REPORT ON THE BANFF WORKSHOP "ANALYSIS AND GEOMETRIC MEASURE THEORY"

Contents

Organizers: A. Granados (University of Washington), H. Pajot (Université de Grenoble I), T. Toro (University of Washington).

1. Presentation of the workshop

1.1. **Objectives.** The workshop has been dedicated to problems where there is strong interplay between analysis (in particular harmonic analysis and complex analysis) and geometric measure theory (in particular rectifiability and variational methods).

Topics to be covered include

(i) Analytic capacity and rectifiability
The classical Painlevé problem consists in finding a geometric characterization for compact sets of the complex plane which are removable for bounded analytic functions. The methods used to study this problem come from complex analysis (analytic capacity), harmonic analysis (Cauchy singular integral operator) and geometric measure theory (rectifiability). In 1998, G. David solved the Vitushkin conjecture which provides an answer to Painlevé's question for sets with finite 1-dimensional Hausdorff measure. His work relied on the ideas of many mathematicians among others M. Christ, P. Jones, P. Mattila, M. Melnikov and J. Verdera. Recently, X. Tolsa proposed a solution for the Painlevé problem in terms of Menger curvature.

Problems to be discussed during the workshop include:

- Discussion of Tolsa's conditions;

- Bilipschitz invariance of the class of removable sets for bounded analytic functions in the complex plane;

- Relationship between analytic capacity and Favard length;

- Harmonic analysis in nonhomogeneous spaces;

- The higher dimensional case, namely the study of removable sets for Lipschitz harmonic functions in $\mathbf{R}^n$ (the main problem is that although there exist analogs of the Menger curvature for $n-1$-dimensional sets, they are not adapted to the study of the Riesz transforms. Hence, only a few basic things about this problem are known).

(ii) Analysis and rectifiability in singular metric spaces
Partially motivated by questions arising in classical differential geometry, several authors have begun developing theories of analysis and rectifiability in metric spaces. To this effect basic tools of geometric function theory, for example Poincaré inequalities or quasi-conformal mappings, have been introduced and studied in general metric spaces. Counterparts to the classical theorems in Euclidean spaces have been proved in metric spaces with bounded geometry. For instance, in 1999 J. Cheeger proved a version of Rademacher's theorem on the differentiability of Lipschitz functions on metric spaces where Poincaré inequalities hold. The tools from non-smooth analysis play a crucial role in understanding limiting phenomena arising from smooth geometry.

Problems to be discussed during the workshop include:

- Geometric analysis (Poincaré inequalities, Sobolev spaces, ..) and applications to PDE and geometry;

- Basic tools of geometric measure theory (Sets of finite perimeter, area and co-area formulas, ..) in metric spaces;

- The Kakeya problem;

- Definitions of rectifiability in metric spaces (for instance, in Carnot groups).

(iii) Mumford-Shah functional
This functional was introduced in connection with image segmentation. Let Ω be a bounded domain in the plane and let g be a bounded function on Ω. The Mumford-Shah functional is given by

$$J(u,K) = \int\int_{\Omega\sim K} |u-g|^2 + \int\int_{\Omega\sim K} |\nabla u|^2 + H^1(K).$$

The existence of minimizers (u, K) (in a reasonable sense) is known, but the main problem consists in studying the geometric properties of the set of singularities K. The Mumford-Shah conjecture states that K should be the finite union of C^1 arcs.

Recent progress have been made by G. David, A. Bonnet, L. Ambrosio, S. Solimini, N. Fusco among others, but the conjecture is still open.
The study of the Mumford-Shah functional in higher dimensions is a vibrant new question which seems to be related to the theory of minimal surfaces.

Problems to be discussed during the workshop include:

- Complete classification of the global minimizers of the 2-dimensional Mumford-Shah functional;

- Study of cracktips (C^1 regularity, calibration,...) for the 2-dimensional Mumford-Shah functional;

- Study of the 3-dimensional Mumford-Shah functional, in particular connexions with minimal surfaces, complete classification of global minimizers.

