# Asymptotics of entropy in quasi-free states of the spin chain

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# Introduction

- Pure states can have mixed marginals: quantum correlations (entanglement)
- Behaviour of entanglement in pure states of infinite quantum systems

  P. Calabrese, J. Cardy, M. Cramer, J. Eisert, M. Fannes, S. Farkas, B. Haegeman, A.R. Its, B.-Q. Jin, J.P. Keating, F. Mezzadri, B. Nachtergaele, A. Kitaev, V.E. Korepin, J.I. Latorre, M.B. Plenio, E. Rico, F. Verstraete, G. Vidal, R.F. Werner, M.M. Wolf, Z. Zimborás,...
- Entropy scaling in ground states of different physical models (fermionic, bosonic, spin chain)
  - area law
  - violation of area law: logarithmic correction
- What are the possible scalings in a pure state of the quantum spin chain?

# Spin chain

- one-site algebra: $\mathcal{C}_0$

  classical spin-$\frac{1}{2}$: $\mathcal{C}_0 = \mathbb{C}^2$ quantum spin-$s$: $\mathcal{C}_0 = \mathcal{B}\left(\mathbb{C}^{2s+1}\right)$

- local algebras: $\mathcal{C}_n := \bigotimes_{k=-n}^{n} \mathcal{C}_0 \hookrightarrow \mathcal{C}_{n+1}$

  $\mathcal{C}_n = \text{span}\,\{\ldots \otimes I \otimes A_{-n} \otimes A_{-n+1} \otimes \ldots \otimes A_n \otimes I \otimes \ldots\}$

- algebra of local observables: $\mathcal{C}_{\text{loc}} := \bigcup_n \mathcal{C}_n$

  infinite chain: $\mathcal{C} := \overline{\mathcal{C}}_{\text{loc}}$

- generators (quantum case): $\{S_j^{(n)} \;:\; j = 1,2,3;\; n \in \mathbb{Z}\}$

$$\left[S_j^{(n)}, S_k^{(n)}\right] = i\,\varepsilon_{jkl}\,S_l^{(n)}, \qquad \left[S_j^{(n)}, S_k^{(m)}\right] = 0$$

- right shift: $\gamma(S_j^{(n)}) := S_j^{(n+1)}$

# States

- positive, normalized functionals on $\mathcal{C}$
- the state space is convex; extremal points: pure states
- shift-invariant states: $\varphi \circ \gamma = \varphi$
- classical chain: pure states are product states $\Rightarrow$ no correlation
- quantum chain: abundance of non-product pure shift-invariant states

  quantum correlations (entanglement)

  Examples:

  Finitely correlated states (Fannes, Nachtergaele, Werner)

  Quasi-free states (Araki)

# Entropy

- local restrictions of a shift-invariant state $\varphi$

$$\varphi_n(a) := \varphi(a) \qquad a \in \mathcal{C}_n$$

$\varphi_n$ state on $\mathcal{C}_n$; $\quad \varphi_{n+k}(a) = \varphi_n(a) \qquad a \in \mathcal{C}_n$

- a pure $\varphi$ is non-product $\iff$ $\varphi_n$ is mixed for all $n$
- measure of mixedness: entropy $\quad S_n := S(\varphi_n) := -\operatorname{Tr} \hat{\varphi}_n \log \hat{\varphi}_n.$
- $0 \le S_1 \le S_2 \le \dots$
- mean entropy: $s(\varphi) := \lim_{n\to\infty} \frac{1}{n} S_n$

  $s(\varphi) > 0$: linear asymptotics of $S_n$

  $s(\varphi) = 0$: sublinear asymptotics of $S_n$

- conjecture: $s(\varphi) = 0$ for pure $\varphi$

# Typical Subspaces

- eigen-decomposition of a local density matrix:

$$\hat{\varphi}_n = \sum_{k=1}^{d^n} \lambda_{k,n} \, |e_{k,n}\rangle\langle e_{k,n}| \qquad \lambda_{1,n} \geq \lambda_{2,n} \geq \dots$$

- $t(n,\varepsilon) := \min \left\{ t \, : \, \sum_{k=1}^{t} \lambda_{k,n} > 1 - \varepsilon \right\}$

  $\varepsilon$-typical subspace: span $\{e_{1,n}, \dots, e_{t(n,\varepsilon),n}\}$
- typical subspace theorem: $\varphi$ ergodic

$$\lim_{n\to\infty} \frac{1}{n} \log t(n,\varepsilon) = s(\varphi)$$

  (Hiai, Petz; Bjelaković, Krüger, Siegmund-Schultze, Szkoła)
- $s(\varphi) > 0$: $t(n,\varepsilon) \sim \exp(S_n) \sim \exp(ns(\varphi))$

  $s(\varphi) < \log d$: data compression
- $s(\varphi) = 0$: $t(n,\varepsilon)$ subexponential

  relation between $t(n,\varepsilon)$ and $S_n$?

