BIRS workshop on braid groups and applications

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The braid groups B_n were introduced by E. Artin in 1926 [1] (see also [2]). They have been of importance in many fields – algebra, analysis, cryptography, dynamics, topology, representation theory, mathematical physics – and many of these aspects were represented in the BIRS workshop. This workshop involved not only leading experts in the field, but also, importantly, a number of young researchers, postdoctoral fellows and several graduate students. This made for an exciting and informative mix of ideas on the subject.

The importance of the braid groups is based, in part, on the many ways in which they can be defined. This is outlined in the following introductory section.

1 Six definitions of the braid groups.

Definition 1: Braids as particle dances. Consider n particles located at distinct points in a plane. To be definite, suppose they begin at the integer points $\{1, \ldots, n\}$ in the complex plane $\mathbb{C}$. Now let them move around in trajectories

$$\beta(t) = (\beta_1(t), \ldots \beta_n(t)), \quad \beta_i(t) \in \mathbb{C}, \quad 0 \leq t \leq 1.$$

A *braid* is then such a time history with the proviso that the particles are noncolliding:

$$\beta_i(t) \neq \beta_j(t) \quad \text{if} \quad i \neq j$$

and end at the spots they began, but possibly permuted:

$$\beta_i(0) = i, \quad \beta_i(1) \in \{1, \ldots, n\}, \quad i = 1, \ldots, n.$$

If one braid can be deformed continuously into another (through the class of braids), the two are considered equivalent – we will say equal.

Braids α and β can be multiplied: one dance following the other, each at double speed. The product is associative but not in general commutative. The identity dance is to stand still, and each dance has an inverse; doing the dance in reverse time. These (deformation classes of) dances form the group B_n.

A braid β defines a permutation $i \to \beta_i(1)$ which is a well-defined element of the permutation group Σ_n. This is a homomorphism with kernel, by definition, the subgroup P_n of *pure* braids. P_n is sometimes called the *colored* braid group, as the particles can be regarded as having identities, or colors. P_n is of course normal in B_n, of index $n!$, and there is an exact sequence

$$1 \to P_n \to B_n \to \Sigma_n \to 1.$$

Definition 2: Braids as strings in 3-D. This is the usual and visually appealing picture. A braid can be viewed as the graph, or timeline, of a braid as in the first definition, drawn in real x, y, t-space, monotone in the t direction. The complex part is described as usual by $x + y\sqrt{-1}$. The product is then a concatenation of braided strings.

This viewpoint provides the connection with knots. A braid β defines a knot or link $\hat{\beta}$, its closure, by connecting the endpoints in a standard way so that no new crossings are introduced. J. W. Alexander showed that all knots arise as the closure of some braid and by a theorem of Markov (see [4] for a discussion and proof) two braids close to equivalent knots if and only if they are related by a finite sequence of moves and their inverses: conjugation in the braid group and a stabilization, which increases the number of strings.

Definition 3: B_n as a fundamental group. In complex n-space $\mathbb{C}^n$ consider the big diagonal

$$\Delta = \{(z_1, \ldots, z_n); \quad z_i = z_j, \quad \text{some} \quad i < j\} \subset \mathbb{C}^n.$$

Using the basepoint $(1, 2, \ldots, n)$, we see that

$$P_n = \pi_1(\mathbb{C}^n \setminus \Delta).$$

In other words, pure braid groups are fundamental groups of complements of a special sort of complex *hyperplane arrangement*, itself a deep and complicated subject.

To get the full braid group we need to take the fundamental group of the *configuration space*, of orbits of the obvious action of Σ_n upon $\mathbb{C}^n \setminus \Delta$. Thus

$$B_n = \pi_1((\mathbb{C}^n \setminus \Delta)/\Sigma_n).$$

Notice that since the singularities have been removed, the projection

$$\mathbb{C}^n \setminus \Delta \longrightarrow (\mathbb{C}^n \setminus \Delta)/\Sigma_n$$

is actually a covering map. As is well-known, covering maps induce injective homomorphisms at the π_1 level, so this is another way to think of the inclusion $P_n \subset B_n$.