1.2. **List of participants.** The following mathematicians

T. Adams (Stanford University, USA)
S. Choi (UCLA, USA)
G. David (University of Paris-Sud, France)
T. De Pauw (University of Paris-Sud, France)
B. Franchi (University of Bologna, Italy)
J. Garnett (UCLA, USA)
F. Germinet (University of Lille I, France)
M. J. Gonzalez (University of Cadiz, Spain)
R. Hardt (Rice University, USA)
S. Keith (University of Helsinki, Finland)
B. Kirchheim (Max Planck Institute Leipzig, Germany)
P. Koskela (University of Jyväskylä, Finland)
I. Laba (University of British Columbia, Canada)
V. Magnani (Scuola Normale Superiore Pisa, Italy)
J. Mateu (Universitat Autonoma de Barcelona, Spain)
P. Mattila (University of Jyväskylä, Finland)
D. Meyer (University of Washington, USA)
T. O'Neil (The Open University, USA)
H. Pajot (University of Grenoble I, France)
S. Pauls (Darmouth College, USA)
C. Rios (Mc Master University, Canada)
R. Serapioni (Universita de Trento, Italy)
N. Shanmugalingam (University of Cincinnati, USA)
Q. Shi (Tsinghua University, China)
J. Tyson (University of Illinois, USA)
Q. Xia (University of Texas at Austin, USA)
N. Zobin (College of William and Mary, USA)

1.3. **Programme.**

Sunday, 27 July 2003

9-10 am, J. Garnett, *Analytic Capacity, Cantor Sets, Menger Curvature, and Bilipschitz maps*
10-10:30 am, Coffee break
10:30-11-30 am, T. De Pauw *The Plateau problem is not yet solved...we're working at it*
11:40-12:40 am, R. Hardt *Rectifiable Scans*
Lunch and free discussions
3:30-4 pm, Coffee break
4-5 pm, T. O'Neil *Dimension of Visible Sets*
5:10-5:40 pm, J. Tyson *Polar Coordinates in Carnot Groups*

Monday, 28 July 2003

9-10 am, G. David *Open Problems on the Mumford-Shah Functional*
10-10:30 am, Coffee break
10:30-11:30 am, F. Germinet *Generalized Fractal Dimensions: Properties and Applications to Quantum Dynamics*
11:40-12:40 am, B. Kirchheim *Rectifiability in The Metric Context and Density of Measures*
Lunch and free discussions
3:30-4 pm, Coffee break
4-4:30 pm, S. Choi *Lower Density Theorem for Harmonic Measure*
4:40-5:10 pm, D. Meyer *Quasiymmetric Embedding of Self-Similar Surfaces*
5:20-5:50 pm, V. Magnani, TBA

Tuesday, 29 July 2003

9-10 am, P. Mattila *Uniqueness of Tangent Measures and Rectifiability in Metric groups*
10-10:30 am, Coffee break
10:30-11:30 am, J. Mateu *Signed Riesz Capacity*
11:40-12:40 am, N. Shanmugalingam *An Introduction to the Dirichlet Problem for the p-Laplacian on Certain Metric Measure Spaces.*
Lunch and free discussions
3:30-4 pm, Coffee break

Wednesday, 30 July 2003

9-10 am, I. Laba TBA
10-10:30 am, Coffee break
10:30-11:30 am, P. Koskela *Sobolev Inequalities in Metric Measure Spaces*
11:40-12:40 am, M. J. Gonzalez *Geometry of Curves and Beltrami-Type Operators*
Lunch and free discussions
3:30-4 pm, Coffee break
4-4:30 pm, N. Zobin *Fourier Analysis on Fock Spaces and Extension Problems for Smooth Functions*
4:40-5:10 pm, S. Pauls *Rectifiability Modeled on Carnot Groups*

5:20-5:50 pm, C. Rios *The L^p-Dirichlet Problem and Nondivergence Harmonic Measure*

Thursday, 31 July 2003

9-10 am, R. Serapioni *Rectifiability in the Heisenberg group*

2. Geometric Function Theory (written by J. B. Garnett)

1. Sunhi Choi: Lower Density Theorem for Harmonic Measure

Let Ω be a simply connected domain in $\mathbb{C}$ and let $\omega(w, \cdot, \Omega)$ denote the harmonic measure on $\partial\Omega$ for $w \in \Omega$. If f is a conformal mapping from the unit disk $\mathbb{D}$ onto Ω with $f(0) = w$, then the angular limit $f(\zeta)$ exists at almost every $\zeta \in \partial\mathbb{D}$ and the harmonic measure of a set $E \subset \partial\Omega$ is the normalized linear measure of $f^{-1}(E) \subset \partial\mathbb{D}$. We have the following theorem:

Theorem 1: $\omega \ll \Lambda_1$ on the set

$$\{x \in \partial\Omega \,|\, \liminf_{r\to 0} \frac{\omega(B(x,r))}{r} > 0\}.$$

Theorem 1 was conjectured by C. J. Bishop in 1991. It has several corollaries.

