The Profile of Relations

Maurice Pouzet

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Overview

Definitions and Simple Examples

A *relational structure* is a realization of a language whose non-logical symbols are predicates.

This is a pair $R := (E, (\rho_i)_{i\in I})$ made of a set E and of a family of m_i-ary relations ρ_i on E. The set E is the *domain* or *base* of R. The family $\mu := (m_i)_{i\in I}$ is the *signature* of R.

The *substructure induced by R on a subset A of E*, simply called the *restriction of R to A*, is the relational structure $R_{\restriction A} := (A, (A^{m_i} \cap \rho_i)_{i\in I})$. Notions of *isomorphism* and *local isomorphism* from a relational structure to an other one are defined in a natural way as well as the notion of *isomorphic type*. In the sequel, $\tau(R)$ stands for the isomorphic type of a relational structure R and Ω_μ stands for the set of isomorphic types of finite relational structures with signature μ.

The *profile* of R is the function φ_R which counts for every integer n the number $\varphi_R(n)$ of substructures of R induced on the n-element subsets, isomorphic substructures being identified.

Clearly, this function only depends upon the set $\mathcal{A}(R)$ of finite substructures of R considered up to an isomorphism, a set introduced by R. Fraïssé under the name of *age* of R .

If the signature μ is finite (in the sense that I is finite), there are only finitely many relational structures with signature μ on an n-element domain, hence $\varphi_R(n)$ is necessarily an integer for each integer n. In order to capture examples coming from algebra and group theory, we cannot preclude I to be infinite. But then, $\varphi_R(n)$ could be an infinite cardinal.

As far as we will be concerned by the behavior of φ_R, we will exclude this case. Indeed, we have:

Fact 1. *Let $n < |E|$. Then*

$$\varphi_R(n) \leq (n+1)\varphi_R(n+1) \tag{1}$$

In particular if $\varphi_R(n)$ is infinite then:

$$\varphi_R(n) \leq \varphi_R(n+1). \tag{2}$$

Except in very few occasions, I make the assumption that φ_R is integer valued, no matter how large I is. With this assumption, *profiles of relational structures with bounded signature are profiles of relational structures with finite signature*, structures that R. Fraïssé call *multirelations*.

Several counting functions are profiles. Here is some simple minded examples.

1. *The binomial coefficient* $\binom{n+k}{k}$. Let $R := (\mathbb{Q}, \leq, u_1, \ldots, u_k)$ where $\leq$ is the natural order on the set $\mathbb{Q}$ of rational numbers, $u_1, \ldots, u_k$ are k unary relations which divide $\mathbb{Q}$ into $k+1$ intervals. Then $\varphi_R(n) = \binom{n+k}{k}$.

2. *The exponential* $n \hookrightarrow k^n$. Let $R := (\mathbb{Q}, \leq, u_1, \ldots, u_k)$, where again $u_1, \ldots, u_k$ are k unary relations, but which divide $\mathbb{Q}$ into k "colors" in such a way that between two rational numbers all colors appear. Then $\varphi_R(n) = k^n$.

3. *The factorial* $n \hookrightarrow n!$. Let $R := (\mathbb{Q}, \leq, \leq')$, where $\leq'$ is an other linear order on $\mathbb{Q}$ such a way that the finite restrictions induce all possible pairs of two linear orders on a finite set (eg take for $\leq'$ an order with the same type as the natural order on the set $\mathbb{N}$ of non-negative integers). Then $\varphi_R(n) = n!$

4. *The partition function* which counts the number $p(n)$ of partitions of the integer n. Let $R := (\mathbb{N}, \rho)$ be the infinite path on the integers whose edges are pairs $\{x, y\}$ such that $y = x + 1$. Then $\varphi_R(n) = p(n)$. The determination of its asymptotic growth is a famous achievement, the difficulties encountered to prove that $p(n) \simeq \frac{1}{4\sqrt{3n}} e^{\pi\sqrt{\frac{2n}{3}}}$ (Hardy and Ramanujan, 1918) suggest some difficulties in the general study of profiles.

Orbital profiles An important class of functions comes from permutation groups. The *orbital profile* of a permutation group G acting on a set E is the function θ_G which counts for each integer n the number, possibly infinite, of orbits of the n-element subsets of E. As it is easy to see, θ_G is the profile of some relational structure $R := (E, (\rho_i)_{i \in I})$ on E. In fact, as it is easy to see:

Lemma 1. *For every permutation group G acting on a set E there is a relational structure R on E such that:*

1. *Every isomorphism f from a finite restriction of R onto an other extends to an automorphism of R.*

2. *$Aut(R) = \overline{G}$ where $\overline{G}$ is the topological adherence of G into the symmetric group $\mathfrak{G}(E)$, equipped with the topology induced by the product topology on E^E, E being equipped with the discrete topology.*

Structures satisfying condition 1) are called *homogeneous* (or *ultrahomogeneous*). They are now considered as one of the basic objects of model theory. Ages of such structures are called *Fraïssé classes* after their characterization by R.Fraïssé in 1954. In many cases, I

is infinite, even if $\theta_G(n)$ is finite. Groups for which $\theta_G(n)$ is always finite are said *oligomorphic* by P.J.Cameron. The study of their profile is whole subject by itself. Their relevance to model theory stems from the following result of Ryll-Nardzewski.

Theorem 2. *Let G acting on a denumerable set E and R be a relational structure such that $AutR = \overline{G}$. Then G is oligomorphic if and only if the complete theory of R is $\aleph_0$-categorical.*

A Sample of Results

The Profile grows

Inequality (1) given in the previous subsection can be substantially improved:

Theorem 3. *If R is a relational structure on an infinite set then φ_R is non-decreasing.*

This result was conjectured with R.Fraïssé. We proved it in 1971; the proof - for a single relation- appeared in 1971 in R.Fraïssé's course in logic, Exercise 8 p. 113. The proof relies on Ramsey theorem. Later on we gave a proof using linear algebra.

Jumps in the Growth of the Profile Beyond bounded profiles, and provided that the relational structures satisfy some mild conditions, there are jumps in the behavior of the profiles: eg. no profile grows as $log\, n$ or $nlog\, n$.

Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ and $\psi : \mathbb{N} \rightarrow \mathbb{N}$. Recall that $\varphi = O(\psi)$ and ψ *grows as fast as* φ if $\varphi(n) \leq a\psi(n)$ for some positive real number a and n large enough. We say that φ and ψ have the *same growth* if φ grows as fast as ψ and ψ grows as fast as φ. The growth of φ is *polynomial of degree* k if φ has the same growth as $n \hookrightarrow n^k$; in other words there are positive real numbers a and b such that $an^k \leq \varphi \leq bn^k$ for n large enough. Note that the growth of φ is as fast as every polynomial if and only if $lim_{n\rightarrow+\infty}\frac{\varphi(n)}{n^k}=+\infty$ for every non negative integer k.

Theorem 4. *Let $R := (E, (\rho_i)_{i\in I})$ be a relational structure. The growth of φ_R is either polynomial or as fast as every polynomial provided that either the signature $\mu := (n_i)_{i\in I}$ is bounded or the kernel $K(R)$ of R is finite.*

The *kernel* of R is the set $K(R)$ of $x \in E$ such that $\mathcal{A}(R_{\restriction E\setminus\{x\}}) \neq \mathcal{A}(R)$. Relational structures with empty kernel are those for which their age has the *disjoint embedding property*, meaning that two arbitrary members of the age can be embedded into a third in such a way that their domains are disjoint. In Fraïssé's terminology, ages with the *disjoint embedding property* are said *inexhaustible* and relational structures whose age is inexhaustible are said *age-inexhaustible*. We will say that relational structures with finite kernel are *almost age-inexhaustible*.

At this point, enough to know that the kernel of any relational structure which encodes an oligomorphic permutation group is finite (this fact immediate: if R encodes a permutation group G acting on a set E then $K(R)$ is the set union of the orbits of the 1-element subsets of E which are finite. Since the number of these orbits is at most $\theta_G(1)$, if G is oligomorphic then $K(R)$ is finite).

Corollary 1. *The orbital profile of an oligomorphic group is either polynomial or faster than every polynomial.*

Groups with orbital profile equal to 1 were described by P.Cameron in 1976 . From his characterization, Cameron obtained that the growth rate of an orbital profile is ultimately constant, or it grows as fast as a linear function with slope $\frac{1}{2}$.

For groups, and graphs, there is a much more precise result than Theorem 4. It is due to Macpherson, 1985.

Theorem 5. *The profile of a graph or a permutation groups grows either as a polynomial or as fast as f_ε, where $f_\varepsilon(n) = e^{n^{\frac{1}{2}-\varepsilon}}$, this for every $\varepsilon > 0$.*

Note that the f_ε are somewhat similar to the partition function. Such growth cannot be prevented. Indeed, the partition function is the orbital profile of the automorphim group of an equivalence relation having infinitely many classes, all being infinite. Such a group is imprimitive. In fact, according to Macpherson 1987

Theorem 6. *If G is primitive then either $\theta_G(n) = 1$ for all $n \in \mathbb{N}$, or $\theta_G(n) > c^n$ for all $n \in N$, where $c := 2^{\frac{1}{5}} - \epsilon$.*

Some hypotheses on R are needed in Theorem 4, indeed

Theorem 7. *For every non-decreasing and unbounded map $\varphi : \mathbb{N} \to \mathbb{N}$, there is a relational* *structure R such that φ_R is unbounded and eventually bounded above by φ.*

More is true.

Let $f : \mathbb{N} \to \mathbb{N}$ be a non-decreasing map such that $1 \leq f(n) \leq n+1$ for all $n \in \mathbb{N}$. Let $A := \{n : f(n') < f(n+1) \text{ for all } n' < n+1\}$. Let $R := (\mathbb{N}, (\rho_n)_{n \in A})$ in which each ρ_n is $n+1$-ary, with $(x_1, \ldots, x_{n+1}) \in \rho_n$ if and only if $\{x_1, \ldots, x_{n+1}\} = \{0, \ldots, n\}$. Then $\varphi_R = f$.

The reader will notice that if f is unbounded then the signature of R is unbounded and also the kernel of R is infinite (equal to $\mathbb{N}$).

The hypothesis about the kernel is not ad hoc. As it turns out, *if the growth of the profile of a relational structure with a bounded signature is bounded by a polynomial then its kernel is finite.*

Theorems 4 and 7 were obtained in 1978. Theorem 7 and a part of Theorem 4 appeared in 1981, with a detailed proof showing that the growth of unbounded profiles of relational structures with bounded signature is at least linear. The notion of kernel is in several papers.

Two proofs of the growth of the profile

Relational structures with bounded profile
Infinite relational structures with profile constant, equal to 1, were called *monomorphic* and characterized by R. Fraïssé who proved that they where *chainable*. Later on, those with profile bounded, called *finimorphic*, were characterized as *almost chainable*.

According to R.Fraïssé who introduced this notion in 1954 in his thesis, a relational structure

$R := (E, (\rho_i)_{i\in I})$ for which $\varphi_R(n) = 1$ for every $n \leq |E|$ is *monomorphic*.

Example 1. *There are eight kinds of monomorphic directed graphs, four made of reflexive directed graphs, four made of irreflexive graphs. For the reflexive ones, there are the chains, the reflexive cliques , the antichains, plus the* 3*-element oriented reflexive cycle. Whereas, for the irreflexive ones, there are the acyclic (oriented) graphs, the cliques, the independent sets, and the* 3*-element oriented irreflexive cycle.*

Fraïssé gave a characterization of infinite monomorphic relational structures by means of his notion of chainability:

A relational structure $R := (E, (\rho_i)_{i\in I})$ is *chainable* if there is a linear ordering $\leq$ on E such that every local isomorphism of $L := (E, \leq)$ is a local isomorphism of R.

Since chains are monomorphic, chainable relational structures are also monomorphic. The converse does not hold, as shown by a 3-element cycle. Fraïssé proved that it holds if the structure is infinite.

