HJM: A Unified Approach to Dynamic Models in Financial Mathematics

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Contents

Equity Market Model

- $\{S_t\}_{t\geq 0}$ price process
- 0 interest rate (discount factor $\beta_t \equiv 1$)
- No dividend

Classical Approach

- Specify dynamics for S_t, e.g. GBM in Black Scholes case

$$dS_t = S_t \sigma_t \, dW_t$$

- Compute prices of derivatives by expectation, e.g.

$$C_0(T, K) = \mathbb{E}\{(S_T - K)^+\}$$

Actively/Liquidly Traded Instrument

Main Assumptions

- At each time $t \geq 0$ we observe $C_t(T, K)$ the market price at time t of European call options of strike K and maturity $T > t$.
- Market prices by expectation

$$C_t(T, K) = \mathbb{E}\{(S_T - K)^+ | \mathcal{F}_t\}$$

for some measure (**not necessarily unique**) $\mathbb{P}$

Empirical Fact

Many observed option price movements cannot be attributed to changes in S_t

- Fundamental market data: **Surface** $\{C_t(T, K)\}_{T,K}$ **instead** of S_t

Remarks

No arbitrage implies

- $C_0(T, K)$ increasing in T
- $C_0(T, K)$ non-increasing and convex in K
- $\lim_{K \nearrow \infty} C_0(T, K) = 0$
- $\lim_{K \searrow 0} C_0(T, K) = S_0$

Realistic Set-Up

- We actually observe

$$C_t(T_i, K_{ij}) \qquad i = 1, \cdots, m, \qquad j = 1, \cdots, n_i$$

More Remarks

- Switch to notation $\tau = T - t$ for **time to maturity**
- Call surface $\{\tilde{C}_t(\tau, K)\}$ of prices $C_t(T, K)$ parameterized by $\tau \geq 0$ and $K \geq 0$.

$$\tilde{C}_t(\tau, K) = \mathbb{E}\{(S_{t+\tau} - K)^+ | \mathcal{F}_t\} = \mathbb{E}^{\mathbb{P}_t}\{(S_{t+\tau} - K)^+\}.$$

$$\tilde{C}_t(\tau, K) = \int_0^\infty (x - K)^+ \, d\mu_{t,t+\tau}(dx)$$

Crucial Fact

For each $\tau > 0$, the **knowledge of all the prices** $\tilde{C}_t(\tau, K)$ completely **determines** the **marginal** distribution $\mu_{t,t+\tau}$ on $[0, \infty)$.

Black-Scholes Formula

Dynamics of the underlying asset

$$dS_t = S_t \sigma dW_t, \qquad S_0 = s_0$$

Wiener process $\{W_t\}_t$, $\sigma > 0$.

Price of a call option

$$\tilde{C}_t(\tau, K) = S_t \Phi(d_1) - K\Phi(d_2)$$

with

$$d_1 = \frac{-\log M_t + \tau\sigma^2/2}{\sigma\sqrt{\tau}}, \qquad d_1 = \frac{-\log M_t - \tau\sigma^2/2}{\sigma\sqrt{\tau}}$$

- $M_t = K/S_t$ moneyness of the option
- Φ *error function*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy, \qquad x \in \mathbb{R}.$$

Implied Volatility

- Classical Black-Scholes framework
- On any given day t fix
  - **maturity** T (or **time to maturity** τ)
  - **strike** K
- price is an **increasing function** of the parameter σ

$$\sigma \leftrightarrows \tilde{C}_t^{(BS)}(\tau, K) \qquad \textbf{one-to-one}$$

Given an option **price quoted** on the market, its **implied volatiltiy** is the unique number $\sigma = \Sigma_t(\tau, K)$ for which $\tilde{C}_t(\tau, K) = C$.
Used by **ALL** market participants as the *currency* for options

*the **wrong** number to put in the **wrong** formula to get the **right** price.*

Implied Volatility Code-Book

$$\{\tilde{C}_t(\tau, K);\ \tau > 0, K > 0\} \leftrightarrows \{\Sigma_t(\tau, K);\ \tau > 0, K > 0\}$$

- **Static** ($t = 0$) "No arbitrage" conditions difficult to formulate
  - (B. Dupire, Derman-Kani, P.Carr, ...)
- **Dynammic** No arbitrage conditions difficult to check in a dynamic framework
  - (Derman-Kani for tree models)