Corollary 1: Let F be a subset of $\partial\Omega$ and assume that there exists a constant $M(F)$ such that

$$\sum \mathrm{rad}(B_i) \leq M(F) < \infty$$

for every disjoint collection of balls $\{B_i\}$ with $\mathrm{center}(B_i) \in F$ and $\mathrm{rad}(B_i) < \mathrm{diam}(\partial\Omega)$. Then, $\omega \ll \Lambda_1$ on F.

Conversely, Theorem 1 can be easily derived from Corollary 1.

The next corollary, first proved by Bishop and Jones by much different methods, resolves a conjecture of ∅ksendal.

Corollary 2: Let F be a subset of a rectifiable curve Γ, then for any simply connected domain Ω, $\omega \ll \Lambda_1$ on $F \cap \partial\Omega$.

The last corollary was also conjectured by Bishop.

Corollary 3: At ω-almost every McMillian twist point $x \in \partial\Omega$,

$$\liminf_{r\to 0} \frac{\omega(B(x,r))}{r} = 0.$$

The corollaries follow from the theorem by covering lemmas, Lebesgue density arguments and theorems of Makarov and Pommerenke relating harmonic measure to linear measure. The proof of the theorem uses extremal length and some explicit constructions of Lipschitz domains.

2. John Garnett: Analytic Capacity, Cantor Sets, Menger Curvature and Bilipschitz Maps

The talk was a survey of the theory of analytic capacity, emphasizing the important recent work of Melnikov and Verdera, of Tolsa and of Volberg.

The *analytic capacity* of a compact plane set E is

$$\gamma(E) = \sup\{|a_1| : f(z) = \frac{a_1}{z} + \cdots \in H^\infty(\mathbb{C} \sim E), ||f||_\infty \leq 1\}.$$

Thus $\gamma(E) = 0$ if and only if there are no non-constant bounded analytic functions on $\mathbf{C} \sim E$. The main question is to give a geometric necessary and sufficient conditions for $\gamma(E) > 0$. In particular if T is a bilipschitz homeomorphism of the plane, is there a constant $C = C(T)$ such that

$$C^{-1}\gamma(E) \leq \gamma(T(E)) \leq C\gamma(E)? \tag{1}$$

It is classical that $\gamma(E) = 0$ if the Hausdorff measure $\Lambda_1(E) = 0$ and $\gamma(E) > 0$ if $\Lambda_\alpha(E) > 0$ for some $\alpha > 0$, i.e. if the Hausdorff dimension of E exceeds 1.

For sets of dimension 1, more recent work of Calderón, Mattila-Melnikov-Verdera, and David, using some ideas of Christ and Jones, show that if $0 < \Lambda_1(E) < \infty$ then the following three conditions are equivalent:

(i) $\gamma(E) > 0$

(ii) there is a rectifiable curve Γ such that $\Lambda_1(E \cap \Gamma) > 0$

(iii) there is $F \subset E$, $\Lambda_1(F) > 0$ and the Cauchy integral

$$Cf(z) = \text{p.v.} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\Lambda_1(\zeta),$$

is bounded $L^2(F, ds) \to L^2(\Gamma, ds)$.

The proof that (iii) implies some local rectifiability hinges on the notion of *Menger curvature*. For three complex numbers named x, y, and z, let $c(x, y, z)$ be the reciprocal of the radius of the circle through x, y and z, and take $c(x, y, z) = 0$ if the points are co-linear. Let μ be a finite positive Borel measure of *linear growth:* $\mu(B(z, R) \leq R,\ \forall z,\ \forall R$. The Menger curvature of μ is defined to be

$$c^2(\mu) = \int\int\int c^2(x, y, z) d\mu(x) d\mu(y) d\mu(z).$$

The connection between Menger curvature and the theorem rests on a remarkable discovery of Melnikov and Verdera: If μ is a positive measure of linear growth, then

$$\frac{c^2(\mu)}{6} = \int \Big| \int \frac{d\mu(w)}{w - z} \Big|^2 d\mu(z) + O(1).$$

Hence by (iii) $\mu = \chi_F \Lambda_1$ has $c^2(\mu) < \infty$, and an argument using the P. Jones β-numbers shows there is some rectifiable curve Γ such that $\Lambda_1(F \cap \Gamma) > 0$.

The assumption (2) was later removed by G. David, and later by Nazarov, Triel and Volberg by a different method.