Finally, we note that the space $(\mathbb{C}^n \setminus \Delta)/\Sigma_n$ can be identified with the space of all complex polynomials of degree n which are monic and have n distinct roots

$$p(z) = (z - r_1) \cdots (z - r_n).$$

This is one way in which the braid groups play a role in classical algebraic geometry, as fundamental group of the space of such polynomials.

Definition 4: The algebraic braid group. B_n can be regarded algebraically as the group presented with generators $\sigma_1, \dots, \sigma_{n-1}$, where σ_i is the braid with one crossing, with the string at level i crossing over the one at level $i+1$ and the other strings going straight across.

These generators are subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1,$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad |i - j| = 1.$$

We can take a whole countable set of generators $\sigma_1, \sigma_2, \dots$ subject to the above relations, to define the infinite braid group B_∞. If we consider the (non-normal) subgroup generated by $\sigma_1, \dots, \sigma_{n-1}$, these algebraically define B_n. Notice that this convention gives "natural" inclusions $B_n \subset B_{n+1}$ and $P_n \subset P_{n+1}$.

Definition 5: B_n as a mapping class group. Going back to the first definition, imagine the particles are in a sort of planar jello and pull their surroundings with them as they dance about. Topologically speaking, the motion of the particles extends to a continuous family of homeomorphisms of the plane (or of a disk, fixed on the boundary). This describes an equivalence between B_n and the mapping class of D_n, the disk D with n punctures (marked points). That is, B_n can be considered as the group of homeomorphisms of D_n fixing ∂D and permuting the punctures, modulo isotopy fixing $\partial D \cup \{1, \dots, n\}$.

Definition 6: B_n as a group of automorphisms. A mapping class $[h]$, where $h : D_n \to D_n$, gives rise to an automorphism $h_* : F_n \to F_n$ of free groups, because F_n is the fundamental group of the punctured disk. Using the interpretation of braids as mapping classes, this defines a homomorphism

$$B_n \to Aut(F_n),$$

which Artin showed to be faithful, i. e. injective.

The generator σ_i acts as

$$x_i \to x_i x_{i+1} x_i^{-1}; \quad x_{i+1} \to x_i; \quad x_j \to x_j, \quad j \neq i, i+1.$$

Thus B_n may be considered a group of automorphisms of $Aut(F_n)$ satisfying a condition made precise by Artin.

2 Representations of braid groups

One of the most active aspects of braid theory is the study of linear representations. A major breakthrough has been the proof in 2000 by S. Bigelow [3] and D. Krammer [12] of the long-standing conjecture that Artin's braid groups B_n are linear groups. That is, there exists a faithful representation of B_n in a finite-dimensional linear group. The Lawrence–Krammer representation that provides a linear representation of B_n has dimension $n(n-1)/2$. After the result was established, considerable efforts have been made to better understand the algebraic underlying socle on which the representations arise. The general question is to identify the non-trivial finite-dimensional quotients of the group algebra $\mathbb{C}B_n$, on the shape of the Iwahori–Hecke algebra investigated in the past decades. The general philosophy is: the bigger the quotient algebra, the better the results. Until recently, the biggest known algebra was the Birman–Murakami–Wenzl algebra [6].

An exciting development presented during the workshop is the description by Stephen Bigelow of a new family of finite-dimensional quotients of the algebra $\mathbb{C}B_n$ that naturally extends the Iwahori–Hecke and the Birman–Murakami–Wenzl algebras. The latter are just the first two steps in the new family. The new algebras, called "Zipper algebras" and denoted $Z_n(q,r)$, depend on two nonzero complex parameters, and they are defined using a diagrammatic approach. The principle is to introduce an additional generator Box_k visualized by a box with $k+1$ input and $k+1$ output strands, and to extend the usual skein relation declaring that a q-twisted combination of opposite crossings is 0 (case of Hecke algebra), or is the 2-2-tangle (case of BMW) into the relation declaring the q-twisted combination of opposite crossings is the new free generator Box_2. Then, inductively, one adds a similar skein relation relating the diagrams with a k-box and the two possible positions of an additional strand with a $k+1$-box. What Bigelow proves so far is that the algebras $Z_n(q,r)$ have finite dimension, and make a proper extension of the BMW algebra. What remains open is the exact dimension of the Zipper algebra, as well as the degeneracy at $q=1$.