Theorem 8. *An infinite relational structure is monomorphic if and only if it is chainable.*

His proof, given for relational structures of finite signature, was based on Ramsey's theorem [0] and the compactness theorem of first order logic. The extension to arbitrary signature requires an other application of the compactness theorem.

Let $R := (E, (\rho_i)_{i \in I})$ be a relational structure and F be a subset of E. The relational structure R is F-*monomorphic* if for every non-negative integer n and every $A, A' \in [E \setminus F]^n$ there is an isomorphism from $R_{\restriction A}$ onto $R_{\restriction A'}$ which extends by the identity on F to an isomorphism of $R_{A\cup F}$ onto $R_{A'\cup F'}$. This relational structure is *F-chainable* if there is a linear order $\leq$ on $E\setminus F$ such that every local isomorphism of $L := (E\setminus F, \leq)$, once extended by the identity on F, is a local isomorphism of R. This relational structure is *almost monomorphic*, resp. *almost chainable*, if it is F-monomorphic, resp. F-chainable for some finite subset F of E.

From Ramsey's theorem, Fraïssé deduced the following lemma.

Lemma 9. *Let R be a relational structure with domain E and F be a finite subset of E. If the signature of R is finite then there is an infinite subset E' of E containing F on which the restriction $R' := R_{\restriction E'}$ is F-chainable.*

Then he applied the compactness theorem of first order logic (in a weaker form, given by his "coherence lemma"). Indeed, from Lemma 9 above, if a monomorphic relational structure

R of finite signature is infinite, it contains an infinite induced substructure R' which is chainable. Since R is monomorphic, each finite substructure of R is isomorphic to some finite substructure of R', hence is chainable. The compactness theorem insures that R is chainable. As said, this conclusion holds if the signature is arbitrary.

An illustration: a proof of Theorem 3

Let $R := (E, (\rho_i)_{i \in I})$ be a relational structure. Suppose E be infinite. Let n be a non negative integer. We claim that $\varphi_R(n) \leq \varphi_R(n+1)$.

Case 1. $\varphi_R(n)$ is infinite. Then, as stated in Fact 1, $\varphi_R(n) \leq \varphi_R(n+1)$ as claimed.

Case 2. $\varphi_R(n)$ is finite. We reduce the claim to the case of an almost monomorphic relational structure.

Claim 1. There is some finite subset I' of I and some infinite subset E' of E such that the reduct $R' := R^{\restriction I'}_{\restriction E'}$ is almost monomorphic and $\varphi_{R'}(n) = \varphi_R(n)$.

Proof of Claim 1. There is some finite subset I' of I such that the reduct $R^{\restriction I'} := (E, (\rho_i)_{i \in I'})$ satisfies $\varphi_{R^{\restriction I'}}(n) = \varphi_R(n)$. Let $m := \varphi_{R^{\restriction I'}}(n)$. Select $F_1, \ldots F_m$ in $[E]^n$ such that the restrictions $R^{\restriction I'}_{\restriction F_1}, \ldots, R^{\restriction I'}_{\restriction F_m}$ are pairwise non-isomorphic. Set $F := F_1 \cup \cdots \cup F_m$. According to Lemma 9, there is an infinite subset E' of E containing F such that the restriction $R' := R^{\restriction I'}_{\restriction E'}$ is F-chainable. This restriction is almost monomorphic. From our construction, $\varphi_{R'}(n) = m$. This proves Claim 1. $\blacksquare$

Claim 2. If an infinite relational structure $R' := (E', (\rho_i)_{i \in I'})$ is almost monomorphic then $\varphi_{R'}$ is non-decreasing.

Proof of Claim 2. Let F be a finite subset of E' such that R' is F-monomorphic. Let n be a non-negative integer. Let $m := \varphi_{R'}(n)$ and let $\tau_1, \ldots, \tau_m$ be the isomorphic types of the n-element restrictions of R'. Select $F_1, \cdots, F_m$ such that for each i, $1 \leq i \leq m$, $R'_{\restriction F_i}$ has type τ_i and $|F \cap F_i|$ is minimum. Pick $x \in E' \setminus (F \cup F_1 \cup \cdots \cup F_m)$ and set $F'_i := F_i \cup \{x\}$ for i, $1 \leq i \leq m$. We claim that the restrictions $R'_{\restriction F'_1}, \ldots, R'_{\restriction F'_m}$ are pairwise non-isomorphic, from which the inequality $\varphi_{R'}(n) \leq \varphi_{R'}(n+1)$ will follow. Indeed, suppose that there is some isomorphism f from $R'_{\restriction F'_i}$ onto $R'_{\restriction F'_j}$. With no loss of generality, we may suppose $|F_i \cap F| \geq |F_j \cap F|$. Then $f(x) \notin F$, otherwise $R'_{\restriction F''_j}$, where $F''_j := F'_j \setminus \{f(x)\}$, has type τ_i and $|F''_j \cap F| < |F_i \cap F|$, contradicting the choice of F_i. Hence $f(x) \in F'_j \setminus F$. Since R' is F-monomorphic, the restriction $R'_{\restriction F'_j \setminus \{f(x)\}}$ and $R'_{\restriction F'_j \setminus \{x\}}$ are isomorphic. Since their types are respectively τ_i and τ_j, we have $i = j$. ∎

The proof based on linear algebra. It says more:

Theorem 10. *If R is a relational structure on a set E having at least $2n+k$ elements then $\varphi_R(n) \leq \varphi_R(n+k)$.*

Meaning that if $|E| := m$ then φ_R increases up to $\frac{m}{2}$; and, for $n \geq \frac{m}{2}$ the value in n is at least the value of the symmetric of n w.r.t. $\frac{m}{2}$.

The result is a straightforward consequence of the following property of incidence matrices.

Let n, k, m be three non-negative integers and E be an m-element set. Let $M_{n,n+k}$ be the matrix whose rows are indexed by the n-element subsets P of E and columns by the $n+k$-element subsets Q of E, the coefficient $a_{P,Q}$ being equal to 1 if $P \subseteq Q$ and equal to 0 otherwise.

