Noncommutative vector-valued L_p spaces

February 16, 2007

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- Entropy $S(x) = -\sum_i x_i \ln x_i = -\|x\|_1 \frac{d}{dp}\|x\|_p\Big|_{p=1}$.

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Theorem 1: $S_{min}(a \otimes b) = S_{min}(a) + S_{min}(b)$.

Lemma: Let $p \leq q$. Then

$$\|a \otimes b : \ell_p^{nm} \to \ell_q^{nm}\|_{1 \to p} = \|a\|_{1 \to p} \|b\|_{1 \to p} .$$

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The reverse inequality is easy. ∎

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Remark: 1) $S_p = L_p(B(\ell_2), tr)$ stands for Schatten p-class.
2) Exercise: $\| \|_p$ is a norm for $1 \leq p \leq \infty$ and

$$\|x + y\|_p^p \leq \|x\|_p^p + \|y\|_p^p$$

for $0 \leq p \leq 1$.

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Literature: $id \otimes tr = tr_2$ partial trace.

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- Exercise: For $q > 2$ the function $f(x) = \varphi(x^q)^{1/q}$ is not convex.

More

- (J.-Xu) For $1 < q < 2$ there is no norm $\| \|_q$ on the space of selfadjoint matrices such that $f(x) = \varphi(x^q)^{1/q}$ such that $f(x) = \|x\|_q$.

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- (Central-limit trick). Let D be the density of φ. Consider the infinite tensor product $\otimes_{n\in\mathbb{N}}\mathbb{M}_k$ and the elements

$$x_k \;=\; D^{1/p} \otimes \cdots \otimes \underbrace{\quad x \quad}_{k\text{-th position}} \otimes D^{\frac{1}{p}} \otimes \cdots \;.$$

Then $\lim_n n^{-1/q}\|(\sum_{k=1}^n x_k^q)^{1/q}\|_p \;=\; tr(D^{1-q/p}x^q)^{1/q}$.

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2) Matrix valued-version of the central limit theorem.

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Remark: 1) If we were to find norms on $\mathbb{M}_{nm}$ satisfying these requirements, then the minimal entropy is additive.
2) For any collection of norms satisfying the first three condition the new expression

$$\widetilde{S}_{\min}(\Phi) \;=\; -\frac{d}{dp}\|\Phi\|_{L_\infty(L_p)}\Big|_{p=1}$$

is additive.

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- A linear map $\Phi : X_1 \to X_2$ is completely bounded if the cb-norm

  $$\|\Phi\|_{cb} \ = \ \sup_n \|id \otimes \Phi : \mathbb{M}_m(X_1) \to \mathbb{M}_n(X_2)\|$$

  is finite.

Operator spaces

Definitions:

- A Banach space X is called operator space if there exists a sequence $\| \; \|_n$ of norms on $\mathbb{M}_n(X)$ such that

  R∞ $\left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$.

  Rcp For every completely positive map $\Phi : \mathbb{M}_n \to \mathbb{M}_n$

$$\|\Phi \otimes id_X : \mathbb{M}_n(X) \to \mathbb{M}_n(X)\| \; \leq \; \|\Phi\| \; .$$

- A linear map $\Phi : X_1 \to X_2$ is completely bounded if the cb-norm

$$\|\Phi\|_{cb} \; = \; \sup_n \|id \otimes \Phi : \mathbb{M}_m(X_1) \to \mathbb{M}_n(X_2)\|$$

  is finite.

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Need to know $L_p(L_q)$ and differentiate!

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Feature: $\|T\|_{cb} = \|id \otimes T : L_p(B(\ell_2); X_1) \to L_p(B(\ell_2); X_2)\|$ for all $1 \leq p \leq \infty$ and $T : X_1 \to X_2$.

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- (algebraic formula 2) Let $p \geq q$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. Then

$$\|x\|_{L_q(\mathbb{M}_n; L_p(\mathbb{M}_m))} = \sup_{\|a\|_{2r} = \|b\|_{2r} = 1} \|(a \otimes 1)x(b \otimes 1)\|_{L_p(\mathbb{M}_{nm})} \, .$$

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- Let $\Phi : L_p \to L_q$ be completely positive. Then

$$\|\Phi \otimes id_X : L_p(X) \to L_q(X)\| \leq \|\Phi\|_{p \to q} .$$

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$$\begin{aligned} & \|\Phi \otimes T : L_1(\mathbb{M}_{nm}) \to L_p(\mathbb{M}_{nm})\| \\ \leq \quad & \|\Phi : L_1 \to L_p\| \, \|T : L_1 \to L_p\|_{cb} \, . \end{aligned}$$

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where $\|T : X_1 \to X_2\|_k = \|id_{\mathbb{M}_k} \otimes T : \mathbb{M}_k(X_1) \to \mathbb{M}_k(X_2)\|$.
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The reverse is easy. ■

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3. Φ has rank 1. Then $\|\Phi\|_{cb} = \|\Phi\|$. Hence

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(see more: Mathias Neufang's talk).

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For $\psi \in \mathbb{C}^{n^2}$ we set $\gamma_{12} \;=\; id \otimes \Phi(|\psi\rangle\langle\psi|)$and $\gamma_2 = id \otimes tr(\gamma) = (id \otimes tr)|\psi\rangle\langle\psi|$.

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Let $\Phi : \mathbb{M}_n \to \mathbb{M}_n$ be a channel. Let $X_\Phi \in \mathbb{M}_{n^2}$ the Choi matrix $[X_\Phi(e_{ij})]_{ij}$. Then

$$\|\Phi : L_1(\mathbb{M}_n) \to L_p(\mathbb{M}_n)\|_{cb} = \sup_{\langle\psi|\psi\rangle=1} \frac{\|id \otimes \Phi(|a\rangle\langle a|)\|_p}{\|id \otimes tr_2(|\psi\rangle\langle\psi|)\|_p} .$$

For $\psi \in \mathbb{C}^{n^2}$ we set $\gamma_{12} = id \otimes \Phi(|\psi\rangle\langle\psi|)$and $\gamma_2 = id \otimes tr(\gamma) = (id \otimes tr)|\psi\rangle\langle\psi|$. Then

$$\frac{d}{dp}\|\Phi : L_1 \to L_p\|_{cb} = \sup_{\psi\in\mathbb{C}^n\otimes\mathbb{C}^n} -S(\gamma_{12}) + S(\gamma_2)$$

Conclusion: $S_{min,cb}(\Phi) \;=\; \inf_{\|\psi\|=1} S(\gamma_{12}) - S(\gamma_2)$ is additive.

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Thanks for your attention!