A graduate student from China, Hao Zheng, also discussed a topological approach to representations of B_n, much in the spirit of Bigelow's earlier work.

One of the best-known (but unfaithful) representations of braid groups is the Burau representation. Jones noted that this representation can be interpreted in terms of probabilities related to a particle jumping from one string to another at a crossing in a braid picture. X. S. Lin expanded on this basic idea to produce a new representation of B_n, using ideas related to probability, which is closely related to the colored HOMFLY polynomial.

The student Holly Hauschild presented a concrete tangle-theoretic approach to the Birman–Murakami–Wenzl algebra. Morton and Wasserman had shown that the BMW algebra is isomorphic to a Kauffman tangle algebra. Haushchild described an extension of this isomorphism to give a similar correspondence between the affine BMW algebra and a corresponding algebra of tangles in the solid torus.

Taking a more abstract approach, Hans Wenzl described how representations of braid groups can be used to construct and classify certain braided tensor categories which are useful in low dimensional topology, physics and operator theory. In particular, he has classified all representations of B_3 up to dimension five.

The talk of Ivan Marin also considered representation theory of braid groups and their generalizations. He discussed representations obtained in a systematic way from the representations of "infinitesimal braids." This approach sheds new light on the decompositions of tensor products and the unitarisability properties of braid representations, as well as the actions of the universal Galois group involved in this setting.

Thus it is fair to say that great strides were made in the BIRS workshop toward the understanding of the representation theory of braid groups and the many applications of these ideas. Of course, much remains to be understood in this important subject.

3 Applications to knot theory and topology

3.1 The Jones polynomial

The most obvious applications of braid theory are to the study of knots. About two decades ago, work of V. Jones [11] established a new powerful knot invariant via representations of B_n. This work led to exciting and unsuspected connections with operator theory, statistical mechanics and other aspects of mathematical physics. It was also generalized to the so-called HOMFLY polynomial, the Kauffman polynomial and a plethora of other knot invariants.

An outstanding open question is whether the Jones polynomial detects the unknot. In other words, if the Jones polynomial $V_K(t)$ of a knot K is trivial, does it imply that K is unknotted? The corresponding question for links of two or more components was settled very recently by Eliahou, Kauffman and Thistlethwaite [9], who displayed infinite families of links with the same Jones polynomial as the unlink, but which are nontrivially linked.

It is also well-known that there are many examples of distinct knots with the same Jones (and HOMFLY) polynomial, using various techniques: Conway mutation, a construction of Kanenobu (producing an infinite family with common Jones polynomial), etc. A new technique was discussed at the workshop by the student Liam Watson, which employs the idea of a braid group action on Conway tangles in a knot diagram to produce distinct knots with the same Jones polynomial, which nevertheless are not Conway mutants. A consequence of his work is that, given any Conway tangle, there exist distinct knots containing that tangle as part of their diagrams, and having the same Jones polynomial. Watson's techniques (unlike Conway mutation) have the possibility of settling the question of whether the Jones polynomial detects the unknot.

Hitoshi Murakami gave a fascinating lecture on the current state of the art of the so-called volume conjecture, which relates the volume of the complement of a hyperbolic knot K with limits of values of the colored Jones polynomial. Originally posed by Kashaev, and following work of J. Murakami and H. Murakami, this conjecture can be made precise:

$$Vol(S^3 \setminus K) = 2\pi \lim_{N\to\infty} \log |J_N(K; \exp(\frac{2\pi\sqrt{-1}}{N})|,$$

where $J_N(K;t)$ denotes the N^{th} colored Jones polynomial of the knot or link K.

This conjecture has been verified for various special cases – the knots 4_1 and 5_2, the Whitehead link and the Borromean rings – by various authors, but remains open in general and is the focus of considerable attention by topologists. Murakami also discussed a complexified version of the formula, in which the absolute value signs in the above equation are removed, and one has the imaginary part of the left-hand side expressed as the Chern-Simons invariant of the complement.