Theorem 11. *If $2n+k \leq m$ then $M_{n,n+k}$ has full row rank (over the the field of rational numbers).*

With this result the proof of Theorem 10 goes as follows:

We suppose that $\varphi_R(n+k)$ is finite (otherwise, from Fact 1, the stated inequality holds). Thus, we may suppose also that E is finite (otherwise, for each isomorphic type τ of $n+k$-element restriction of R we select a subset Q of E such that $R_{\restriction Q}$ has type τ and we replace E by the union of the $Q's$). We consider the matrix whose rows are indexed by the isomorphic types τ of the restrictions of R to the n-element subsets of E and columns by the n-element subsets P of E, the coefficient $a_{\tau,P}$ being equal to 1 if $R_{\restriction P}$ has type τ and equal to 0 otherwise. Trivially, this matrix has full row rank, hence if we multiply it (from the left) with $M_{n,n+k}$ the resulting matrix has full row rank. Thus, there are $\varphi(n)$ linearly independent colums. These columns being distinct, the restrictions of R to the corresponding $(n+k)$-element subsets have diff erent isomorphic types, hence $\varphi_R(n) \leq \varphi_R(n+k)$.

We proved Theorem 10 in 1976 (MZ). The same conclusion was obtained first for orbits of finite permutation groups by Livingstone and Wagner, 1965, and extended to arbitrary permutation groups by Cameron, 1976. His proof uses the dual version of Theorem 11. Later on, he discovered a nice translation in terms of his age algebra.

Theorem 11 is in W.Kantor 1972, with similar results for affine and vector subspaces of a vector space. Over the last 30 years, it as been applied and rediscovered many times; recently, it was pointed out that it appeared in a 1966 paper of D.H.Gottlieb. Nowadays, this is one of the fundamental tools in algebraic combinatorics. A proof, with a clever argument leading to further developments, was given by Fraïssé in the 1986 edition of his book, Theory of relations.

Polynomial Growth

It is natural to ask:
Problem 1. *If the profile of a relational structure R with finite kernel has polynomial growth, is $\varphi_R(n) \simeq cn^{k'}$ for some positive real c and some non-negative integer k'?*

The problem was raised by P.J.Cameron for the special case of orbital profiles. Up to now, it is unsolved, even in this special case.

An example, pointed out by P.J.Cameron, suggests that a stronger property holds.

Let G' be the wreath product $G' := G \wr \mathfrak{S}_\omega$ of a permutation group G acting on $\{1, \ldots, k\}$ and of $\mathfrak{S}_\omega$, the symmetric group on ω. Looking at G' as a permutation group acting on $E' := \{1, \ldots, k\} \times \omega$, then - as observed by Cameron- $\theta_{G'}$ is the *Hilbert function* $h_{Inv(G)}$ of the subalgebra $Inv(G)$ of $\mathbb{C}[x_1, \ldots, x_k]$ consisting of polynomials in the k indeterminates $x_1, \ldots, x_k$ which are invariant under the action of G. The value of $h_{Inv(G)}(n)$ is, by definition, the dimension $dim(Inv_n(G))$ of the subspace of homogeneous polynomials of degree n. As it is well known, the *Hilbert series* of $Inv(G)$,

$$\mathcal{H}(Inv(G), x) := \sum_{n=0}^{\infty} h_{Inv(G)}(n) x^n$$

is a rational fraction of the form

$$\frac{P(x)}{(1-x)\cdots(1-x^k)} \tag{3}$$

with $P(0) = 1$, $P(1) > 0$, and all coefficients of P being non negative integers.

Problem 2. *Find an example of a permutation group G' acting on a set E with no finite orbit, such that the orbital profile of G' has polynomial growth, but is not the Hilbert function of the invariant ring $Inv(G)$ associated with a permutation group G acting on a finite set.*

Let us associate to a relational structure R whose profile takes only finite values its *generating series*

$$\mathcal{H}_{\varphi_R} := \sum_{n=0}^{\infty} \varphi_R(n) x^n$$

Problem 3. *If R has a finite kernel and φ_R is bounded above by some polynomial, is the series $\mathcal{H}_{\varphi_R}$ a rational fraction of the form*

$$\frac{P(x)}{(1-x)(1-x^2)\cdots(1-x^k)}$$

with $P \in \mathbb{Z}[x]$?

Under the hypothesis above we do not know if $\mathcal{H}_{\varphi_R}$ is a rational fraction.

It is well known that if a generating function is of the form $\frac{P(x)}{(1-x)(1-x^2)\cdots(1-x^k)}$ then for n large enough, a_n is a *quasi-polynomial* of degree k', with $k' \leq k-1$, that is a polynomial $a_{k'}(n)n^{k'} + \cdots + a_0(n)$ whose coefficients $a_{k'}(n), \ldots, a_0(n)$ are periodic functions. Hence, a subproblem is:

Problem 4. *If R has a finite kernel and φ_R is bounded above by some polynomial, is $\varphi_R(n)$ a* quasi-polynomial *for n large enough?*

Remark 2. *Since the profile is non-decreasing, if $\varphi_R(n)$ is a quasi-polynomial for n large enough then $a_{k'}(n)$ is eventually constant. Hence the profile has polynomial growth in the sense that $\varphi_R(n) \sim cn^{k'}$ for some positive real c and $k' \in \mathbb{N}$. Thus, in this case, Problem 1 has a positive solution.*

In the theory of languages, one of the basic results is that the generating series of a regular language is a rational fraction. This result is not far away from our considerations. Indeed, if A is a finite alphabet, with say k elements, and A^* is the set of words over A, then each word can be viewed as a finite chain coloured by k colors and A^* can be viewed as the age of the relational structure made of the chain $\mathbb{Q}$ of rational numbers divided into k colors in such a way that, between two distinct rational numbers, all colors appear. This structure was Example (2) in Subsection 1.1.

Problem 5. *Does the members of the age of a relational structure with polynomial growth can be coded by words forming a regular language?*

Problem 6. *Extend the properties of regular languages to subsets of Ω_μ.*

Morphology of Relational Structures with Polynomial Growth We only have a partial description of relational structures with polynomial growth.

