CELESTIAL MECHANICS

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There were 22 participants to the First BIRS Celestial Mechanics Workshop, organized by Florin Diacu (Canada) and Donald Saari (USA). In alphabetical order, they were:

Albouy, Alain (Paris, France)
Belbruno, Ed (Princeton, USA)
Buck, Gregory (Saint Anselm College, USA)
Chenciner, Alain (Paris, France)
Corbera, Montserrat (Universitat de Vic, Spain)
Cushman, Richard (Utrecht, Holland and Calgary, Canada)
Diacu, Florin (Victoria, Canada)
Gerver, Joseph (Rutgers, USA)
Hampton, Marshall (Minneapolis, USA)
Kotsireas, Ilias (Wilfried Laurier, Waterloo, Canada)
Lacomba, Ernesto (UAM-I, Mexico)
Llibre, Jaume (Barcelona, Spain)
McCord, Chris (Cincinatti, USA)
Meyer, Kenneth (Cincinatti, USA)
Montgomery, Richard (Santa Cruz, USA)
Offin, Dan (Queens, Canada)
Palacian, Jesus (Universidad Publica de Navarra, Pamplona, Spain)
Perez, Ernesto (UAM-I, Mexico)
Roberts, Gareth (College of the Holy Cross, USA)
Saari, Donald (Irvine, USA)
Santoprete, Manuele (Irvine, USA)
Xia, Zhihong (Jeff) (Northwestern, Evanston, USA)
Yanguas, Patricia (Universidad Publica de Navarra, Pamplona, Spain)

The programme of the workshop consisted of 45-minute talks followed by 15 minutes of discussions. Often those discussions have been continued in the evening in smaller groups. The schedule was planned by the two organizers as follows:

Sunday, April 18

8:30-9:15, Alain Chenciner: Rotating eights
9:30-10:15, Ernesto A. Lacomba: Symbolic dynamics in the rectilinear 3-body problem
11:00-11:45, Montserrat Corbera: Global existence of subharmonic orbits in the Sitnikov Problem
14:30-15:15, Richard Montgomery: Fitting hyperbolic pants to a 3-body problem
16:00-16:45, Dan Offin: Variational-stability method for some n-body problems

Monday, April 19

8:30-9:15, Ernesto Perez-Chavela: Symmetrical central configurations in the 4-body problem
9:30-10:15, Ken Meyer: Elliptic central configuration solutions of the n-body problem
11:00-11:45, Patricia Yanguas: Dynamics of charged particles in planetary magnetospheres
14:30-15:15, Alain Albouy: New hidden symmetries in the Kepler problem and Lambert's theorem
16:00-16:45, Gareth Roberts: Linear stability analysis of the figure-eight orbit

Tuesday, April 20

8:30-9:15, Jaume Llibre: On the families of periodic orbits in the Sitnikov Problem
9:30-10:15, Chris McCord: Collinear blow-ups and integral manifolds of the spatial n-body problem
11:00-11:45, Joe Gerver: Infinite spin and non-collision singularities
14:30-15:15, Manuele Santoprete: Qualitative properties of 2-body problems with anisotropic potentials
16:00-16:45, Ilias Kotsireas: Symmetries of polynomial and differential equations

Wednesday, April 21

8:30-9:15, Jeff Xia: Action-minimizing periodic and quasi-periodic solutions of the n-body problem
9:30-10:15, Richard Cushman: Monodromy in the swing spring
11:00-11:45, Marshall Hampton: New central configurations in the 5-body problem
The afternoon was reserved for trips

Thursday, April 22

8:30-9:15, Don Saari: Analysing central configurations
9:30-10:15, Ed Belbruno: Chaos with weak ballistic capture, and low energy lunar transfer

The workshop was opened on Sunday by Alain Chenciner of Paris. During the past few years, Alain has focused on studying the existence of choreographic solutions using variational methods. His proof of the existence of the Figure Eight solution together with Richard Montgomery [1] has revolutionized the field. Many researchers have adopted their methodology, seeking new periodic solutions in the n-body problem and for related systems of differential equations.

In his talk, Alain presented some recent work written with Jacques Féjoz and Richard Montgomery. He showed the existence of three families of relative periodic solutions which bifurcate out of the Figure Eight solution of the equal-mass three-body problem : the planar Hénon family, the spatial Marchal P_{12} family and a new spatial family. Each family corresponds to a different breaking of the $D_6 \times Z_2$ symmetry of the Eight solution in 3-space. Alain described this result as well as some of its developments.