3.2 Three-dimensional manifolds and TQFT's

One of the most important new tools in the study of 3-manifolds is the Casson invariant $\lambda(M)$, defined by A. Casson for any integral homology 3-sphere M. The original definition by Casson in 1984 involved counting $SU(2)$ representations of the fundamental group of M. Greg Kuperberg and Dylan Thurston showed, in 1999, how to express $\lambda(M)$ as a configuration space integral. A very interesting new approach was explained in the BIRS workshop by Christine Lescop. She showed that $6\lambda(M)$ is the algebraic intersection of three codimension 2 manifolds in the 6-dimensional space of two-point configurations of M, for any integral homology sphere M. Lescop went on to show it extends to the Walker generalisation of the Casson invariant to rational homology spheres, giving a topological characterisation of the Walker invariant.

Partly inspired Jones' lead in connecting braid theory with mathematical physics, and subsequent work by Atiyah, Witten and many others, topological quantum field theories have become an important new field of study. Several of the lectures in the BIRS workshop concentrated on aspects of TQFT's and their application to 3-manifolds.

Gregor Masbaum discussed joint work with P. Gilmer regarding naturally defined lattices in the vector spaces associated to surfaces, by the SO(3) TQFT at an odd prime. These lattices, whose existence comes from the fact that the associated quantum invariants of 3-manifolds are algebraic integers, form an "Integral TQFT" in an appropriate sense. Masbaum defined an explicit basis for this lattice.

In a talk entitled "Braids and hypergeometric integrals," Toshitake Kohno discussed two approaches to braid group representations: the homological approach of Laqwrence, Krammer, Bigelow, et. al. and a more physically motivated approach involving monodromy of flat connections. The latter involves, in particular, solutions to the Khizhnik-Zamolodchikov equations and conformal blocks, as well as the Drinfel'd approach using quantum groups. In his lecture, Kohno related the theory of conformal blocks to certain hypergeometric integrals. This deep subject promises to enrich both the theory of braid representations, as well as questions of interest to mathematical physicists.

Related to the above, the postdoctoral fellow Alissa Crans discussed new methods for finding solutions to the Zamolodchikov tetrahedral equations. In particular, after a discussion of 2-categories, she showed that, just as any Lie algebra gives a solution of the Yang-Baxter equation, any Lie 2-algebra gives a solution of the Zamolodchikov tetrahedron equation.

3.3 Braids and homotopy theory

Fred Cohen spoke on some striking connections between braid theory and deep questions of homotopy theory. He related questions some elementary constructions in the pure braid groups, such as string doubling and forgetting strands, with open questions in homotopy theory. As an example, the homotopy groups $\pi_N(S^2)$ have not been calculated for high values of N, and settling questions regarding the constructions on pure braids would determine those groups.

4 Braids, combinatorics and algorithms

A very active area which was well-represented at the conference concerns ideas surrounding Garside's 1969 solution to the word and conjugacy problems in the braid groups [10]. Three talks (by Gonzalez-Meneses, by Gebhardt and by Krammer) related directly to this circle of ideas, with Gonzalez-Meneses and Gebhardt focussing on ways to understand and simplify the combinatorics, while Krammer's efforts were directed toward extending it to surface mapping class groups. The discussions that followed these talks were broadly based, because at least some number of the other participants (e.g. Dehornoy, Paris, Michel, Birman and Brendle) had themselves made important contributions to what have become known as "Garside structures", so that the workshop was a major event for workers in the area.

Another very exciting development was presented by Daan Krammer. Building on the seminal work by Garside, many authors have developed a general theory of Garside groups, which are groups of fractions of monoids in which divisibility has a lattice structure. The braid groups have several Garside structures, namely (at least) the one originally defined by Garside, and the one associated with the recent Birman–Ko–Lee monoid. Krammer proposes new developments that seem to go far beyond the previous attempts. The point is to weaken the condition that the group is the group of fractions of a lattice into the weaker one that the group acts on a lattice, by an action that need not be transitive.

An equivalent way of describing the framework is to introduce the notion of a Garside groupoid (small catgeory where all arrows are invertible). Technically, an extended Garside structure is specified by axiomatizing the intervals $[a, a\Delta]$ of a Garside monoid, where Δ is a Garside element. The main interest of this extended framework is to make it possible to define completely new Garside structures on braid groups — and, possibly, on more general mapping class groups, but this remains a conjecture. The construction starts with considering the braid group B_n as acting on a disk with n punctures, as in Definition 5 above.

