HEEGNER POINTS AND COHOMOLOGICAL MODULAR FORMS

MATT GREENBERG

Techniques used to study units in number fields can be used to study rational points on elliptic curves. Elliptic units are analogous to Heegner points. Stark units are analogous to the Stark-Heegner points introduced by Darmon.

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. Let $K/\mathbb{Q}$ be a quadratic field. Let $\mathcal{O} \subset K$ be an order. Suppose that $(\operatorname{disc} \mathcal{O}, N) = 1$. Here $\operatorname{disc} \mathcal{O} = (\operatorname{disc} K)(\operatorname{cond} \mathcal{O})^2$. Let $H_{\mathcal{O}}$ be the ring class field. Let $\chi\colon \operatorname{Gal}(H_{\mathcal{O}}/K) \to \mathbb{C}^\times$ be a character. Let $S_{N,K} = \{\text{infinite places of } K\} \cup \{\ell \mid N : \ell \text{ inert in } K/\mathbb{Q}\}$. Let $\operatorname{sign}(N, K) = (-1)^{\# S_{N,K}}$. Then $L(E/K, \chi, s) = \operatorname{sign}(N, K) L(E/K, \chi, 2 - s)$; note that $\operatorname{sign}(N, K)$ is independent of χ. If $\operatorname{sign}(N, K) = -1$, then $L(E/K, \chi, 1) = 0$ for all χ, and then BSD predicts that $E(H_{\mathcal{O}})^\chi$ has rank ≥ 1 for all χ.

Philosophy: In such a situation, there should be a systematic construction of points in E explaining this (as in the classical case of Heegner points).

Assume that K is imaginary. Then there is a map from a Shimura curve X to E, and the CM-points on X corresponding to $\mathcal{O}$ map to points of $E(H_{\mathcal{O}})$.

Generalization: If E is defined over a totally real field F, and K is a CM extension of F, then we get the same picture.

Main ingredients of the Heegner point construction:

(1) modularity
(2) CM theory (Darmon's construction bypasses this, but is conjectural)

Darmon's construction: Assume that $K/\mathbb{Q}$ is real quadratic. So we want $\# S_{N,K}$ to be odd. Darmon considers the case $N = pM$ with $p \nmid M$ and p inert in K, and all $\ell \mid M$ split in K.

Key ingredients in Darmon's construction:

(1) Measure-valued modular symbols attached to E.
(2) p-adic integration

Define

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[1/p]) : M \mid c \right\}.$$

Let $\mathcal{M}$ be the set of $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$ with total measure 0. Define $\rho\colon \mathcal{M} \to \mathbb{Z}$ by $\rho(\mu) = \mu(\mathbb{Z}_p)$. This map is $\Gamma_0(N)$-equivariant, since $\Gamma_0(N)$ stabilizes the ball $\mathbb{Z}_p$ in $\mathbb{P}^1(\mathbb{Q}_p)$.

Let $\Delta := \operatorname{Div} \mathbb{P}^1(\mathbb{Q})$. Let $\Delta^0 := \operatorname{Div}^0 \mathbb{P}^1(\mathbb{Q})$.

Proposition 0.1. *There exist $\phi^\pm \in \operatorname{Hom}(\Delta^0, \mathcal{M})$ such that*

$$(\rho_* \phi^\pm)(D)(\mathbb{Z}_p) = \frac{1}{\Omega_E^\pm} \int_D \omega_{f_E}^\pm.$$

Date: June 5, 2007.

p-adic 2-cocycles: Fix $\tau \in \mathcal{H}_p = \mathbb{C}_p - \mathbb{Q}_p$. Fix a cusp $r \in \mathbb{P}^1(\mathbb{Q})$. Define a 2-cocycle by the *multiplicative integral*

$$\kappa_\tau^\pm(\gamma_1, \gamma_2) := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{x - \gamma_1^{-1}\tau}{x - \tau} \, d\phi^\pm(\{\gamma_2 r\} - \{r\}) \in Z^2(\Gamma, \mathbb{C}_p^\times).$$

The class $\kappa_\tau^\pm \in H^2(\Gamma, \mathbb{C}_p^\times)$ is independent of choices.

Theorem 0.2 (Darmon, Greenberg-Stevens)**.** *There exists a lattice $\Lambda \subset \mathbb{C}_p^\times$ commensurable with the Tate lattice of E such that κ_τ splits mod Λ.*

There exists $\eta_\tau \in C^1(\Gamma, \mathbb{C}_p^\times/\Lambda)$ such that $d\eta_\tau \equiv \kappa_\tau \pmod{\Lambda}$. Let $\Gamma_\tau \subseteq \Gamma$ be the stabilizer of τ. Let $\xi_\tau := \eta_\tau|_{\Gamma_\tau}$, so $\xi_\tau \in Z^1(\Gamma_\tau, \mathbb{C}_p^\times/\Lambda) = \operatorname{Hom}(\Gamma_\tau, \mathbb{C}_p^\times/\Lambda)$. Let τ be a fixed point of an optimal embedding

$$\psi \colon \mathcal{O} \to \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : M \mid c \right\};$$

this exists because of the hypothesis $S_{N,K} = \{p, \infty_1, \infty_2\}$. We have $\Gamma_\tau \simeq \mathbb{Z}$. So $\xi_\tau \in \operatorname{Hom}(\Gamma_\tau, \mathbb{C}_p^\times/\Lambda) = \mathbb{C}_p^\times/\Lambda$. We get $P_\tau = \operatorname{Tate}(\xi_\tau) \in E(\mathbb{C}_p)$.

