Combinatorics of Words

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PARTITIONS

L a set.

L-**partition** of a set X:

- distinct block containing a for each $a \in L$

- free blocks

WORDS

Think of L as an alphabet.

Words over L:

$$a_0 a_1 \cdots a_n$$

$a_0, a_1, \ldots, a_n \in L$.

Infinite words over L:

$$a_0 a_1 \cdots a_n \cdots$$

$a_n \in L$ $(n \in \omega)$

VARIABLE WORDS

Fix an infinite list of variables:

$$v_0, v_1, v_2, \ldots, v_n, \ldots$$

Families of variable words:

- $W(L, m, n)$
- $W(L, m)$
- $W(L, m, n) \uparrow$

Allow n or m or both to be ω.

We can make the following identification

$W(L, n, m) =$ L-partitions of m with n free blocks

THE HALES-JEWETT THEOREM

is the following generalization of van der Waerden's Theorem on arithmetic progressions:

For any finite alphabet L and any positive integer c, there is an integer n so large that for any coloring

$$W(L, 0, n) = C_1 \cup C_2 \cup \cdots \cup C_c$$

there is a monochromatic *line* i.e. a monochromatic set of the form

$$\{w(a) \mid a \in L\}$$

where $w \in W(L, 1, n)$.

HIGHER DIMENSIONAL HALES-JEWETT

THEOREM. (Hales-Jewett) For any finite alphabet L and any positive integers c and m, there is an integer n so large that for any coloring

$$W(L, 0, n) = C_1 \cup C_2 \cup \cdots \cup C_c$$

there is a monochromatic subspace of $W(L, 0, n)$ of dimension m i.e. a monochromatic set of the form

$$\{w(a_0 a_1 \cdots a_{m-1}) \,|\, a_0 a_1 \cdots a_{m-1} \in W(L, 0, m)\}$$

where $w \in W(L, m, n)$.

Remark: w can be chosen in $W(L, m, n) \uparrow$.

GRAHAM-ROTHSCHILD PARAMETER SET THEOREM

THEOREM (Graham-Rothschild) For any finite alphabet L and any positive integers c, m and k there is an integer n so large that for any coloring

$$W(L,k,n) = C_1 \cup C_2 \cup \cdots \cup C_c$$

there is a monochromatic subset of $W(L,k,n)$ of dimension m of the form

$$\{w(a_0a_1\cdots a_{m-1}) \,|\, a_0a_1\cdots a_{m-1} \in W(L,k,m)\}$$

where $w \in W(L,m,n)$.

Remark: The set in the conclusion is not quite a subspace by the conditions on the variables. However, notice that e.g.

$$v_0v_1\cdots v_{k-1}a_k\cdots a_{m-1} \in W(L,k,m)$$

for all $a_k,\ldots,a_{m-1} \in L \cup \{v_0,\ldots,v_{k-1}\}$.

AN INFINITARY VERSION OF THE GRAHAM-ROTHSCHILD THEOREM

THEOREM. (with Simpson) For any finite alphabet L, any positive integers c and k and any coloring

$$W(L, k, \omega) = C_1 \cup C_2 \cup \cdots \cup C_c$$

where each C_i is Borel there is a monochromatic subset of $W(L, k, \omega)$ of the form

$$\{w(a_0 a_1 \cdots a_m \cdots) \,|\, a_0 a_1 \cdots a_m \cdots \in W(L, k, \omega)\}$$

where $w \in W(L, \omega, \omega)$.

The case when $L = \emptyset$ is known at the **Dual Ramsey Theorem**.

STRENGTH?

PROBLEM: Determine the strength of the Dual Ramsey Theorem along with various restricted versions.

Some partial results:

(Slaman) The Dual Ramsey Theorem can be proved in $\Pi^1_1 - CA_0$.

The Dual Ramsey Theorem for open colorings and partitions with $k+1$ blocks implies Ramsey's Theorem for sets of size k.

(Miller-Solomon) (RCA_0) The Dual Ramsey Theorem restricted to open colorings and paritions with 4 blocks implies ACA_0.

(Miller-Solomon) WKL_0 does not imply the Dual Ramsey Theorem for open colorings and partitions with 3 blocks.