Now, the new ingredient is to add q marked points on the boundary circle. By considering certain cell decompositions of such "bi-punctured" disks (punctures in the interior and on the boundary) up to isotopy, one obtains a lattice and, under a convenient version of Dehn's half-twist in which the boundary punctures are shifted, one obtains an action of the braid group B_n on that lattice. In the case $q = 2$ (only the North and the South poles of the disk are marked), the action is simply transitive, and one obtains the standard Garside structure of B_n. For $q \geq 3$, the action is not transitive, and one obtains a completely new structure. In particular, for $q = 3$ (3 punctures on the boundary disk), the lattice can be described explicitly, and, surprisingly enough, the famous MacLane pentagon shows up, and, more generally, the intervals $[a, a\Delta]$ are closely related with the Stasheff associahedra. This opens a new, fascinating connection between Artin's braid group and Richard Thompson's groups, and certainly much more is still to come.

The word and conjugacy problems in the braid groups have importance for their role in public key cryptography. It is well known that the complexity of the word problem in the braid group B_n is $(|W|^2 n)$, where $|W|$ is word length and n is braid index, whereas all solutions to the conjugacy problem known at this time are exponential. Codes have been designed which are based on the assumption that the conjugacy problem is fundamentally exponential, so a polynomial solution to the conjugacy problem would be of major importance.

A new idea was to apply the partial solutions to the same problems by Thurston, by treating braids which are"reducible, finite order and pseudo-Anosov" separately. This proved to be very fruitful as regards the combinatorics of Garside's work in the braid groups. Since Thurston's ideas apply to all mapping class groups, not just to the braid groups, it was then very interesting when Daan Krammer presented his fascinating talk, which aimed to go the other way and introduce Garside-like combinatorics into the study of surface mapping class groups.

It can be mentioned that a different connection between Artin's braid group and Richard Thompson's groups was discussed in Dehornoy's talk in the workshop, devoted to Bar Natan's parenthesized braids. The latter can be made into a group which contains both the braid groups and the Thompson groups, and some new results about self-distributive operations on that new group are quite intriguing. This group also enjoys a left-invariant ordering extending the well-known ordering of B_n.

Another connection with combinatorics was given by Christian Kassel. In joint work with Christophe Reutenauer, they considered the classical idea of Sturmian sequences of two symbols, which occur in fields such as number theory, ergodic theory, dynamical systems, computer science and crystallography. They show that the class of special Sturmian sequences (a submonoid of $Aut(F_2)$) can be realized naturally as a submonoid of the four strand braid group B_4. As an application, this leads to a new criterion for determining when two words form a basis for the free group F_2.

The Markov theorem without stabilization (MTWS) of J. Birman and W. Menasco established a calculus of braid isotopies that can be used to move between closed braid representatives of a given oriented link type without having to increase the braid index by stabilization. Although the calculus is extensive there are three key isotopies that were identified and analyzed—destabilization, exchange moves and braid preserving flypes. One of the critical open problems left in the wake of the MTWS is the *recognition problem*— determining when a given closed n-braid admits a specified move of the calculus. Bill Menasco described an algorithmic solution to the recognition problem for three isotopies of the MTWS calculus—destabilization, exchange moves and braid preserving flypes. The algorithm is "directed" by a complexity measure that can be *monotonic simplified* by the application of *elementary moves* on a modified braid presentation.

5 Generalizations of the braid groups

Because of the many definitions of the braid groups, there are various natural ways to generalize them, some of which have far-reaching applications. Several such generalizations were considered in the BIRS workshop, namely Artin groups (an algebraic generalization), mapping class groups (also known as modular groups), configuration spaces and their algebraic properties

5.1 Artin groups

Deligne [7] and Brieskorn-Saito [5], introduced a family now referred to as Artin groups, which generalizes the braid groups and is also closely related to the so-called Coxeter groups which arise in the study of Lie groups and symmetries of Euclidean space. For a fixed positive integer n, consider an n by n matrix $M = \{m_{ij}\}$, where m_{ij} is a positive integer or ∞, with the assumption that $m_{ij} = m_{ji} \geq 2$ and $m_{ii} = 1$. The corresponding Artin group has a presentation with generators $x_1, \ldots, x_n$ and, for each pair i, j

there is a relation:

$$x_i x_j x_i \cdots = x_j x_i x_j \cdots$$

where the product on each side has length m_{ij} ($m_{ij} = \infty$ indicates no relation is present). If one adjoins relations $x_i^2 = 1$, the result is the so-called Coxeter group corresponding to the given matrix.

