^{-\boldsymbol{i}} \text{ in } \boldsymbol{V}^*.$$

We write $\boldsymbol{\alpha}_{\mathbf{1}}^* \approx \boldsymbol{\alpha}_{\mathbf{2}}^*$ if $\mathbf{st}(\boldsymbol{\alpha}_{\mathbf{1}}^* - \boldsymbol{\alpha}_{\mathbf{2}}^*) = \mathbf{0}$. Using overspill, we can show

$$\forall \boldsymbol{\alpha}^{\mathrm{s}} \in \mathbb{R}^{\mathrm{s}} \exists \boldsymbol{b}^* \in \mathbb{Q}^*\ \ \mathbf{st}(\boldsymbol{b}^*) = \boldsymbol{\alpha}^{\mathrm{s}}.$$

We can do Reverse Mathematics for some typical non-standard statements.

Theorem 6. The following are equivalent over **ns-BASIC**.

1. **ns-$\mathbf{WKL_0}$**.
2. For any $\alpha^* \in \mathbb{R}^*$,
$$\exists K^{\mathrm{s}} \in \mathbb{N}^{\mathrm{s}}\ |\alpha^*| < K^{\mathrm{s}} \longrightarrow \exists \beta^{\mathrm{s}} \in \mathbb{R}^{\mathrm{s}}\ \mathbf{st}(\alpha^*) = \beta^{\mathrm{s}}.$$

Next, we consider compactness of complete separable metric spaces.

Theorem 7. The following are equivalent over **ns-BASIC**.

1. **ns-$\mathbf{WKL_0}$**.
2. For any totally bounded complete separable metric space $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ in V^{s}, there exist $A^* \supset A^{\mathrm{s}}$ and $d^* \supset d^{\mathrm{s}}$ in V^*,
$$\forall x^* \in \hat{A}^* \exists x^{\mathrm{s}} \in \hat{A}^{\mathrm{s}}\ \mathbf{st}(x^*) = x^{\mathrm{s}}.$$

Proposition 8 (ns-BASIC). Let $\langle \boldsymbol{A}^{\mathrm{s}}, \boldsymbol{d}^{\mathrm{s}} \rangle \in \boldsymbol{V}^{\mathrm{s}}$ and $\langle \boldsymbol{A}^{*}, \boldsymbol{d}^{*} \rangle \in \boldsymbol{V}^{*}$. If $\forall \boldsymbol{x}^{*} \in \hat{\boldsymbol{A}}^{*} \exists \boldsymbol{x}^{\mathrm{s}} \in \hat{\boldsymbol{A}}^{\mathrm{s}}\, \mathbf{st}(\boldsymbol{x}^{*}) = \boldsymbol{x}^{\mathrm{s}}$, then $\langle \boldsymbol{A}^{\mathrm{s}}, \boldsymbol{d}^{\mathrm{s}} \rangle$ is Heine-Borel compact.

Corollary 9. **ns-WKL$_0$** $\vdash$ (Heine-Borel theorem)$^{\mathrm{s}}$.
Thus, **WKL$_0$** $\vdash$Heine-Borel theorem.

Next, we consider continuous functions.

Definition 2 (ns-BASIC). Let $\boldsymbol{f}^{*}$ be a continuous function in $\boldsymbol{V}^{*}$. $\boldsymbol{f}^{*}$ is said to be s-continuous if

$$\mathbf{st}(\boldsymbol{\alpha}^{*}) = \mathbf{st}(\boldsymbol{\beta}^{*}) \in \mathbb{R}^{\mathrm{s}} \longrightarrow \mathbf{st}(\boldsymbol{f}^{*}(\boldsymbol{\alpha}^{*})) = \mathbf{st}(\boldsymbol{f}^{*}(\boldsymbol{\beta}^{*})) \in \mathbb{R}^{\mathrm{s}}.$$

$\boldsymbol{f} \in \boldsymbol{V}^{\mathrm{s}}$ is said to be a standard part of $\boldsymbol{f}^{*}$ ($\mathbf{st}(\boldsymbol{f}^{*}) = \boldsymbol{f}$) if

$$\boldsymbol{f}(\mathbf{st}(\boldsymbol{\alpha}^{*})) = \mathbf{st}(\boldsymbol{f}^{*}(\boldsymbol{\alpha}^{*})).$$

Theorem 10. The following are equivalent over **ns-BASIC**.