Search for another Option Code-Book

$$dS_t = S_t \sigma_t \, dW_t, \qquad S_0 = s_0$$

If $t > 0$ is fixed, for any τ_1 and τ_2 such that $0 < \tau_1 < \tau_2$, then for any convex function ϕ on $[0, \infty)$ we have (Jensen)

$$\int_0^\infty \phi(x) \mu_{t,t+\tau_1}(dx) \leq \int_0^\infty \phi(x) \mu_{t,t+\tau_2}(dx)$$

Or

$$\mu_{t,t+\tau_1} \preceq \mu_{t,t+\tau_2}$$

- $\{\mu_{t,t+\tau}\}_{\tau>0}$ non-decreasing in the **balayage order**
- Existence of a Markov martingale $\{Y_\tau\}_{\tau\geq 0}$ with marginal distributions $\{\mu_{t,t+\tau}\}_{\tau>0}$.
- **NB** $\{Y_\tau\}_{\tau\geq 0}$ contains more information than the mere marginal distributions $\{\mu_{t,t+\tau}\}_{\tau>0}$

Local Volatility Code-Book

On Wiener space (in Brownian filtration)
Martingale Property implies

$$Y_\tau = Y_0 + \int_0^\tau Y_s a(s)\, dB_s$$

Markov Property implies

$$a(s, \omega) = a_t(s, Y_s(\omega))$$

At each time t, I choose surface $\{a_t(\tau, K)\}_{\tau>0, K>0}$ as an alternative code-book for $\{\tilde{C}(\tau, K)\}_{\tau>0, K>0}$.

$\{a_t(\tau, K)\}_{\tau>0, K>0}$ was introduced in a static framework (i.e. for $t = 0$) simultaneously by Dupire and Derman and Kani **local volatility surface**

PDE Code, I

Assume

$$dY_\tau = Y_\tau a_t(\tau, Y_\tau) d\tilde{B}_\tau, \quad \tau > 0$$

with initial condition

$$Y_0 = S_t$$

and $\mu_{t,t+\tau}$ has density $g_t(\tau, x)$.

Breeden-Litzenberger argument (specific to the *hockey-stick* pay-off)

$$\tilde{C}_t(\tau, K) = \int_0^\infty (x - K)^+ g_t(\tau, x) dx$$

Differentiate both sides twice with respect to K

$$\frac{\partial^2}{\partial K^2} \tilde{C}_t(\tau, K) = g_t(\tau, K). \tag{1}$$

PDE Code, II

$$(Y_\tau - K)^+ = (Y_0 - K)^+ + \int_0^\tau \mathbf{1}_{[K,\infty)}(Y_s) dY_s + \frac{1}{2}\int_0^\tau \delta_K(Y_s)\, d[Y,Y]_s$$

and taking $\mathbb{E}_t$ - expectations on both sides using the fact that Y is a martingale satisfying $d[Y,Y]_s = Y_s^2 a_t(s,Y_s)^2 ds$, we get:

$$\begin{aligned}\tilde{C}_t(\tau,K) &= (S_t - K)^+ + \frac{1}{2}\int_0^\tau \mathbb{E}_t\{\delta_K(Ys)Y_s^2 a_t(s,Y_s)^2\}\, ds \\ &= (S_t - K)^+ + \frac{1}{2}\int_0^\tau K^2 a_t(s,K)^2 g_t(s,K)\, ds.\end{aligned}$$

Take derivatives with respect to τ on both sides

$$\frac{\partial \tilde{C}(\tau,K)}{\partial \tau} = \frac{1}{2} K^2 a_t(\tau,K)^2 g_t(\tau,K).$$

PDE Code, III

Equate both expressions of $g_t(\tau, K)$

$$a_t(\tau, K)^2 = \frac{2\partial_\tau \tilde{C}(\tau, K)}{K^2 \partial^2_{KK} \tilde{C}(\tau, K)}$$

Smooth **Call Prices** $\hookrightarrow$ **Local Volatilities**

PDE Code IV

From local volatility surface $\{a_t(\tau, K)\}_{\tau,K}$ to call option prices $\{\tilde{C}_t(\tau, K)\}_{\tau,K}$ solve PDE (Dupire's PDE)

$$\partial_\tau \tilde{C}(\tau, K) = \frac{1}{2}K^2\, a^2(\tau, K)\partial^2_{KK}\tilde{C}(\tau, K), \qquad \tau > 0,\ K > 0$$

$$\tilde{C}(0, K) = (S_t - K)^+$$

$$\{\tilde{C}_t(\tau, K);\ \tau > 0, K > 0\} \leftrightarrow \{a_t(\tau, K);\ \tau > 0, K > 0\}$$

Why is that better?