In this context, the $n+1$ by $n+1$ matrix with entries equal to 3 just above and below the diagonal, and 2 in entries farther from the diagonal, corresponds exactly to the braid group B_n; in this case the Coxeter group is the symmetric group Σ_n. The Artin groups for which the corresponding Coxeter group is finite are an important subclass, referred to as "spherical." As with the braid groups, Artin groups of spherical type correspond to fundamental groups configuration spaces associated to hyperplane arrangements.

Fundamental to the understanding of semisimple Lie groups is the well-known classification of finite Coxeter groups into several infinite families and certain sporadic types E_6, E_7, E_8, F_4, etc. These Coxeter groups are well known to be distinct, but the corresponding question for the associated Artin groups had been open until now. This question was finally settled by L. Paris, as announced in the BIRS workshop. He used various group-theoretic invariants to establish that the spherical Artin groups of (apparently) different type really are non-isomorphic.

In a different approach to the subject, Dan Margalit discussed embeddings of three infinite families of Artin groups (modulo their centers) as finite index subgroups of the mapping class group of a punctured sphere. As a corollary Margalit, in joint work with Bob Bell, was able to classify all injections of these Artin groups into each other.

5.2 Reflection groups

The finite Coxeter groups can be considered as groups of reflections of $\mathbb{R}^n$, acting on configuration spaces, as described in Definition 3 for the case of the braid groups. Several talks focussed on this aspect, as well as natural generalizations to complex reflection groups

The lecture of Gus Lehrer dealt with the cohomology of these configuration spaces, with local coefficients. For the case of the braid groups, this calculation was accomplished by Arnol'd in 1969, with further progress made by Brieskorn, F. Cohen, Orlik-Solomon and others. In particular, the rank of the cohomology in various dimensions is encoded in a Poincaré polynomial, a sort of generating function. Lehrer's lecture gave a method of calculating these polynomials using Z-functions, defined using centralizers, and related this work to varieties defined over number fields.

In the lecture "Hurwicz action on euclidean reflections," Jean Michel discussed a theorem of Dibrovin and Mazocco that, if the Hurwicz action of the braid group on a triple of Euclidean reflections in R^3 has a finite orbit, then the group generated by these reflections is finite. Michel extended this result to the case of R^n, correcting an erroneous proof which had appeared recently in the literature and simplifying the Dibrovin and Mazocco proof as well.

5.3 Mapping class groups

The mapping class group $Mod(S)$ of an orientable surface S is well-known to be generated by Dehn twists about simple closed curves in S. An important subgroup of this is the Torelli subgroup, consisting of (classes of) homeomorphisms which induce the identity on the homology of S. In particular, the subgroup K of $Mod(S)$ generated by twists along separating curves of S, called the Johnson kernel, lies in the Torelli subgroup. Tara Brendle, in joint work with Dan Margalit, outlined a proof that the abstract commensurator of K satisfies $Comm(K) = Aut(K) = Mod(S)$, thus verifying a conjecture of Benson Farb.

In the interpretation of braid groups as mapping class groups of a punctured disk, one notes that the homeomorphisms involved may be taken to be smooth and area-preserving. Thus B_n is related to the group G of area-preserving diffeomorphisms of the disk. The study of G is also important in understanding flows related to magnetic fields in the solid torus. This was the subject of a fascinating talk by Elena Kudryavtseva, which concentrated on the so-called Calabi invariant, the averaged linking number for pairs of orbits of the magnetic flow in the solid torus. Her main result is that any C^1-smooth function on G is, in fact, a function of the Calabi invariant. This has the consequence that higher-order knot and braid invariants cannot be generalized to invariants of magnetic fields in the solid torus.

Thus we have three classes of groups: the braid groups, mapping class groups of more general surfaces and Artin groups, and while it has been known since the early 1970's that they are interrelated, the full richness of the interrelationship is just now beginning to be made clear. The last word on this fascinating subject does not appear to have been said.

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