The end-of-the-talk discussions revolved around questions related to the nature of the symmetries. For example, Jeff Xia pointed out that he had already obtained some more general results relative to one type of symmetry, but admitted that they did not contain the other rotation types. Alain's talk was very well received and imposed a very high standard for the entire workshop. A preliminary version of the paper, [2], is available at:

http://www.imcce.fr/Equipes/ASD/person/chenciner/chen_preprint.html

The second talk of the morning was given by Ernesto Lacomba. He talked about symbolic dynamics in the rectilinear restricted 3-body problem. This joint work with Sam Kaplan extended some ideas Sam had developed in his doctoral thesis in connection with a 2-body problem with a bumper.

They generalized these results to a symmetric rectilinear restricted 3-body problem for which the equal mass primaries perform elliptic collisions, while the infinitesimal body moves in the line between the primaries. Symbolic dynamics could be applied to mark the time between two consecutive elliptic collisions. Since any nonhomothetic solution performs binary collisions, the basic behavior of the solution can be studied through its successive intersections with 2 two-dimensional strips, corresponding to regularized binary collisions. Thus Ernesto and Sam obtained a singular global Poincar section. In this way, they were able to describe all possible itineraries an orbit may have.

The last talk of the morning was that of Montserrat (Montse) Corbera of the Vic University near Barcelona. She presented several new results on the global existence of subharmonic orbits in the Sitnikov Problem. This consists of the motion of 3 bodies, two of them of equal mass, moving in a plane on circular or elliptic orbits of eccentricity e, and the third of infinitesimal mass, moving on the z axis perpendicular to the plane of the other two that passes through their centre of mass. A solution is said to be an (m, n)-orbit if it is $2m\pi$-periodic and there are exactly $2n$ zeroes of z in the time interval $[0, 2m\pi)$.

Montse wrote this paper in collaboration with Jaume Llibre (present at the meeting) and Pedro Torres. The main result of the talk was that for all m natural numbers, there exist at least two $(m, 1)$-orbits for any value of the eccentricity $0 < e < 1$. Moreover, for all m, n natural numbers, there is a positive number $e_{m,n}$ such that the problem has at least an (m, n)-orbit for any value of the eccentricity $e < e_{m,n}$. The proof used a version of the Poincaré-Birkhoff theorem proved in [3]. The discussions focused on the central theorem and on possible generalizations.

The first talk of the afternoon session was that of Richard Montgomery of the University of California in Santa Cruz. The title of his talk was "Fitting hyperbolic pants to a three-body problem," and his results were inspired by the the Figure Eight solution he and Chenciner had discovered a few years earlier [1]. He considered bounded zero-angular momentum solutions to the $1/r^2$ potential (not Newton's $1/r$) three-body problem. He showed that upon modding out by the symmetries of scaling, translation and rotation, this problem is equivalent to the geodesic flow for a certain metric on the pair of pants, namely the thrice-punctured two-sphere. The sphere is the shape sphere. The punctures are collisions. The metric is the Jacobi-Maupertuis metric at zero energy. It is complete and noncompact. His main result was that if all masses are equal, then the Gaussian curvature of the metric is everywhere negative, except at two points, the Lagrange points. A number of dynamical consequences directly follow, such as the uniqueness of the $1/r^2$ figure-eight solution, and the existence of a complete symbolic dynamics description (symbols are syzygies [4]) for the non-collision bounded solutions. Other papers relevant to his talk are [5], [6], [7].

It is interesting to note that the excellent internet connection in Room 159 at BIRS was of great help during Richard's talk. He pointed at his website (which anyone with a laptop could access) and at several papers, including the preprint of the present talk. These were great additions to the talk and helped deepening the understanding of his results. The discussions that followed showed the clear necessity of an ad-hoc session on the Figure Eight solution. This took place with 6 participants on Tuesday night, after dinner.

The last talk of the day was that of Dan Offin of Queens University. He talked about the variational-stability method for some N-body problems. He showed showed how the variational method can be extended to the variational-stability method for existence and stability type of periodic solutions in certain subsystems of the N-body problem. These include, the isosceles 3-body problem, and the equal mass symmetric 4-body problem. Then he showed that the instability of absolutely minimizing periodic orbits in these systems has implications for the existence of mountain pass critical orbits and orbits homoclinic to minimizing-type orbits.

The variational method has long traditions in celestial mechanics since Poincaré introduced it in 1896 to obtain periodic orbits in what we call today Manev-type potentials $(1/r + 1/r^2)$. The proof of the existence of the Figure Eight solution, of a few more choreographic solutions as well as the numerical discovery of hundreds of periodic orbits in the last few years, have led to some intense research, and several of the researchers present at this meeting work in this direction. Therefore the discussions that followed after Dan's talk focused on technical aspects related to the variational method.