The remaining case, E of Hausdorff dimension 1 but $\Lambda_1(E) = \infty$ was recently resolved by X. Tolsa, who proved that $\gamma(E) > 0$ if and only if E supports a positive measure of linear growth and finite Menger curvature and if and only if E supports a positive measure of linear growth and bounded Cauchy potential $\int \frac{d\mu(w)}{z-w}$.

The speaker described his result with Verdera that (1) holds for all bilipschitz images of planar Cantor sets, a new stronger theorem of Tolsa that proved (1) for *all* compact plane sets, and the generalizations by Volberg (without Menger curvature!)

to the case of Lipschitz harmonic capacity in $\mathbb{R}^n$. In particular, Volberg has proved that for $E \subset \mathbb{R}^n$ compact, if

$$\Gamma_n(E) = \sup\big\{ | < \Delta f, 1 > | : f \text{ is harmonic off } E, ||\nabla f||_\infty \leq 1 \big\}$$

and

$$\Gamma_n^+(E) = \sup\big\{ \mu(E) : \mu > 0, \int_E \frac{d\mu(y)}{|x-y|^{n-2}} = f(x) \in \mathrm{Lip}_1,\ ||\nabla f||_\infty \leq 1 \big\},$$

then $\Gamma_n^+ \leq \Gamma_n \leq C_n \Gamma_n^+$.

3. Marie Jose Gonzalez: Geometry of Curves and Beltrami-Type Operators

A rectifiable plane curve Γ is called a *chord-arc* curve (or Lavrientiev curve) if each sub-arc $\gamma \subset \Gamma$ with endpoints a and b has length

$$\ell(\gamma) \leq C|b-a|.$$

We assume that $\infty \in \Gamma$, and that Γ is chord-arc. Then $\Gamma = \varphi(\mathbb{R})$ where φ is a bilipschitz homeomorphism of the plane to itself and if Φ is a conformal map from the upper halfplane to either component of $\mathbb{C} \sim \Gamma$, then $\log(\Phi') \in BMO$.

This talk gives the latest news on three related problems:

Problem 1: If $\epsilon > 0$ and if f is a bilipschitz homeomorphism of the plane, can f be factored

$$f = f_1 \circ f_2 \circ \cdots \circ f_n$$

where each f_j and f_j^{-1} has Lipschitz constant bounded by $1+\epsilon$?

Problem 2: If Γ is a chord-arc curve, is there a deformation from Γ to $\mathbb{R}$ through which $\log(\Phi')$ varies continuously in BMO?

Problem 3: Is the subset $\big\{\log(\Phi') : \Gamma \text{ chord} - \text{arc}\big\} \subset BMO$ connected?

A combination of theorems by Astala and Zinsmeister, MacManus, and Bishop and Jones shows that when Γ is a quasicircle the following are equivalent:

(i) $\log(\Phi') \in BMO$;

(ii) $\Gamma = \rho(\mathbb{R})$ where ρ is a quasiconformal map for which $\frac{|\mu^2|}{y}$ is a Carleson measure in the upper half plane and $\mu = \frac{\rho_{\bar z}}{\rho_z}$ is the Beltrami coefficient of ρ.

(iii) Γ contains big pieces of chord-arc curves.

Because of (iii) such curves are called "BJ curves". It follows from (ii) that the set of BJ curves is connected in the $\log \Phi' - BMO$ topology. An important related result is the 1988 theorem of Semmes: The quasicircle Γ is chord-arc if $\frac{|\mu^2|}{y}$ has small Carleson measure constant.

The speaker discussed her two recent theorems with K. Astala.

Theorem 1: Γ is a BJ curve if and only if there exists a quasiconformal map ρ such that $\Gamma = \rho(\mathbb{R})$, ρ has Beltrami coeffecient μ and $I - \mu S$ is bounded on $L^2(\frac{dxdy}{y})$, where S is the Beurling transform

$$Sf(z) = \frac{1}{\pi} \int \int \frac{f(w)}{(w-z)^2} dA(w).$$

Theorem 2: Γ is chord arc if and only if there exists such ρ and μ such that $I - \mu S$ is invertible on $L^2(\frac{dxdy}{y})$.

The speaker also discussed further connections between $I - \mu S$ and the Semmes result above.

