Geometry and Analysis on Cauchy Riemann Manifolds

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One of the famous open questions in several complex variables is the following:

Is every 5 dimensional strongly pseudoconvex CR manifold locally embeddable?

This is a remarkably subtle question relating geometry, complex analysis and partial differential equations. To shed some light on this question, we will briefly recall the definitions and known results.

Let M be a smooth oriented $2n+1$-manifold. An *almost CR structure* (for Cauchy-Riemann) on M is a complex subbundle $H_{(1,0)}M$ of the complexified tangent bundle such that $H_{(0,1)}M := \overline{H_{(1,0)}M}$ is everywhere transverse to $H_{(1,0)}M$. The almost CR structure is a *CR structure* if in addition $H_{(0,1)}M$ is integrable as a complex subbundle of the complexified tangent bundle $T_{\mathbb{C}}M$.

Let η be a real one-form annihiliating $H_{(1,0)}M$; it is determined up to multiplication by a nonvanishing function. We choose the sign of η such that the orientation determined by η and the natural orientation for $H_{(1,0)}M$ agrees with the orientation for M. We define the *Levi form* associated to η to be the Hermitian form on $H_{(1,0)}M$ determined by

$$\mathcal{L}(Z,\overline{W}) = -id\eta(Z,\overline{W})$$

for all $Z, W \in H_{(1,0)}M$. If the Levi form is positive definite, then the CR structure is said to be *strongly pseudoconvex*. This condition is independent of the choice of η (except for its sign).

Remark 1 *Let U be a smoothly bounded strongly pseudoconvex open set in a complex manifold X. Then the complex structure from X restricts to ∂U as a strongly pseudoconvex CR structure; the Levi form defined here is the same as the usual Levi form defined in complex analysis.*

A *CR function* on M is a function $f : M \to \mathbb{C}$ such that $\bar{Z}(f) = 0$ for every $\bar{Z} \in H_{(0,1)}M$. These are the *tangential Cauchy-Riemann equations.* We denote this $\bar{\partial}_b f = 0$.

One of the most basic questions in the study of CR manifolds is the following:

Question 2 *Given a strongly pseudoconvex CR manifold $(M, H_{(1,0)}M)$, is it embeddable? That is, does there exist a smooth embedding $X : M \hookrightarrow \mathbb{C}^N$ such the the components of X are CR functions: $\bar{\partial}_b X = 0$.*

Notice that this is strictly a question in PDE's – the solvability of a linear PDE.

We now recall some well known results concerning the embeddability of strongly pseudoconvex manifolds. The first is a result by Boutet de Monvel.

Theorem 3 (Boutet de Monvel [Bou]) *Let M be a compact strongly pseudoconvex CR manifold of dimension $(2n+1) \geq 5$. Then M is embeddable.*

The idea of the proof is as follows. Associated to $\bar{\partial}_b$ is a natural subelliptic Laplacian $\Box_b$. Using standard techniques, one solves the $\bar{\partial}_b$ equations with weights in order to specify the differentials at a point, and the value at two points.

The situation is entirely different in three dimensions.

Example 4 (Rossi [R]) *There is a real analytic deformation of the standard structure on S^3 which is nonembeddable.*

The basic idea is that the mapping $\Phi : (z,w) \mapsto (z^2 + \epsilon\bar{w}^2/r^4, zw - \epsilon\bar{z}\bar{w}/r^4, w^2 + \epsilon\bar{z}^2/r^4)$ maps $\mathbf{C}^2 \setminus \{0\} \to \mathbf{C}^3$ as a $2:1$-cover of the quadric $XZ - Y^2 = \epsilon$. The induced CR structure on S^3 is strongly pseudoconvex, but for $\epsilon \neq 0$ all CR functions descend to the quotient, hence do not separate points.

In fact, the space of CR structures on S^3 near the standard one is locally a Hilbert space, and the subspace of embeddable structures is a Hilbert subspace of infinite dimension and codimension.

It is natural to ask the same question for local embeddability.

First, it is worth pointing out that for the local question, the answer is always positive in the real analytic case. This follows easily from Cauchy-Kowalewski. This already provides an indication that the question in the smooth case is likely to be either easy or delicate; the latter turns out to be the case.

Here we have the following results.

Theorem 5 (Kuranishi [Kur]) *Let M be a strongly pseudoconvex manifold (not necessarily compact) of dimension $2n+1 \geq 9$. Then M is locally embeddable.*

This result was extended to cover the seven dimensional case by Akahori, and simplified by Webster.

Theorem 6 (Akahori [Aka], Webster [Web1] [Web2]) *Let M be a strongly pseudoconvex manifold (not necessarily compact) of dimension $(2n+1) \geq 7$. Then M is locally embeddable.*

However, local embeddability fails dramatically in the 3 dimensional case. (See, for example, the example by Nirenberg [Nir]).

This leaves the following famous open question.

Question 7 *Does local embeddability hold for 5 - dimensional strongly pseudoconvex CR manifolds?*

This question is remarkably delicate, and resolving it is not simply a matter of obtaining better estimates. Indeed, on the infinitesimal level, there is no obstruction. On the other hand, we have the following example.

Example 8 (Nagel-Rosay, [NR]) *On S^5, we consider the one-form*

$$\omega := (\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1) / (|z_1|^2 + |z_2|^2)^2 .$$

This is $\bar{\partial}$-closed (Bochner-Martinelli), but

$$\omega \wedge dz^1 \wedge dz^2$$

is twice the volume form on the 3 spheres $[z_3 = c] \cap S^5$. In particular, it is not in the range of $\bar{\partial}_b$. While it is not smooth when $|z_3| = 1$ (that is, $z_1 = z_2 = 0$), it can be approximated by smooth forms, giving "approximate cohomology"; in particular, it eliminates the possibility of a homotopy formula.

Webster also identifies the special difficulty which arises in dimension 5. One approach to solving the $\bar{\partial}_b$-equations is to use the integral kernels of Henkin. In this case, the $\bar{\partial}_b$ equation is always solvable if there exists a homotopy formula of the form $\alpha = \bar{\partial}_b P\alpha + Q\bar{\partial}_b\alpha$. However, Webster shows that in the 5-dimensional case, the local formula becomes $\alpha = \bar{\partial}_b P\alpha + Q\bar{\partial}_b\alpha + H(\alpha)$; the final term H represents possible obstructions to embeddability. The subtlety of the problem is indicated by the fact that at the infinitesimal level, the obstruction identified by H vanishes for integrable structures.

One of the new ingredients to shed light on the analysis of the situation is the geometry which arises from the compact three dimensional case. This situation can be understood geometrically. If the structure is embeddable, then it is embeddable as the boundary of a convex set in $\mathbf{C}^{n+1}$. By taking slices of the convex set with complex hyperplanes parallel to the tangent complex hyperplane at a point, we obtain a (singular) foliation of M by embeddable CR 3-spheres. On the other hand, one can choose a foliation of M by 3-spheres, and first try to normalize the CR structure on M in such a way that the 3-spheres are CR 3-spheres. Then, roughly speaking, M is embeddable if and only if all of the 3-spheres in the foliation are embeddable.

This research problem and the approach outlined was the subject of the RIT.

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