It was a fortunate decision to have Dan talk on Sunday since the same night his wife gave birth to healthy son in Kingston, Ontario, and Dan had to leave on the first flight he could book in Calgary. We missed him, but such circumstances need no further comment.

Monday, the second day of the meeting, had several talks on central configurations. The subject is extremely important in celestial mechanics, it showed up in almost every talk, and two more presentations in different days were dedicated to it.

The first talk of the day was that of Ernesto Pérez-Chavela of Departamento de Matemáticas UAM-Iztapalapa, Mexico City, now on sabbatical leave at the University of Victoria. He talked about symmetrical central configurations in 4-body problems. More precisely, he studied planar central configurations with an axis of symmetry containing two of the particles. The central configurations can be concave or convex, depending if one mass is in the interior of the convex hull of the other three or not. If three of the masses are equal (of unit mass), the axis of symmetry contains the mass m, and we find the total number of central configurations. If two of the masses are equal, and not taking into account the permutations between the equal masses, then there is exactly one convex central configuration. Ernesto also proved the existence of several concave central configurations. The references relevant to his talk are [8], [9] and [10].

The discussions focused on the question of existence of infinitely many central configurations for N given masses. This is an open problem left about 60 years ago. Ernesto as well as Gareth Roberts have shown that for certain types of potentials and/or for negative masses, there exists a continuous set of solutions, but in the Newtonian case the problem is open for more than 4 masses. However, recently important progress was done in this direction, as will become clear from other talks.

The second talk of the day was that of Ken Meyer of the University of Cincinnati. He talked about elliptic central configuration solutions of the N-body problem. This was a paper written in collaboration with Dieter Schmidt and Klaatu. At the beginning of the talk, Ken presented a few scenes from a science-fiction movie: "The Day the Sun Stood Still," released in 1951. The movie is about an alien who comes to the Earth and tries to save the earthlings from a collision with a comet. One of the scenes shows the alien (as a middle-aged man) with a boy, knocking at the door of a famous professor. On the blackboard in the professor's study are written the equations of motion of the 3-body problem. It was amusing to listen to the dialog and see that it was not totally nonsensical relative to mathematics, as it usually is in such movies.

Then Ken got into the real mathematics and showed how a planar central configuration of the N-body problem gives rise to a solution where each particle moves on a specific Keplarian orbit while the totality of the particles move on a homothety motion. The totality of such solutions forms a 4-dimensional symplectic subspace. He gave a symplectic coordinate system which is adapted to this subspace and its symplectic complement. If the Keplerian orbit is elliptic, the solution of the N-body problem is called an elliptic central configuration solution. In his coordinate system, the linear variational equations of such a solution decouple into three subsystems. One subsystem simply gives the motion of the center of mass, another is Kepler's problem and the third determines the non-trivial characteristic multipliers. Using these coordinates, Ken studied the linear stability of for several cases when $N = 3, 4, 5$. The discussions focused on the stability question, which is fundamental in celestial mechanics. More about this when discussing Gareth Robert's talk.

The last talk of the morning session was given by Patricia Yanguas of Pamplona, Spain. She had obtained the results she presented together with her husband, Jésus Palacián (also present at the meeting) as well as with the colleagues M. Iñarrea, V. Lanchares, A. I. Pascual and J. P. Salas. The title of the talk was "Dynamics of Charged Particles in Planetary Magnetospheres: Periodic Orbits, Two-Dimensional Tori and Bifurcations," and it presented a study of the dynamics of a charged particle orbiting around a rotating magnetic planet. The system is modelled by the Hamiltonian of the two-body problem perturbed by an axially-symmetric function which goes to infinity as soon as the particle approaches the planet. The perturbation consists in a magnetic dipole field and a corotational electric field. When the perturbation is weak compared to the Keplerian part of the Hamiltonian, the authors averaged the system with respect to the mean anomaly up to first order in terms of a small parameter defined by the ratio between the magnetic and the Keplerian interactions. After truncating higher-order terms, they used invariant theory to reduce the averaged system by virtue of its continuous and discrete symmetries, determining also the successive reduced phase spaces. Once the original system is reduced, they studied the flow of the resulting system in the most reduced phase space describing all equilibria and their stability, as well as the different classes of bifurcations. Finally, they connected the analysis of the flow on these reduced phase spaces with the one corresponding to the original system. More details about this work can be found in [11]. Other relevant references are: [12], [13] and [14]. Since Richard Cushman had done some work in this direction, an interesting discussion about the main results took place at the end of the talk.