NEED ONLY POSITIVITY for no arbitrage

Dupire Formula

If

$$dS_t = S_t \sigma_t \, dW_t$$

for some Wiener process $\{W_t\}_t$ and some adapted non-negaitve process $\{\sigma_t\}_t$, then

$$a_t(\tau, K)^2 = \mathbb{E}_t\{\sigma_{t+\tau}^2 | S_{t+\tau} = K\}.$$

HJM Prescription

- Compute $a_0(\tau, K)$ from **market call prices** (**Initial condition**)
- Define a dynamic model by defining the **dynamics of the local volatility surface**

$$da_t(\tau, K) = \alpha_t(\tau, K)dt + \beta_t(\tau, K)dW_t$$

Consistency

- **Question** Under what conditions do the Call Prices computed from the dynamics of $a_t(\tau, K)$ come from a model of the form of the form

$$dS_t = S_t \sigma_t dB_t^1$$

with initial condition $S_0 = s$ the underlying instrument?

- **Answer**

$$\sigma_t = a_t(0, S_t)$$

- **Question** Under what conditions on the dynamics of $a_t(\tau, K)$ are the call prices (local) martingales?
- **Answer**

$$(\alpha + \frac{\|\beta\|^2}{2}) \cdot \frac{\partial^2}{\partial K^2} C + \frac{\partial}{\partial t} \langle a, \frac{\partial^2}{\partial K^2} C \rangle_t = \frac{\partial}{\partial T} a \cdot \frac{\partial^2}{\partial K^2} C$$

Recall classical HJM drift condition

$$\alpha(t, T) = \beta(t, T) \cdot \int_t^T \beta(t, s) ds = \sum_{j=1}^d \beta^{(j)}(t, T) \int_t^T \beta^{(j)}(t, s) ds.$$

Main Result Statement

The dynamic model of the local volatility surface given by the system of equations

$$d\tilde{a}_t(\tau, K) = \tilde{\alpha}_t(\tau, K)dt + \tilde{\beta}_t(\tau, K)dW_t, \qquad t \geq 0, \tag{2}$$

is **consistent** with a spot price model of the form

$$dS_t = S_t \sigma_t dB_t$$

for some Wiener process $\{B_t\}_t$, and **does not allow for arbitrage** if and only if a.s. for all $t > 0$:

$$\bullet \tilde{a}_t(0, S_t) = \sigma_t \tag{3}$$

$$\bullet \partial_\tau \tilde{a}_t(\tau, K) \partial^2_{KK} \tilde{C}_t(\tau, K) = \tag{4}$$

$$\left(\tilde{a}_t(\tau, K)\tilde{\alpha}_t(\tau, K) + \frac{\|\beta_t(\tau, K)\|^2}{2} \right) \partial^2_{KK} \tilde{C}_t(\tau, K) + \frac{d}{dt} \langle \tilde{a}_\cdot(\tau, K)^2, \partial^2_{KK} \tilde{C}_\cdot(\tau, K) \rangle_t$$

$\langle \cdot \; \cdot \rangle_t$ quadratic covariation of two semi-martingales.

Practical Monte Carlo Implementation

- Start from a model for $\beta_t(\tau, K)$ (say a stochastic differential equation);
- Get S_0 and $C_0(\tau, K)$ from the market and compute $\partial^2_{KK} C_0$, a_0 and β_0 from its model;
- Loop: for $t = 0, \Delta t, 2\Delta t, \cdots$
  1. Get $\alpha_t(\tau, K)$ from the drift condition (**??**);
  2. Use Euler to get
     - $a_{t+\Delta t}(\tau, K)$ from the dynamics of the local volatility given by (**??**);
     - $S_{t+\Delta t}$ from S_t Dynamics;
     - $\beta_{t+\Delta t}$ from its own model;

Markovian Spot Models ($\beta \equiv 0$)

$$\tilde{\alpha}_t(\tau, K) = \frac{d}{dt}\tilde{a}_t(\tau, K).$$

Drift condition reads

$$\partial_\tau \tilde{a}_t(\tau, K) = \tilde{\alpha}_t(\tau, K)$$

Hence

$$\partial_\tau \tilde{a}_t(\tau, K) = \frac{d}{dt}\tilde{a}_t(\tau, K)$$

which shows that for fixed K, $\tilde{a}_t(\tau, K)$ is the solution of a transport equation whose solution is given by:

$$\tilde{a}_t(\tau, K) = \tilde{a}_0(\tau + t, K)$$

and the consistency condition forces the special form

$$\sigma_t = a_0(t, S_t)$$

of the spot volatility. Hence we proved:

The local volatility is a process of bounded variation for each τ and K fixed if and only if it is the deterministic shift of a constant shape and the underlying spot is a Markov process.