1. **ns-WKL$_0$**.
2. If f^* is an $\mathbf{s}$-continuous continuous function in V^*, then there exists a continuous function $f^{\mathbf{s}}$ in $V^{\mathbf{s}}$ such that $\mathbf{st}(f^*) = f^{\mathbf{s}}$.
3. For any continuous function $f^{\mathbf{s}}$ on $[0, 1]$ in $V^{\mathbf{s}}$, there exists a piecewise linear $\mathbf{s}$-continuous continuous function f^* on $[0, 1]$ in V^* such that $\mathbf{st}(f^*) = f^{\mathbf{s}}$.

Proposition 11 (ns-BASIC). If $\mathbf{st}(f^*) = f^{\mathbf{s}}$ and $\alpha^* \approx \beta^* \longrightarrow f^*(\alpha^*) \approx f^*(\beta^*)$, then f is uniformly continuous.

Corollary 12. **ns-WKL$_0$** $\vdash$(every continuous function on $[0, 1]$ is uniformly continuous)$^{\mathbf{s}}$.
Thus, **WKL$_0$** $\vdash$(every continuous function on $[0, 1]$ is uniformly continuous).

Next, we consider sequential compactness. For this, the transfer principle is very useful.

We usually use $\mathbf{\Sigma^0_1 TP}$ for reals and $\mathbf{\Sigma^1_1 TP}$ for continuous functions, but they are equivalent.

Theorem 13. The following are equivalent over $\textbf{ns-WKL}_\mathbf{0}$.

1. $\textbf{ns-ACA}_\mathbf{0}$.
2. $\mathbf{\Sigma^0_1 TP}$.
3. $\mathbf{\Sigma^0_1 TP}$ for real numbers in $\boldsymbol{V}^{\mathbf{s}}$.
4. $\mathbf{\Sigma^1_1 TP}$ for continuous functions in $\boldsymbol{V}^{\mathbf{s}}$.

Theorem 14 (ns-$\mathbf{ACA_0}$).

1. Let $\mathcal{A}^{\mathrm{s}} : \mathbb{N}^{\mathrm{s}} \longrightarrow \mathbb{R}^{\mathrm{s}}$ be a real sequence on $[0, 1]$ in V^{s}, and let $H^* \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$. Then, $\mathbf{st}(\surd(\mathcal{A}^{\mathrm{s}})(H^*))$ is an accumulation value of $\mathcal{A}^{\mathrm{s}}$.
2. Let $\mathcal{F}^{\mathrm{s}}$ be a sequence of continuous functions on $[0, 1]$ in V^{s}, and let $H^* \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$. If $\mathbf{st}(\surd(\mathcal{F}^{\mathrm{s}})(H^*))$ exists, then it is an accumulation value of $\mathcal{F}^{\mathrm{s}}$.

Question: Can we prove the converse?

Proposition 15 (ns-$\mathbf{ACA_0}$). If $\mathcal{F}^{\mathrm{s}}$ is uniformly bounded and equicontinuous, then $\surd(\mathcal{F}^{\mathrm{s}})(H^*)$ is s-continuous.

Corollary 16. ns-$\mathbf{ACA_0}$ $\vdash$ (Ascoli's lemma)$^{\mathrm{s}}$.
Thus, $\mathbf{ACA_0}$ $\vdash$Ascoli's lemma.

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We will show a version of Riemann mapping theorem within $\mathbf{WKL_0}$ as an example of an application of ns-systems to Reverse Mathematics.

Theorem 17 (**$\mathbf{WKL_0}$**: RMT for Jordan regions)**.**

JRMT: Let γ be a Jordan curve on $\mathbb{C}$ and D be the interior of γ. Then, there exists a biholomorphic map $h : \Delta(1) \to D$.

$(\Delta(1) = \{ z \in \mathbb{C} : |z| < 1 \}.)$

Proof.

By Corollary 5, we only need to show **ns-$\mathbf{WKL_0}$** $\vdash$(JRMT)$^{\mathrm{s}}$.
Thus, we reason within **ns-$\mathbf{WKL_0}$**.
Let γ^{s} be a Jordan curve on $\mathbb{C}^{\mathrm{s}}$. Then, there exists a piecewise linear Jordan curve $\gamma^* \in V^*$ such that $\mathrm{st}(\gamma^*) = \gamma^{\mathrm{s}}$.
Let D^* be the interior of γ^*.

Lemma 18 ($\mathbf{RCA_0}$: RMT for polygonal regions (Horihata-Y)).

PRMT: Let γ be a piecewise linear Jordan curve on $\mathbb{C}$ and D be the interior of γ. Then, there exists a biholomorphic map $h : \Delta(1) \longrightarrow D$.