4. Pekka Koskela: Metric Sobolev Spaces

This talk gives an approach to Sobolev spaces in metric spaces based on point-wise Lipschitz constants. The *point-wise Lipschitz constant* of u is

$$\mathrm{Lip}u(x) = \limsup_{r \to 0} \sup_{d(y,x)<r} \frac{|u(y) - u(x)|}{r}.$$

(X, d, μ) is a *doubling space* if (X, d) is a metric space and μ is a Borel measure on X such that for constant C_d,

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)).$$

We say X supports a $p-$*Poincaré inequality* if there exist constants C_p and $\lambda \geq 1$ such that

$$⨍_B |u - u_B| \, d\mu \leq C_p \operatorname{diam}(B) \left(⨍_{\lambda B} (\mathrm{Lip}u(x))^p \, d\mu \right)^{1/p}, \tag{1}$$

for all balls B and for each Lipschitz function u.

This should perhaps be called a weak Poincaré inequality, but it turns out that the Poincaré inequality always improves itself to a (p, p)-inequality, perhaps with larger C and λ. Indeed, even a (q, p)-inequality follows with an optimal $q > p$. Also, the constant λ can often be taken to be 1 by enlarging C. This holds if the metric d is a *path metric* (i.e. infimum of lengths of paths joining the points) and *geodesic*: in this case the geometry of balls can be controlled and one can iterate the Poincaré inequality so as to decrease λ. We call such a metric a *length metric* and the corresponding space a *length space*. If we assume that X is *proper* (i.e. all closed balls are compact), then it follows from the Poincaré inequality that we can replace the metric d with a bi-Lipschitz equivalent length metric.

In $\mathbb{R}^n$ every Sobolev function has a gradient almost everywhere. In our general situation a version of this persists, if we use the concept of upper gradient.

Let $u : A \to \overline{R}$, $A \subset X$. Any Borel function $g : A \to [0, \infty]$ such that for each rectifiable path $\gamma : [0, l] \to A$

$$|u(\gamma(l)) - u(\gamma(0))| \leq \int_\gamma g ds$$

is called an *upper gradient* of u on A. We now define, for given $1 \leq p \leq \infty$,

$$N^{1,p}(X) = \{u \in L^p(X) : \ u \text{ has an upper gradient } g \in L^p(X)\},$$

where the L^p-spaces are taken with respect to our measure μ and the concept of an upper gradient is with respect to our metric d. The norm on $N^{1,p}$ is

$$||u||_{1,p} = ||u||_p + \inf_{g_u} ||g_u||_p,$$

where the infimum is taken over all upper gradients of u, and as usual one needs to consider equivalence classes in order to obtain a normed vector space. In the Euclidean setting

$$N^{1,p}(R^n) = W^{1,p}(R^n)$$

when both the metric and the measure are the usual Euclidean ones.

Theorem 1: Suppose that (X, d, μ) is a doubling length space that supports a p-Poincaré inequality. Let B be a ball and $u \in N^{1,p}(B)$. Suppose that $\mu(B(x,r)) \geq C_b(r/\operatorname{diam}(B))^s \mu(B)$ whenever $B(x,r) \subset B$.

(1) If $p < s$, then

$$||u - u_B||_{L^{p^*}(B)} \leq C \operatorname{diam}(B) \mu(B)^{1/p^* - 1/p} \|g\|_{L^p(B)}, \tag{2}$$

where $p^* = ps/(s-p)$.

(2) If $p = s$, then

$$\fint_B \exp\left(\frac{C_1 \mu(B)^{1/s} |u - u_B|}{\operatorname{diam}(B)\|g\|_{L^s(B)}}\right)^{s/(s-1)} d\mu \leq C_2.$$

(3) If $p > s$, then $|u(x) - u_B| \in L^\infty(B)$ and

$$\|u - u_B\|_{L^\infty(B)} \leq C \operatorname{diam}(B) \mu(B)^{-1/p} \|g\|_{L^p(B)}.$$

Here $C_i = C_i(\lambda, s, C_p, C_b, C_d)$.

Theorem 2: Let X be a proper doubling space that supports a p-Poincaré inequality, $p \geq 1$. Then $N^{1,p}(X)$ consists precisely of those functions in $L^p(X)$ that are L^p-limits of sequences of Lipschitz functions for which also the sequence of the point-wise Lipschitz norms converges in $L^p(X)$. Moreover, when $p > 1$, the space $N^{1,p}(X)$ is reflexive.

The approximation result here is essentially due to Shanmugalingam and the reflexivity is due to Cheeger.