Let us say that a relational structure $R := (E, (\rho_i)_{i\in I})$ is *almost multichainable* if there is a finite subset F of E and an enumeration $(a_{x,y})_{(x,y)\in V\times L}$ of the elements of $E \setminus F$ by a set $V \times L$, where V is finite and L is a linearly ordered set, such that for every local isomorphism f of L the map $(1_V, f)$ extended by the identity on F is a local isomorphism of R (the map $(1_V, f)$ is defined by $(1_V, f)(x, y) := (x, f(y))$).

Note that if L is infinite, $K(R)$, the kernel of R, is a subset of F. Thus we have:

Fact 2. *An almost multichainable relational structure has a finite kernel.*

The profile of an almost multichainable relational structure is not necessarily bounded above by a polynomial.

Problem 7. *If the profile of an almost multichainable relational structure is not bounded* *above by a polynomial, does his profile has exponential growth? Is the generating series a rational fraction?*

Theorem 12. *If the profile of a relational structure R with bounded signature or finite kernel is bounded above by a polynomial then R is almost multichainable.*

There are two cases, in fact opposite cases, for which the profile of an almost multichainable relational structure is bounded above by a polynomial.

1. **Case 1.** $(1_V, f)$ extended by the identity on F is an automorphism of R for every permutation f of L.

2. **Case 2.** For every family $(f_x)_{x\in V}$ of local isomorphisms of L, the map $\cup\{f_x : x \in V\}$ extended by the identity on F is a local

isomorphism of f (the map $\cup\{f_x : x \in V\}$ associates $(x, f_x(y))$ to (x, y)).

A relational structure for which there are F and $(a_{x,y})_{(x,y)\in V\times L}$ such that Case 1 holds is *cellular*. This notion was introduced by Schmerl [0].

The Case of Graphs A *directed graph* is a pair $G := (E, \rho)$ where ρ is a binary relation on E. Ordered sets and tournaments are special case of directed graphs. We will use the term *graph* if ρ is irreflexive and symmetric. In this case ρ is identified with the set $\mathcal{E}$ of pairs $\{x, y\}$ of members of E such that $x\rho y$, G is identified with $(E, \mathcal{E})$; the members of E and $\mathcal{E}$ are the *vertices* and *edges* of G. We denote by $V(G)$, resp. $E(G)$, the set of vertices, resp. edges, of G.

In terms of profile, the class of graphs provides interesting examples.

Examples 13. *1.* $\varphi_G(n)$ *is constant, equal to* 1*, for every* $n \leq |V(G)|$*, if and only if* $\varphi_G(2) \leq$ 1*, that is* G *is a clique or an independent set (trivial).*

2. φ_G *is bounded if and only if* G *is "almost constant" in the Fraïssé's terminology, that is there is a finite subset* F_G *of vertices such that two pairs* $\{x, y\}$ *and* $\{x', y'\}$ *of vertices having the same intersection on* F_G *are both edges or both non-edges.*

3. If G *is the direct sum of infinitely many edges, or the direct sum* $K_\omega \oplus K_\omega$ *of two infinite cliques, then* $\varphi_G(n) = \lfloor \frac{n}{2} \rfloor + 1$*, whereas* $\mathcal{H}_{\varphi_G} = \frac{1}{(1-x)(1-x^2)}$.

4. Let G *be the direct sum* $K_{(1,\omega)} \oplus \overline{K}_\omega$ *of an infinite wheel and an infinite independent set, or the direct sum* $K_\omega \oplus \overline{K}_\omega$ *of* *an infinite clique and an infinite independent set. Then $\varphi_G(n) = n$. Hence $\mathcal{H}_{\varphi_G} = 1 + \frac{x}{(1-x)^2}$, that we may write $\frac{1-x+x^2}{(1-x)^2}$, as well as $\frac{1+x^3}{(1-x)(1-x^2)}$.*

5. *Let G be the direct sum of infinitely many k-element cliques or the direct sum of k infinite cliques. Then $\varphi_G(n) = p_k(n) \simeq \frac{n^{k-1}}{(k-1)!k!}$ and $\mathcal{H}_{\varphi_G} = \frac{1}{(1-x)\cdots(1-x^k)}$.*

6. *If G is either the direct sum of infinitely many infinite cliques -or an infinite path- then $\varphi_G(n) = p(n)$ the partition function.*

7. *Let $C := (E, \leq)$ be a chain and $K_{C,\frac{1}{2}}$ be the graph whose vertex set is $2 \times E$ and the edge set is $\{\{(0,i),(1,j)\} : i < j \text{ in } C\}$. Such a graph is an* half-complete bipartite graph. *If C is infinite, then $2^{n-2} \leq$*

$\varphi_{K_{C,\frac{1}{2}}}(n) \leq 2^{n-1}$ *[0], hence its growth is exponential. In fact, one can check that:* $\mathcal{H}_{K_{C,\frac{1}{2}}} = \frac{1-2x-x^2+3x^3-x^4}{(1-x)(1-2x)(1-2x^2)} = 1+x+2x^2+3x^3+6x^4+10x^5+20x^6+36x^7+72x^8+136x^9+O(x^{10})$.

8. *Let* $\tilde{K}_{C,\frac{1}{2}}$ *be the graph obtained from* $K_{C,\frac{1}{2}}$ *by adding all possible edges between vertices of the form* $(1,i)$, *for* $i \in E$. *Then* $\varphi_{\tilde{K}_{C,\frac{1}{2}}}(n) = 2^{n-1}$.

Theorem 14. *The profile of a graph is bounded by a polynomial if and only if this graph is cellular.*

A straightforward computation shows that the profile of a cellular graph is bounded by a polynomial. The converse follows directly from Theorem 12 and Lemma 15 below. A self-contained proof will hopefully appear in a joint work with S. Thomassé and R. Woodrow.