DUAL GALVIN-PRIKRY THEOREM

THEOREM. (with Simpson) For any positive integer c and any coloring

$$W(\emptyset, \omega, \omega) = C_1 \cup \cdots \cup C_c$$

where each C_i is Borel there is $w \in W(\emptyset, \omega, \omega)$ such that $\{w(u) \mid u \in W(\emptyset, \omega, \omega)\}$ is monochromatic.

The Dual Galvin-Prikry Theorem easily implies the Dual Ramsey Theorem and the Galvin-Prikry Theorem.

LARGE PARTITIONS

$w \in W(\emptyset, k, n)$ is **large** if k is greater than the least occurrence of v_1.

COROLLARY of Dual G-P Thm. For any positive integers m, k and c, there is an integer n so large that for all colorings

$$W(\emptyset, k, n) = C_1 \cup \cdots \cup C_c$$

there exists $m' \geq m$ and large $w \in W(\emptyset, m', n)$ such that $\{w(u) \,|\, u \in W(\emptyset, k, m')\}$ is monochromatic.

The corollary easily implies the Paris-Harrington Theorem, so it can't be proved in ACA_0.

COMBINATORIAL CORE

The infinitary theorems discussed to this point are proved in two stages:

1. Establish the *combinatorial core* of the theorem.

2. Establish the full topological version by a fusion argument like that used to establish Ellentuck's Theorem.

All of the infinitary theorems to this point have the same combinatorial core:

For any coloring of $W(L, 0)$ with finitely many colors there is a $w \in W(L, \omega, \omega)$ such that the collection of *initial parts* of $w(u)$ ($u \in W(L, \omega, \omega)$) is monochromatic.

REDUCTION RELATIONS

For u a variable word over L and $\vec{w} = w_0, \ldots, w_n$ a sequence of variable words over L define

$$u \leq \vec{w}$$

iff $u = w_0(t_0) \cdots w_n(t_n)$ for some variable words $t_0, \ldots, t_n$. For $\vec{u} = u_0, u_1, \ldots$ and $\vec{v} = w_0, w_1, \ldots$ infinite sequences of variable words define

$$\vec{u} \leq \vec{w}$$

iff there is an infinite subsequence $\vec{w}'$ of $\vec{w}$ such that $\vec{w}'$ can be written as $\vec{w}'_0 * \vec{w}'_1 * \cdots$ where $u_n \leq \vec{w}'_n$ for all $n \in \omega$.

For $e : \omega \to \omega$, $\mathcal{S}(L, e)$ is the set of infinite sequences

$$w_0, w_1, \ldots, w_n, \ldots$$

such that $w_n \in W(L, e(n))$.

STRONGER THEOREMS

THEOREM. For any finite L, all $e : \omega \to \omega$ and any coloring

$$\mathcal{S}(L, e) = C_1 \cup \cdots \cup C_c$$

where each C_i is Borel there is $\vec{w} \in \mathcal{S}(L, e)$ such that the set of all $\vec{u}$ in $\mathcal{S}(L, e)$ with $\vec{u} \leq \vec{v}$ is monochromatic.

This easily implies the Galvin-Prikry Theorem.

The combinatorial core is:

(CC) For any finite L, any positive integer n and any coloring

$$W(L, n) = C_1 \cup \cdots \cup C_c$$

there is $\vec{w} \in \mathcal{S}(L, \langle n, n+1, \ldots \rangle)$ such that the set of $\vec{u}(0)$ where $\vec{u} \in \mathcal{S}(L, \langle n, n+1, \ldots \rangle)$ and $\vec{u} \leq \vec{w}$ is monochromatic.

For fixed n, this implies Ramsey's Theorem for sets of size n. Lower bounds for the case $n = 1$?

ULTRAFILTERS

The previous theorem uses the general theory of *families of idempotent ultrafilters* (e.g. see recent work with Hindman and Strauss and their text *Algebra in the Stone-Cech Compactification*). What is the status of this theory in terms of reverse mathematics? Hirst, Simpson and Mummert have some results in this direction.

EXAMPLE. There exist ultrafilters V on $W(L,0)$ and U on $W(L,1)$ such that

- $V * V = V$
- $h_a(U) = V$ for all $a \in L$
- $V * U = U * V = U$
- $U * U = U$

The example is strong enough to prove the theorem on $\mathcal{S}(L, \langle 1, 1, 1, \ldots \rangle)$.