The first talk of the afternoon session was that of Alain Albouy, who presented his results about Alain Albouy some (possibly) new "hidden symmetries" in the Kepler problem and Lambert's theorem. The so-called SO(4) symmetry of the Kepler problem is usually associated to the Gyorgyi-Moser correspondence of this problem with the geodesic flow on the sphere. This correspondence is not time-preserving. Alain showed that there is another symmetry which does not change the time, and discussed the relation with the classical Lambert theorem. Alain pointed at the study [15], which inspired his research. Several questions occurred during the talk and the end-of-the-talk discussions tried to find answers to some of those questions.

The last talk of the day was that of Gareth Roberts of the College of the Holy Cross, near Boston. He talked about some work in progress about the linear stability of the Figure-Eight orbit [1]. This is an interesting topic, which has preoccupied him since Carles Simó came up with numerical evidence that the Figure Eight solution has a very small zone of stability [17]. Gareth had done previous work on the linear stability of the elliptic Lagrangean triangle solutions of the 3-body problem, so he wanted to use this experience in this new case [16]. At the time of his presentation he had no definite results, but he was able to point out the directions and the plan of his research as well as directions he had tried and which seemed to lead nowhere. The discussions focused on the evaluation of the chances of the possible directions of attack.

The first talk of Tuesday morning was that of Jaume Llibre of Barcelona, Spain. He presented some new results he had obtained on certain families of periodic orbits of the Sitnikov problem. This was a continuation of previous work he had done with Montse Corbera and P.J. Torres, [19], [18]. The main goal of this talk was to present a study of the families of symmetric periodic orbits of the elliptic Sitnikov problem for all values of the eccentricity in the interval [0,1). The basic tool for proving our results was the global continuation method of the zeros of a function depending on one-parameter provided by Leray and Schauder and based in the Brouwer degree.

Since the 1960s, when Sitnikov came up with his problem in order to prove the existence of oscillatory solutions in the 3-body problem, the equations of motions have been studied intensively. Jaume's work presented the latest in this direction. The end-of-the-talk discussions were related to technical details in proving the main result.

The second talk of the day was that of Chris McCord of Cincinnati, who presented his latest results on collinear blow-ups and the integral manifolds of the spatial n-body problem. For the past decade, Chris and Ken Meyer had been exploring the dependence of the topology of the integral manifolds (as measured by their homology groups) on the energy and angular momentum. They had analyzed various special cases: spatial 3-body for all energies; planar n-body for all energies; spatial n-body for positive energy. In addition to whatever intrinsic interest these studies may have had, they had also served to isolate the obstacles to solving the general problem: the spatial n-body problem for all energies. It has emerged that all of the obstacles center around the collinear configurations. By introducing a blow-up of the configuration space at the collinear configurations, Chris was able to understand how the discontinuities at the collinear configurations change the behavior of the integral manifolds. This in turn allowed him to develop Morse-theoretic formulae for the homology of the spatial integral manifolds. The discussions focused on the perspectives these results open to the understanding of the global dynamics of the n-body problem.

The last talk of the morning was that of Joe Gerver from Rutgers University. He talked about infinite spin and noncollision singularities. It is well known that as n bodies approach a collision in a system with Newtonian potential, they must approach the central configuration manifold. It is not known whether they must approach a single point on this manifold. In particular, a central configuration remains central if it is rotated, and it is an open question whether a set of bodies can revolve an infinite number of times as it approaches a collision in a Newtonian system. (This can of course occur with an inverse squared potential.) Joe presented a possible model for a Newtonian infinite spin collision in the case when the collision is not isolated; instead other bodies, which are involved in a noncollision singularity which occurs simultaneously with the collision, approach the colliding bodies arbitrarily closely, but keeping moving away again. Joe's results in this direction follow his previous work on noncollision singularities [20] and [21]. The discussions focused on other possibilities of achieving such a scenario.

The first talk of the afternoon was that of Manuele Santoprete of the University of California at Irvine, who presented his results as well as some he obtained with Florin Diacu and Ernesto Pérez-Chavela on the qualitative properties of the anisotropic Manev problem. Manuele had just received his Ph.D. degree at the University of Victoria under the supervision of Florin Diacu and the day before the meeting started he learned that he had been awarded the Governor General's Gold Medal at the University of Victoria, for the best dissertation presented in 2003. Anisotropic problems, describing the interaction of two bodies, started to arouse a good deal of interest in the 1970s, when Martin Gutzwiller proposed the Anisotropic Kepler Problem to study connections between classical and quantum mechanics. In recent years other anisotropic potentials have been introduced, as for example the anisotropic Manev problem and the Kepler problem with anisotropic perturbations. In this talk Manuele described some qualitative properties of the anisotropic Manev problem and of the Kepler problem with anisotropic perturbations. In particular he studied collisions, near collision solutions, and the mechanisms responsible for the appearance of chaos. His techniques are a nice combination of dynamical and variational techniques. Papers relevant to his talk are [22] and [23]. The discussions focused on the differences and similarities of the anisotropic and nonisotropic cases as well as on the unusual case of a disconnected infinity manifold