Lemma 15. *The growth of the profile of almost multichainable graph which is not cellular is at least exponential*

Indeed, let G be an almost multichainable graph. The sets F, V and L which appear in the definition of the almost multichainability of G satisfy the following conditions: F, V are finite, $V(G) = F \cup V \times L$ and:

$$\{a,(x,i)\} \in E(G) \text{ if and only if } \{a,(x,j)\} \in E(G) \tag{4}$$

for all $\quad a \in F, x \in V, j \in L$

$$\{(x,i),(y,j)\} \in E(G) \text{ if and only if } \{(x,i'),(y,j')\} \in E \tag{5}$$

for all $\quad x, y \in V, i, j, i', j' \in L$ such that $i\rho j$ and $i'\rho j'$ where ρ is either the equality relation on L or the strict order $<$ on L.

If G is not cellular then there is some permutation f of L such that $(1_V, f)$ extended by the identity on F is not an automorphism of G. The map f does not preserve the order on L, hence, there are $i_0, j_0 \in L$ and $x, y \in V$ such that $\{(x, i_0), (y, j_0)\} \in E(G)$ and $\{(x, j_0), (y, i_0)\} \notin E(G)$.

Let $H := G_{\restriction \{x,y\} \times L}$. This graph is multichainable, hence it is entirely determined by the edges belonging to $[\{x, y\} \times \{i_0, j_0\}]^2 \setminus \{(x, j_0), (y, j_0)\}$. There are 16 possible graphs. But, if L is infinite, these graphs yield only two distinct ages, namely the age of $K_{C,\frac{1}{2}}$ and the age of $\tilde{K}_{C,\frac{1}{2}}$, two graphs described in (7) and (8) of Examples 13. Hence, they yield at most two distinct profiles. Their growth rates, as computed in Examples 13, are exponential, hence the growth rate of φ_G is at least exponential as claimed.

We do not know if Problem 1 has a positive answer for cellular graphs. Problem 3 has a positive answer for a special class of relational structures described in the following subsection.

Relational Structures Admitting a Finite Monomorphic Decomposition

A *monomorphic decomposition* of a relational structure R is a partition $\mathcal{P}$ of E into blocks such that for every integer n, the induced structures on two n-elements subsets A and A' of E are isomorphic whenever the intersections $A \cap B$ and $A' \cap B$ over each block B of $\mathcal{P}$ have the same size.

This notion was introduced with N. Thiéry.

If an infinite relational structure R has a monomorphic decomposition into finitely many blocks, whereof k are infinite, then the profile is bounded by some polynomial, whose degree itself is bounded by $k-1$. Indeed, as one may immediately see:

$$\varphi_R(n) \leq \sum_{s\leq r} \binom{r}{s}\binom{n+k-1-s}{k-1} \leq 2^r\binom{n+k-1}{k-1} \tag{6}$$

where r is the cardinality of the union of the finite blocks.

One can say more:

Theorem 16 ([0]). *Let R be an infinite relational structure R with a monomorphic decomposition into finitely many blocks $(E_i, i \in X)$, k of which being infinite. Then, the generating series $\mathcal{H}_{\varphi_R}$ is a rational fraction of the form:*

$$\frac{P(x)}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Corollary 2 ([0]). *Let R a relational structure as above, then φ_R has a polynomial growth and in fact $\varphi_R(n) \sim an^{k'}$ for some positive real a, some non-negative integer k'.*

Recently, with N.Thiéry, we proved:

Lemma 17. *If k is the least number of infinite blocks that a monomorphic decomposition of R may have then $\varphi_R(n) \sim an^{k-1}$.*

The proof of this result relies on Proposition 1 below for which we introduce the following definition. Let R be a relational structure on E; a subset B of E is a *monomorphic part* of R if for every integer n and every pair A, A' of n-element subsets of E the induced structures on A and A' are isomorphic whenever $A \setminus B = A' \setminus B$.

Proposition 1 ([0]). *1. For every $x \in E$, the set-union $R(x)$ of all the monomorphic parts of R containing x is a monomorphic part, the largest monomorphic part of R containing x.*

2. The largest monomorphic parts form a monomorphic decomposition of R of which every *monomorphic decomposition of R is a refinement.*

We will call *canonical* the decomposition of R into maximal monomorphic parts. This decomposition has the least possible number of parts.

Despite the apparent simplicity of relational structures admitting a finite monomorphic decomposition, there are many significant examples.

Quasi-Symmetric Polynomials

Let $x_1, \ldots, x_k$ be k-indeterminates and $n_1, \ldots, n_l$ be a sequence of non-negative integers, $1 \leq l \leq k$. The polynomial

$$\sum_{1 \leq i_1 < \cdots < i_l \leq k} x_{i_1}^{n_1} \ldots x_{i_l}^{n_l}$$

is a *quasi-monomial* of degree n, where $n =: n_1 + \cdots + n_l$. The vector space spanned by the quasi-monomials forms the space $\mathcal{QS}_k$ of *quasi-polynomials* as introduced by I. Gessel. As in the example above, the Hilbert series of $\mathcal{QS}_{k+1}$ is defined as

$$\mathcal{H}_{\mathcal{QS}_k} := \sum_{n=0}^{\infty} dim \mathcal{QS}_{k,n} x^n.$$

As shown by F. Bergeron, C. Retenauer, see [0], this series is a rational fraction of the form $\frac{P_k}{(1-x)(1-x^2)...(1-x^k)}$ where the coefficients P_k are non negative. Let R be the poset product of a k-element chain by a denumerable antichain. More formally, $R := (E, \rho)$ where $E := \{1, \ldots, k\} \times \mathbb{N}$ and $\rho := \{((i,n),(j,m)) \in E$ such that $i \leq j\}$. Each isomorphic type of an n-element restriction may be identified to a quasi-polynomial, hence the generating series associated to the profile of R is the Hilbert series defined above. Since R decomposes into

k monomorphic components, the rationality of this series is a special case of Theorem 16. The reason for which the coefficients of this fraction are non-negative was elucidated only recently by Garsia and Wallach [0]. They proved that $\mathcal{QS}_k$ is Cohen-Macaulay.