A First Parametric Family

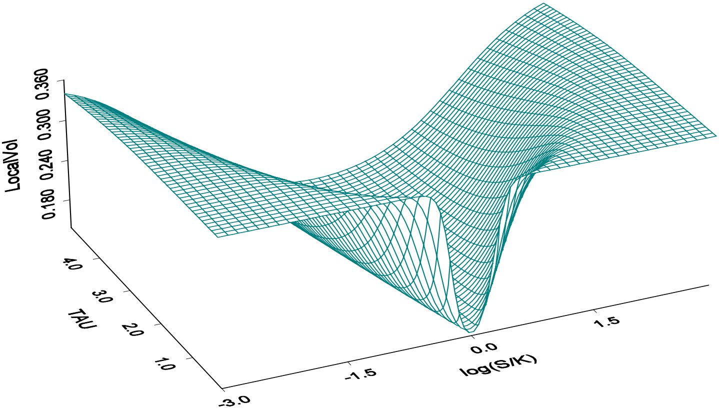
$$a^2(\tau, x, \Theta) = \frac{\sum_{i=0}^{2} p_i \sigma_i e^{-x^2/(2\tau\sigma_i^2) - \tau\sigma_i^2/8}}{\sum_{i=0}^{2} (p_i/\sigma_i) e^{-x^2/(2\tau\sigma_i^2) - \tau\sigma_i^2/8}}$$

for

$$\Theta = (\sigma_0, \sigma_1, \sigma_2, p_1, p_2)$$

- Mixture of Black-Scholes Call surfaces for 3 different volatilities
- **Singularity** when $\tau \searrow 0$

Numerical Evidence of Singularity



A Second Parametric Family

- Still a mixture of Black-Scholes Call surfaces for 3 different volatilities
- Each volatility is time dependent $t \hookrightarrow \sigma_i(t)$
- $\sigma_0(0) = \sigma_1(0) = \sigma_2(0)$

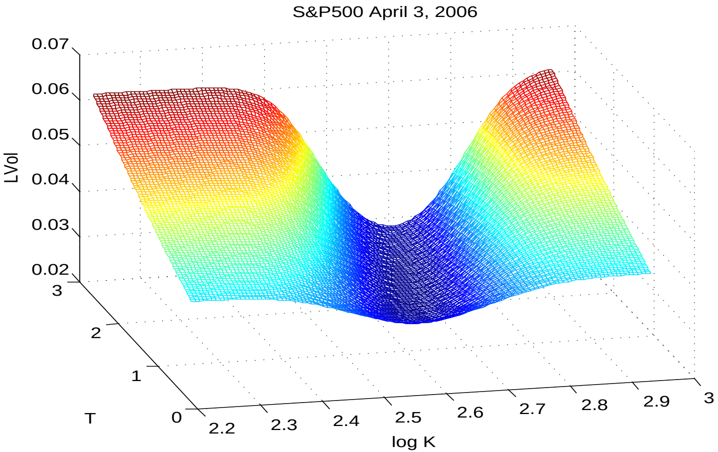
$$a^2(\Theta,\tau,x) = \frac{(1-(p_1+p_2)\tau)\,\sigma e^{-d^2(\sigma)/2} + p_1\tau\sigma_1 e^{-d^2(\sigma_1)/2} + p_2\tau\sigma_2 e^{-d^2(\sigma_2)/2}}{(1-(p_1+p_2)\tau)\,\frac{1}{\sigma} e^{-d^2(\sigma)/2} + p_1\tau\frac{1}{\sigma_1} e^{-d^2(\sigma_1)/2} + p_2\tau\frac{1}{\sigma_2} e^{-d^2(\sigma_2)/2}}$$

where

$$d(\sigma) = \frac{s - x + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

$$\Theta = (p_1, p_2, \sigma, \sigma_1, \sigma_2, s, r)$$

Fit to Real Data



Stochastic Volatility Models

$$dS_t = \sigma_t S_t dW_t$$

with

$$d\sigma_t^2 = b(\sigma_t^2)dt + a(\sigma_t^2)d\tilde{W}_t$$

where

$$d\langle W, \tilde{W}\rangle_t = \rho dt.$$

Usually

$$b(\sigma^2) = -\kappa(\sigma^2 - \overline{\sigma^2})$$

Special cases:

$$a(\sigma^2) = \gamma, \quad \text{(Hull-White)} \qquad a(\sigma^2) = \gamma\sqrt{\sigma^2} \quad \text{(Heston)}$$