Conjecture 0.3. The point P_τ is algebraic over $H_{\mathcal{O}}$.

Cohomologification:

$$0 \to \Delta^0 \to \Delta \to \mathbb{Z} \to 0$$

gives

$$0 \to \mathcal{M} \to \operatorname{Hom}(\Delta, \mathcal{M}) \to \operatorname{Hom}(\Delta^0, \mathcal{M}) \to 0$$

and

$$\operatorname{Hom}(\Delta^0, \mathcal{M})^\Gamma \xrightarrow{d} H^1(\Gamma, \mathcal{M}).$$

(A) Describe Darmon's construction in terms of $\psi^\pm$.
(B) Construct $\psi^\pm$ formally.

For (A), use the standard resolution $F_\cdot$ of $\mathbb{Z}$; i.e., with $F_n := \mathbb{Z}[\Gamma]^{\otimes(n+1)}$. Fix $\tau \in \mathcal{H}_p$. Given $\phi \in C^n(\Gamma, \mathcal{M})$, define

$$\kappa_\tau(\phi)(g_1 \otimes \cdots \otimes g_{n+1}) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{x - g_0\tau}{x - g_1\tau} \, d\phi(g_1 \otimes \cdots \otimes g_{n+1}) \in C^{n+1}(\Gamma, \mathbb{C}_p^\times),$$

where again this is a multiplicative integral.

Proposition 0.4. *$d\kappa_\tau(\phi) = \kappa_\tau(d\phi)$.*

Corollary 0.5. *$\kappa_\tau \colon H^n(\Gamma, \mathcal{M}) \to H^{n+1}(\Gamma, \mathbb{C}_p^\times)$ is independent of τ.*

$\kappa_\tau \rightsquigarrow \xi_\tau \rightsquigarrow P_\tau$
For (B), define $\mathbb{T}^{(p)} := \mathbb{Z}[\{T_\ell\}_{\ell \nmid N}, \{U_\ell\}_{\ell \mid M}]$. Let $\lambda_E \colon \mathbb{T}^{(p)} \to \mathbb{Z}$ correspond to E.

Proposition 0.6. rank $H^1(\Gamma, \mathcal{M})^{\lambda_E} = 2$.

Let $\psi^\pm$ be eigenvectors in $H^1(\Gamma, \mathcal{M})^{\lambda_E}$ for complex conjugation.

Proof of Proposition. Let $\mathcal{B}$ be the set of compact open balls in $\mathbb{P}^1(\mathbb{Q}_p)$. By strong approximation, $\mathcal{B} = \Gamma\mathbb{Z}_p \cup \Gamma(\mathbb{P}^1(\mathbb{Q}_p) - \mathbb{Z}_p)$. Let $\mathcal{B}_0 := \Gamma\mathbb{Z}_p = \Gamma/\Gamma_0(N)$. We have

$$\mathcal{M} \hookrightarrow \text{functions}(\mathcal{B}_0, \mathbb{Z}) \simeq \text{Hom}_{\Gamma_0(N)}(\mathbb{Z}[\Gamma], \mathbb{Z})$$

and

$$0 \to \mathcal{M} \to \text{Hom}_{\Gamma_0(N)}(\mathbb{Z}[\Gamma], \mathbb{Z}) \xrightarrow{T} \text{Hom}_{\Gamma_0(M)}(\mathbb{Z}[\Gamma], \mathbb{Z})^2 \to 0.$$

We can prove that T is surjective by interpreting T in terms of the Bruhat-Tits tree of $\text{PGL}_2(\mathbb{Q}_p)$. Take Galois cohomology and apply Shapiro's lemma:

$$0 \to \mathbb{Z} \oplus \mathbb{Z}/(p+1) \to H^1(\Gamma, \mathcal{M}) \to H^1(\Gamma_0(N), \mathbb{Z})_{p\text{-new}} \to 0.$$

Since the $\mathbb{T}^{(p)}$-action on $\mathbb{Z}\oplus\mathbb{Z}/(p+1)$ is Eisenstein, rank $H^1(\Gamma_0(M), \mathbb{Z})^{\lambda_E} = 2$, so rank $H^1(\Gamma, \mathcal{M})^{\lambda_E} = 2$. □

Let $K/\mathbb{Q}$ be real quadratic and suppose that $\text{sign}(N, K) = -1$ with $N = pDM$ where all $\ell \mid pD$ inert in K and all $\ell \mid M$ split, where D has an even number of prime divisors.

Let $B/\mathbb{Q}$ be the quaternion algebra of discriminant D. Let $R \subset B$ be a $\mathbb{Z}$-Eichler order of level pM. Let $\Gamma_0(N)$ be the image in $B^\times/\mathbb{Q}^\times$ of $R_1^\times$. Let Γ be the image in $B^\times/\mathbb{Q}^\times$ of $(R \otimes \mathbb{Z}[1/p])_1^\times$.