Tournaments

With Y.Boudabbous [0] we proved:

Theorem 18. *[0] The growth of the profile of a tournament T is either polynomial, in which case T is a lexicographic sum of acyclic tournaments indexed by a finite tournament, or it is at least exponential.*

Here is an outline:

Let T be a tournament, a subset A of $V(T)$ is *autonomous* if for every $x \notin A$, $y, y' \in V(T)$, $(x, y) \in E(T)$ if and only if $(x, y') \in E(T)$. A tournament is *acyclically prime* if no acyclic autonomous subset has more than one element. (this notion was recently introduced by J.F.Culus and B.Jouve).

Proposition 2. *Every tournament T is a lexicographical sum of acyclic tournament indexed by an acyclically prime tournament and up to isomorphy this tournament is unique.*

We denote it by $\check{T}$.

Hint:An *acyclic component* of a tournament T is a subset of $V(T)$ which is maximal w.r.t. inclusion among the acyclic autonomous subsets of $V(T)$. Clearly, every acyclic autonomous subset is contained into an acyclic component. As it is easy to see, the acyclic components of a tournament form a partition of its vertex set.

From this result, a tournament is a lexicographic sum of acyclic tournaments indexed by a finite tournament iff it contains no infinite acyclically prime tournament.

We proved:

Theorem 19. *There are twelve infinite acyclically prime tournaments such that each infinite acyclically prime tournament contains a copy of one of them.*

To conclude it was enough to prove that the profiles of these tournaments are exponential.

Enough to say that to an acyclic tournament α, where α is either the acyclic tournament $\underline{n}$ on n-vertices, or the tournament ω on the set $\mathbb{N}$ of integers or the dual tournament ω^*, we associate a set $\mathfrak{B}_\alpha$ consisting of (at most) six tournaments denoted respectively $C_3.\alpha, V_\alpha, T_\alpha, H_\alpha, U_\alpha, K_\alpha$ and we set $\mathfrak{B} = \mathfrak{B}_\omega \cup \mathfrak{B}_{\omega*}$. Not all members of $\mathfrak{B}$ are acyclically prime (eg: K_ω and U_ω). But, all the $\check{X}_\alpha \in \mathfrak{B}$ are infinite. They form the list. The proof of Theorem 19 is based on a separation lemma and Ramsey Theorem.

Our result has a finitary version.

For that, let n be a non-negative integer, set $\check{\mathfrak{B}}_{\underline{n}} := \{\check{T} : T \in \mathfrak{B}_{\underline{n}}\}$.

Theorem 20. *For every non-negative integer n there is an integer $a(n)$ such that every finite tournament of size at least $a(n)$ which is acyclically prime embeds a member of $\check{\mathfrak{B}}_{\underline{n}}$.*

An upper bound for $a(n)$ can be deduced from a careful analysis of the proof of Theorem 19. An existence proof is readily obtained by means of the compactness theorem of first order logic.

The Age Algebra of Cameron P. J. Cameron associates to $\mathcal{A}(R)$, the age of a relational structure R, its *age algebra*, a graded commutative algebra $\mathbb{K}.\mathcal{A}(R)$ over a field $\mathbb{K}$ of characteristic zero. He shows that if φ_R takes only finite values, then the dimension of $\mathbb{K}.\mathcal{A}(R)_n$, the homogeneous component of degree n of $\mathbb{K}.\mathcal{A}(R)$, is $\varphi_R(n)$. Hence, in this case, the generating series of the profile is simply the Hilbert series of $\mathbb{K}.\mathcal{A}(R)$.

P.J Cameron mentions several interesting examples of algebras which turn to be age algebras. The most basic one is the *shuffle algebra* on the set A^* of words on a finite alphabet A [0]. Indeed, as mentionned at the end of Subsection 2.3, A^* is the age of the relational structure $(\mathbb{Q}, (U_a)_{a\in A})$ where the U_a's are unary relations forming a coloring of $\mathbb{Q}$ into distinct colors, in such a way that between two distinct rational numbers, all colors appear. And the shuffle algebra is isomorphic to the age algebra of $(\mathbb{Q}, (U_a)_{a\in A})$.

The Set Algebra

Let E be a set and let $[E]^{<\omega}$ be the set of finite subsets of E (including the empty set $\emptyset$). Let $\mathbb{K}$ be a field and $\mathbb{K}^{[E]^{<\omega}}$ be the set of maps $f : [E]^{<\omega} \to \mathbb{K}$. Endowed with the usual addition and scalar multiplication of maps, this set is a vector space over $\mathbb{K}$. Let $f, g \in \mathbb{K}^{[E]^{<\omega}}$ and $Q \in [E]^{<\omega}$. Set

$$fg(Q) = \sum_{P \in [Q]^{<\omega}} f(P)g(Q \setminus P) \qquad (7)$$

. With this operation added, the above vector space becomes a $\mathbb{K}$-algebra. This algebra is commutative and it has a unit, denoted by 1. This is the map taking the value 1 on the empty set and the value 0 everywhere else. The *set algebra* is the subalgebra made of the maps such that $f(P) = 0$ for every $P \in [E]^{<\omega}$ with $|P|$ large enough. This algebra is graded, the homogeneous component of degree n being made of maps which take the value 0 on every subset of size different from n.

Let $\equiv$ be an equivalence relation on $[E]^{<\omega}$. A map $f : [E]^{<\omega} \rightarrow \mathbb{K}$ is $\equiv$-*invariant*, or briefly, *invariant*, if f is constant on each equivalence class. Invariant maps form a subspace of the vector space $\mathbb{K}^{[E]^{<\omega}}$.

if R is a relational structure with domain E, set $F \equiv_R F'$ for $F, F' \in [E]^{<\omega}$ if the restrictions $R \restriction_F$ and $R \restriction_{F'}$ are isomorphic. The resulting equivalence on $[E]^{<\omega}$ is such that the invariant maps form a subalgebra . Let $\mathbb{K}.\mathcal{A}(R)$ be the intersection of the subalgebra of $\mathbb{K}^{[E]^{<\omega}}$ made of invariant maps with the set algebra. This is a graded algebra, the *age algebra* of Cameron.