# Examples

- Finitely correlated states:

$$\begin{aligned} \dim \mathsf{supp}\ \hat{\varphi}_n &\leq d^2 \\ S_n &\leq 2\log d \end{aligned}$$

  (Fannes, Nachtergaele, Werner; 1994)

- Ground state of the XX-model: quasi-free state

$$S_n \sim \log n$$

  (Jin, Korepin; 2003)

- What other asymptotic behaviour can we have?

  Can we have arbitrary sublinear asymptotics of the entropy?

  Examples with quasi-free states.

# CAR

- $\mathcal{H}$ Hilbert space, $\mathcal{A}_0$ $C^*$-algebra, $c : \mathcal{H} \to \mathcal{A}_0$ antilinear

$$\{c(x), c(y)\} = 0\,, \qquad \{c(x), c^*(y)\} = \langle x, y\rangle \mathbf{1}$$

canonical anticommutation relations

$$\mathcal{A} := CAR(\mathcal{H}) := C^*\{c(x) \,:\, x \in \mathcal{H}\}$$

- $CAR(\mathcal{H})$ is unique up to isomorphism
- $\mathcal{H}$ separable $d$ dimensional ($d \in \{1, 2, \ldots, \infty\}$):

  $CAR(\mathcal{H}) \cong \otimes_{k=1}^{d} \mathcal{M}_2$ Jordan-Wigner isomorphism
- Fock representation: $\mathcal{F}(\mathcal{H}) := \oplus_{n=0}^{\infty} \wedge^n \mathcal{H}$

$$x_1 \wedge \ldots \wedge x_n \;:=\; \frac{1}{\sqrt{n!}} \sum_{\pi \in S_n} \varepsilon(\pi)\, x_1 \otimes \ldots \otimes x_n$$

$$c^*(y)\, x_1 \wedge \ldots \wedge x_n \;:=\; y \wedge x_1 \wedge \ldots \wedge x_n$$ creation operator

# Quasi-free Automorphisms

- parity automorphism: $\kappa(c(x)) := c(-x)$

  even part: $\mathcal{A}_+ := \{a \in \mathcal{A} \,:\, \kappa(a) = a\}$ (subalgebra)

  odd part: $\mathcal{A}_- := \{a \in \mathcal{A} \,:\, \kappa(a) = -a\}$

  $\mathcal{A} = \mathcal{A}_+ + \mathcal{A}_-$
- a state $\varphi$ is even if $\varphi \circ \kappa = \varphi$, i.e. $\varphi(a) = 0 \;\; \forall a \in \mathcal{A}_-$
- Let $\mathcal{H} := l^2(\mathbb{Z})$ with its standard base $\{e_k \,:\, k \in \mathbb{Z}\}$

  $c_k := c(e_k)$
- shift: $\gamma(c_k) := c_{k+1}$
- filtration: $\mathcal{A}_n := C^*\{c_k \,:\, k = -n, \ldots, n\}$

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \ldots \qquad \mathcal{A} = \overline{\cup_{n\in\mathbb{N}} \mathcal{A}_n}$$

# Quasi-free States

- Let $Q \in \mathcal{B}(\mathcal{H})$, $\quad 0 \leq Q \leq I$; $\quad$ symbol of $\varphi$

$$\varphi\left(c_{i_1}^* \dots c_{i_n}^* c_{j_m} \dots c_{j_1}\right) = \delta_{m,n} \det\{Q_{i_k,j_l}\}$$

- $\varphi$ is even
- $\varphi$ is shift-invariant $\iff$ $Q$ is shift-invariant, i.e. $\exists\, q \in l^2(\mathbb{Z})$

$$Q_{kl} = q(k-l) \qquad \text{Toeplitz operator}$$

Fourier transformation: $F : \mathcal{H} \mapsto L^2(\mathbb{T})$

$$F\,Q\,F^{-1} = M_{\hat{q}}; \qquad \hat{q}(\theta) = \sum_{k \in \mathbb{Z}} q(k) e^{ik\theta}$$