Using this lemma, PRMT holds in V^{s}. Then, by $\mathbf{\Sigma^1_2 equiv}$, PRMT holds in V^*. Thus, there exists a biholomorphic function $h^* : \Delta(1) \longrightarrow D^*$.
By the Schwarz lemma, $h^{*\prime}$ is bouded by $K_i^{\mathrm{s}} \in \mathbb{N}^{\mathrm{s}}$ on $\Delta(1 - 2^{-i})$ for any $i \in \mathbb{N}^{\mathrm{s}}$. Thus, h^* is s-continuous on $\Delta(1)$.
Then we can easily show that $h^{\mathrm{s}} = \mathbf{st}(h^*)$ is a desired biholomorphic function in V^{s}.
Hence **ns-WKL$_0$** $\vdash$(JRMT)$^{\mathrm{s}}$. □

Similarly, we can show the following.

Theorem 19.

1. **ns**-$\mathbf{WKL_0}$ $\vdash$(Jordan curve theorem)$^{\mathbf{s}}$.
   Thus, $\mathbf{WKL_0}$ $\vdash$Jordan curve theorem. (Sakamoto-Y)
2. **ns**-$\mathbf{ACA_0}$ $\vdash$(Riemann mapping theorem)$^{\mathbf{s}}$.
   Thus, $\mathbf{ACA_0}$ $\vdash$Riemann mapping theorem.

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ns-system for $\mathsf{WWKL_0}$?

We consider the next property.

$$\mathbf{LMP}: \forall \boldsymbol{H}^* \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}\ \forall \boldsymbol{T}^* \subseteq \mathbf{2}^{<\boldsymbol{H}^*}$$
$$\mathbf{st}\left(\frac{\mathbf{card}(\{\boldsymbol{\sigma}^* \in \boldsymbol{T}^* \mid \mathbf{lh}(\boldsymbol{\sigma}^*) = \boldsymbol{H}^*\})}{\mathbf{2}^{\boldsymbol{H}^*}}\right) > \mathbf{0}$$
$$\to \exists \boldsymbol{\sigma}^* \in \boldsymbol{T}^* \mathbf{lh}(\boldsymbol{\sigma}^*) = \boldsymbol{h}^* \wedge \boldsymbol{\sigma}^* \cap \mathbb{N}^{\mathrm{s}} \in \boldsymbol{V}^{\mathrm{s}}.$$

$\mathbf{LMP}$ is a principle for Loeb Measure theory.

Proposition 20. ns-**BASIC** $+ \mathbf{LMP} \vdash (\mathsf{WWKL_0})^{\mathrm{s}}$.

ns-system for $\mathbf{WWKL_0}$?

Question 1: Is **ns-BASIC** $+ \mathrm{LMP}$ a conservative extension of $\mathbf{WWKL_0}$?

Question 2: Let $(M, S) \models \mathbf{WWKL_0}$ be a countable model. Then, is there $\bar{S} \supseteq S$ such that $(M, \bar{S}) \models \mathbf{WKL_0}$ and for any binary tree $T \in \bar{S}$,

$$\lim_{i \to \infty} \frac{\mathrm{card}(\{\sigma \in T \mid \mathrm{lh}(\sigma) = i\})}{2^i} > 0$$

$\rightarrow \exists f \in S\ f$ is a path of T.

We can show that 2 implies 1.

other ns-systems?

$$\textsf{ns-ATR}_0 = \textsf{ns-ACA}_0 + (\textsf{ATR}_0)^{\mathrm{s}} + \Sigma^1_2\mathrm{TP}?$$

$$\textsf{ns-}\Pi^1_1\textsf{CA}_0 = \textsf{ns-ACA}_0 + (\Pi^1_1\textsf{CA}_0)^{\mathrm{s}} + \Sigma^1_2\mathrm{TP} + \Sigma^1_1\mathrm{overspill}?$$

Question 3: Are there any good principles for ns-systems? (saturation principles?)

H. Jerome Keisler. Nonstandard arithmetic and reverse mathematics. The Bulletin of Symbolic Logic, 12(1):100.125, 2006.

Ali Enayat. From bounded arithmetic to second order arithmetic via automorphisms. the proceedings of the workshop and conference Logic, Algebra, and Arithmetic, volume 26 of Lecture Notes in Logic, 2006.

Kazuyuki Tanaka. The self-embedding theorem of $\mathbf{WKL_0}$ and a non-standard method. Annals of Pure and Applied Logic, 84:41-49, 1997.

Y. Formalizing non-standard arguments in second order arithmetic. preprint.