Local Volatility of SV Models

$$a^2(\tau, K) = \frac{2\partial_\tau C}{K^2 \partial^2_{KK} C} = \sigma_0^2 \sqrt{1-\rho^2} \cdot \frac{\mathbb{E}\left\{ S \frac{\tilde{\sigma}_T^2}{\bar{\sigma}_T} e^{-\frac{d_1^2}{2}} \right\}}{\mathbb{E}\left\{ \frac{S}{\bar{\sigma}_T} e^{-\frac{d_1^2}{2}} \right\}}$$

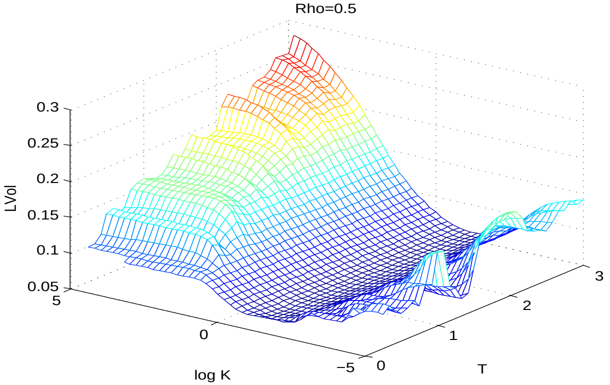
where $\tilde{\sigma}_T = \frac{\sigma_T}{\sigma_0}$, and $\bar{\sigma}_T = \sqrt{\frac{1}{T}\int_0^T \tilde{\sigma}_s^2 ds}$

$$S = s_0 \exp\left(\frac{\rho\sigma_0}{\hat{\sigma}} (\tilde{\sigma}_\tau - 1) - \frac{1}{2}\sigma_0^2 \rho^2 \bar{\sigma}_\tau^2 \tau \right)$$

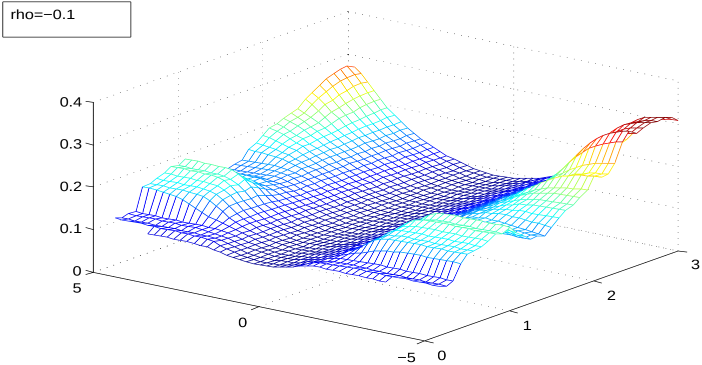
and

$$d_1 = \frac{\log(s_0) - \log(K) + \frac{\rho\sigma_0}{\hat{\sigma}}(\tilde{\sigma}_\tau - 1) + (\frac{1}{2} - \rho^2)\sigma_0^2 \bar{\sigma}_\tau^2 \tau}{\sqrt{1-\rho^2}\,\sigma_0 \bar{\sigma}_\tau \sqrt{\tau}}$$

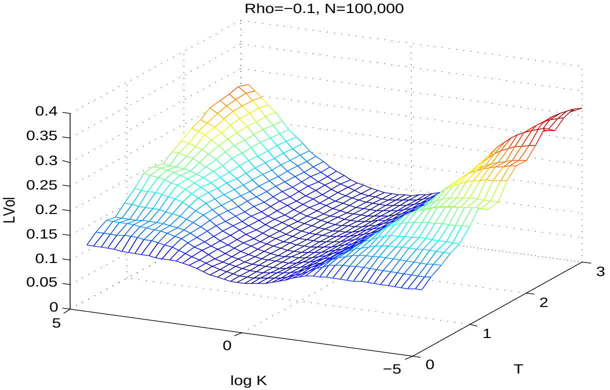
First Example: $\rho = 0.5$



Second Example: $\rho = -0.1$



Third Example: $\rho = -0.75$



Comparing SV Models