The last talk of the day was that of Ilias Kotsireas of the Wilfried Laurier University, who presented his results about symmetries of polynomial and differential equations. Many systems of polynomial and differential equations arising incelestial mechanics, exhibit various kinds of symmetries that can best be described group theoretically by finite and Lie group actions. Computational approaches to solving systems with symmetries are available for both the polynomial and the differential case. The eigenvalue method for solving polynomial systems is an ideal paradigm to study the effect of the symmetries on the complexity of the method. The eigenvalue method is simplified considerably in the presence of symmetries, in the sense that the sizes of the matrices involved are diminished considerably. Certain aspects of the interplay between methods for solving polynomial and differential systems can be exploited effectively via a well-known polynomial/differential morphism. Ilias's talk was a tour-de-force on how very complicated computations can be performed in celestial mechanics using a computer. The essential references to his talk are [24], [25] and [26].

On Wednesday the first talk was that of Jeff Xia of Northwestern University. He presented his latest results on action-minimizing periodic and quasi-periodic solutions of the n-body problem. These solutions extend the classic Euler and Moulton relative equilibria. This interesting new development can be found in detail in [27] and [28]. The discussions focused on the perspectives this research opens for further investigations.

The second talk of the day was that of Richard Cushman of Utrecht, Holland and University of Calgary. He presented his latest results on monodromy in the swing spring. Richard discussed the three degree of freedom classical mechanical system of an elastic pendulum which is tuned to be in 1:1:2 resonance. This explains the following motion: start the pendulum springing in the vertical direction. After a while it begins to swing in a plane and then returns to the springing motion. During successive cycles the swing planes make the same angle with the vertical direction. Richard's explanation used of the concept of monodromy for a Liouville integrable system. The references relevant to his talk are: [29], [30], [31], [32] and [32].

The last talk of the morning and of the day (since the afternoon was dedicated to trips and relaxation) was that of Marshall Hampton of the University of Minnesota at Minneapolis. He presented his latest results on new central configurations in the 5-body problem. Marshall showed the existence of a class of planar 5-body central configuration which contradict an assertion of W. L. Williams in his 1938 paper, "Permanent configurations in the problem of five bodies", in which he claims that there are no central configurations with positive masses and which have 2 masses in the interior of a triangle. His methodology combined ingenious geometrical, algebraic and analytical techniques, see [33], citeSma and [35]. The discussions focused on these results as well as on the recent proof of Rick Moeckel of Minneapolis on the finiteness of the central configurations in the 4-body problem with positive masses.

Thursday, the last day of the meeting, started with Don Saari's talk on analysing central configurations. Don, who is now at the University of California at Irvine, has written a few decades ago a famous paper on the role and properties of central configurations, and in his talk he used a geometric approach to see how central configurations are described. Many traditional results followed from his approach, and it appeared that several new results are forthcoming. The discussions focused on the potential of this new approach.

The meeting was closed by Ed Belbruno of Princeton, who showed how the theoretical results most of the members of this group have obtained can be used in space science. Ed's talk was about the existence of chaos associated with weak ballistic capture and about low energy lunar transfer. A theory to achieve low energy transfers using ballistic capture (with no fuel required), called weak stability boundary theory, was successfully used in 1991 to resurrect a Japanese lunar mission and successfully bring the spacecraft, Hiten, to the Moon using a new type of lunar transfer. This was one of the more spectacular applications of celestial mechanics, and although well known in the aerospace community, was not as known in the dynamical systems/celestial mechanics community, until much more recently. This is because the underlying mathematics of the dynamics of the capture and the transfer itself were not really understood and were understood more from a numerical point of view. A proof has recently been obtained which explains, in part, the capture process. This is accomplished by two ingredients: one is to estimate a special region near the secondary mass point (Moon) in the restricted three-body problem where "weak capture" occurs, and the other is to prove that there exists a hyperbolic invariant set within this region, instrumental in the capture process. This result solves a problem investigated by Alekseev in 1981 in his last published paper. Ed also mentioned a number of applications in astrodynamics and dynamical astronomy. The relevant references to his talk are: [36], [37] and [38].

Overall this was a highly stimulating meeting of which all participants benefitted greatly. Everybody has been impressed with the facilities at BIRS, the efficiency of the staff and with the way the meeting was run.

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