The name comes from the fact that this algebra depends only upon the age of R.

If φ_R takes only integer values, $\mathbb{K}.\mathcal{A}(R)$ identifies with the set of (finite) linear combinations of members of $\mathcal{A}(R)$. This explain the fact that, in this case, $\varphi_R(n)$ is the dimension of the homogeneous component of degree n of $\mathbb{K}.\mathcal{A}(R)$. In a special case, we have

Theorem 21. *[0] If R has a monomorphic decomposition into finitely many blocks $E_1, \ldots, E_k$, all infinite, then the age algebra $\mathbb{K}.\mathcal{A}(R)$ is a polynomial algebra, isomorphic to a subalgebra $\mathbb{K}[x_1, \ldots, x_k]^R$ of $\mathbb{K}[x_1, \ldots, x_k]$, the algebra of polynomials in the indeterminates $x_1, \ldots, x_k$.*

Behavior of the Profile

In the frame of its age algebra, Cameron gave the following proof of the fact that the profile does not decrease.

Let R be a relational structure on a set E, let $e := \sum_{P \in [E]^1} P$ (that we could view as the sum of isomorphic types of the one-element restrictions of R) and U be the subalgebra generated by e. Members of U are of the form

$\lambda_m e^m + \cdots + \lambda_1 e + \lambda_0 1$ where 1 is the isomorphic type of the empty relational structure and $\lambda_m, \ldots, \lambda_0$ are in $\mathbb{K}$. Hence U is graded, with U_n, the homogeneous component of degree n, equals to $\mathbb{K}.e^n$.

Theorem 22. *If R is infinite then for every $u \in \mathbb{K}.\mathcal{A}(R)$, $eu = 0$ if and only if $u = 0$*

This innocent looking result implies that φ_R is non decreasing. Indeed, the image of a basis of $\mathbb{K}.\mathcal{A}(R)_n$ by multiplication by e^m is an independent subset of $\mathbb{K}.\mathcal{A}(R)_{n+m}$.

Finite generation

If a graded algebra A is finitely generated, then, since A is a quotient of the polynomial ring $\mathbb{K}[x_1, \ldots, x_k]$, its Hilbert function is bounded above by a polynomial. In fact, as it is well known, its Hilbert series is a fraction of form

$\frac{P(x)}{(1-x)^d}$, thus of the form given in (3) of subsection . Moreover, one can choose a numerator with non-negative coefficients whenever the algebra is Cohen-Macaulay. Due to Problem 3, one could be tempted to conjecture that these sufficient conditions are necessary in the case of age agebras. Indeed, from Theorem 22 one deduces easily:

Theorem 23. *The profile of R is bounded if and only if $\mathbb{K}.\mathcal{A}(R)$ is finitely generated as a module over U. In particular, if one of these equivalent conditions holds, then $\mathbb{K}.\mathcal{A}(R)$ is finitely generated*

But this case is exceptional. Indeed, on one hand, as we have mentionned in , there are tournaments whose profile has arbitrarily large polynomial growth rate. On an other hand, with N.Thiery we proved:

Theorem 24. *The age algebra of a tournament is finitely generated if and only if the profile is bounded.*

The Behavior of the Hilbert Function; a Conjecture of Cameron

Cameron [0] made an important observation about the behavior of the Hilbert fonction.

Theorem 25. *Let A be a graded algebra over an algebraically closed field of characteristic zero. If A is an integral domain the values of the Hilbert function h_A satisfy the inequality*

$$h_A(n) + h_A(m) - 1 \leq h_A(n+m) \qquad (8)$$

for all non-negative integers n and m.

In 1981, he made the conjecture that if R codes a permutation groups with no finite orbits then the age algebra over C is an integral domain. I solved it positively in a slightly more general setting:

Theorem 26. *Let R be a relational structure with possibly infinitely many non isomorphic* *types of n-element substructures. If the kernel of R is empty, then $\mathbb{K}.\mathcal{A}(R)$ is an integral domain.*

Since the kernel of a relational structure R encoding a permutation group G is the union of its finite orbits, if G has no finite orbit, the kernel of R is empty. Thus from this result, $\mathbb{K}.\mathcal{A}(R)$ is an integral domain, as conjectured by Cameron.

At the core of the solution is this lemma:
Lemma 27. *Let m, n be two non negative integers. There is an integer t such that for every set E, every field $\mathbb{K}$ with characteristic zero, every pair of maps $f : [E]^m \to \mathbb{K}$, $g : [E]^n \to \mathbb{K}$, if $fg(Q) := \sum_{P \in [Q]^m} f(P)g(Q \setminus P) = 0$ for every $Q \in [E]^{m+n}$ but f and g are not identically zero, then f and g are zero on $[E \setminus S]^m$ and $[E \setminus S]^n$ where S is a finite subset of E with size at most t*

If the age is inexhaustible, then in order to prove that there is no zero divisor, the only part of the lemma we need to apply is the assertion that S is finite.

The fact that S can be bounded independently of f and g, and the value of the least upper bound $\tau(n, m)$, seem to be of independent interest. The only exact value we know is $\tau(1, n) = 2n$, a fact which amounts to a weighted version of Theorem 11. Our existence proof of $\tau(m, n)$ yields astronomical upper bounds. For example, it gives $\tau(2, 2) \leq 2(R^2_k(4) + 2)$, where $k := 5^{30}$ and $R^2_k(4)$ is the Ramsey number equals to the least integer p such that for every colouring of the pairs of $\{1, \ldots, p\}$ into k colors there are four integers whose all pairs have the same colour. The only lower bound we have is $\tau(2, 2) \geq 7$ and more generally $\tau(m, n) \geq (m+1)(n+1)-2$. We cannot preclude a extremely simple upper bound for $\tau(m, n)$, eg quadratic in $n + m$.

For example, our 1971 proof of Theorem 3 consisted to show that $\varphi_R(n) \leq \varphi_R(n+1)$ provided that E is large enough, the size of E being bounded by some Ramsey number, whereas, according to Theorem 11, $2n+1$ suffices [0].

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