- $\varphi$ pure $\iff$ $Q$ is a projection $\iff$ $q = 1_K \qquad K \subset [0, 2\pi]$

# Transfer of states

- the Jordan-Wigner isomorphism is not compatible with the shift and the filtration
- extension of $\mathcal{A}$ (crossed product)

$$N_- := \sum_{k<0} c_k^* c_k \,, \qquad T := (-1)^{N_-}; \qquad T^* = T^2 = T \notin \mathcal{A}$$

$$\tilde{\mathcal{A}} := C^*(\mathcal{A}, T) = \mathcal{A} + T\mathcal{A}$$

- $\mathcal{C} \cong \mathcal{A}_+ + T\mathcal{A}_-$

  $W : a \mapsto a_+ + Ta_-$ linear isomorphism between $\mathcal{A}$ and $\mathcal{C}$ compatible with the shift and the filtration
- $\hat{\varphi} := \varphi \circ W^{-1}$ is a state on $\mathcal{C}$ for any even state $\varphi$ on $\mathcal{A}$
- $\varphi_N$ and $\hat{\varphi}_N$ have the same density matrices

  $\varphi_N$ is shift-invariant $\Rightarrow$ $\hat{\varphi}_N$ is shift-invariant

  $\varphi$ pure $\Rightarrow$ $\hat{\varphi}$ pure

# Examples

- ground state of the XX-model

$$H_n := -\frac{1}{2}\sum_{k=-n}^{n-1}\left(\sigma_x^k\sigma_x^{k+1}+\sigma_y^k\sigma_y^{k+1}\right) - h\sum_{k=-n}^{n}\sigma_z^k$$
$$= -\sum_{k=-N}^{N-1}\left(a_k^* a_{k+1} + a_{k+1}^* a_k\right) + 2h\sum_{k=-N}^{N} a_k^* a_k - h(2n+1)I.$$

- $h \geq 1$: $K = \emptyset$: Fock state (vacuum) all spins $|\uparrow\rangle$
- $h \leq -1$: $K = [0, 2\pi]$: anti-Fock state all spins $|\downarrow\rangle$
- $h \in (0,1)$: $K = [\arccos(-h), 2\pi - \arccos(-h)]$ single interval

# Entropy

- Szegő's theorem: $s = \frac{1}{2\pi} \int_0^{2\pi} \eta(\hat{q}(\theta)) d\theta = 0$ for pure states

$$\eta(x) := -x \log x - (1-x) \log(1-x)$$

- $S_n = \mathrm{Tr}\, \eta\,(Q_n)$; $\quad (Q_n)_{k,l} = Q_{k,l}$; $\quad 0 \leq k, l \leq n-1$

  difficult to compute explicitly

- approximation:

$$\mathrm{Tr}\; Q_n(\mathbf{1} - Q_n) \leq S_n \leq 1 + c\,(\log n)\,\mathrm{Tr}\; Q_n(\mathbf{1} - Q_n)$$

- $\mathrm{Tr}\; Q_n(\mathbf{1} - Q_n) = \dfrac{n}{4\pi^2} \displaystyle\int d\theta\, k_n(\theta)\, |K \setminus (K+\theta)|$

  $k_n(\theta) := \frac{1}{n} \frac{\sin^2 n\theta}{\sin^2 \theta}$: Dirichlet kernel

# Results

- subadditivity: $K = A \cup B, \qquad A \cap B = \emptyset$

$$S_{n,K} \leq S_{n,A} + S_{n,B}$$

- $K$ is a union of finitely many intervals: $S_n \sim \log n$
- $K$ Cantor type: $S_n \sim n^{\alpha}; \quad \alpha \in (0,1)$

  (Fannes, Haegeman, Mosonyi)
- any non-product pure finitely correlated state has at least logarithmic entropy growth
- For any $f : \mathbb{N} \to \mathbb{R}_+$ sublinear there exists a $K \subset [0, 2\pi]$ s.t.

$$S_n \geq f_n \qquad \forall n \in \mathbb{N}$$

  sharpness of the zero-entropy conjecture

  $K$ is a union of intervals

  (Farkas, Zimborás)

# Further questions

- Can we have arbitrarily slow entropy growth?
- asymptotics of $t(n, \varepsilon)$ ?

  numerics: $\log t(n, \varepsilon) \sim c(\varepsilon) \log n$ for a single interval

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