It is often convenient to know that the Poincaré inequality can be characterized by a point-wise inequality. We recall that for every $R > 0$ the *restricted maximal operator* is

$$M_R u(x) = \sup_{0<r<R} \fint_{B(x,r)} |u(x)| d\mu,$$

where u is a measurable function. Because the proof of the point-wise inequality is somewhat easier when $p > 1$ and works for pairs of functions, not only pairs of functions and upper gradients, we first only state this case.

Lemma 1: Let (X, d, μ) be a doubling space, u be locally integrable and $g \geq 0$ measurable. If $p > 1$, then the following conditions are quantitatively equivalent:

(1) There exist $C > 0$ and $\lambda \geq 1$ such that

$$\fint_B |u - u_B| \, d\mu \leq C \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, d\mu\right)^{1/p} \tag{3}$$

for every ball B.

(2) There exist $C > 0$ and $\tau > 0$ such that

$$|u(x) - u_B| \leq C \operatorname{diam}(B) \left(M_{\tau \operatorname{diam}(B)} g^p(x)\right)^{1/p}$$

for every ball B and a.e. $x \in B$.

(3) There exist $C > 0$ and $\sigma > 0$ such that

$$|u(x) - u(y)| \leq C d(x,y) \left(M_{\sigma d(x,y)} g^p(x) + M_{\sigma d(x,y)} g^p(y)\right)^{1/p}$$

for almost every $x, y \in X$.

Moreover, even when $p = 1$, condition 1 implies condition 2 which yields condition 3.

Lemma 2: Let (X, d, μ) be a proper doubling space. Then the three conditions of Lemma 2. are quantitatively equivalent for functions $u \in N^{1,p}_{loc}(X)$ and their upper gradients.

We say that X is *quasiconvex* if there exists a constant $C \geq 1$ such that each pair $x, y \in X$ can be joined with a rectifiable curve γ such that

$$\mathrm{length}(\gamma) \leq Cd(x, y).$$

Lemma 3: Assume that (X, d, μ) is a doubling space that supports a p-Poincaré inequality and that X is proper. Then X is quasiconvex.

The previous result allows one to replace the metric of a proper space that supports a Poincaré inequality with a bi-Lipschitz equivalent path metric.

Corollary: Suppose that (X, d, μ) supports a p-Poincaré inequality and that X is proper. Define $\hat{d}(x, y) = \inf_\gamma \mathrm{length}(\gamma)$, where the infimum is taken over all curves that join x and y. Then $\hat{d}$ is a geodesic metric and there exists a constant C so that

$$d(x, y)/C \leq \hat{d}(x, y) \leq Cd(x, y)$$

for all $x, y \in X$.

There is yet another way to characterize the Poincaré inequality. Following Semmes we define, for given $\epsilon > 0$ and measurable $u : X \to \overline{R}$,

$$D_\epsilon u(x) = \sup_{y \in B(x,\epsilon)} \frac{|u(x) - u(y)|}{\epsilon},$$

for every $x \in X$. The following result is due to Keith and Rajala.

Theorem 4: Let X be a proper doubling space. Then X supports a p-Poincaré inequality if and only if there are constants C and λ so that

$$⨍_B |u - u_B| \, d\mu \leq C \operatorname{diam}(B) \left(⨍_{\lambda B} (D_\epsilon u)^p \, d\mu \right)^{1/p}, \tag{4}$$

for each ϵ and each ball $B \subset X$ of diameter at least 2ϵ and all u.

We have mentioned that the Poincaré inequality is not destroyed by by bi-Lipschitz changes of the metric. The Poincaré inequality turns out also to persist under convergence of spaces. if we use the notion of (pointed) measured Gromov-Hausdorff convergence.

Theorem 5: Suppose that $(X_i, x_i, \mu_i, d_i)_i$ is a sequence of geodesic, pointed, proper doubling spaces so that each space is doubling with the same constant C_d and so that each of them supports a p-Poincaré inequality with fixed constants C_P, λ. If this sequence converges in the pointed, measured Gromov-Hausdorff sense to a proper space (X, x, d, μ), then (X, d, μ) is a doubling space that supports a p-Poincaré inequality. Moreover, (X, d, μ) is geodesic.

5. Joan Mateu: Signed Riesz Capacities

This talk represents joint work with Laura Prat and Joan Verdera. If $K \subset \mathbb{R}^n$ is a compact set and $0 < \alpha < n$ we define

$$\gamma_\alpha(K) = \sup |T(1)|,$$

where the supremum is over all distributions T supported on K such that for $1 \leq i \leq n$,

$$||T * \frac{x_i}{|x|^{1+\alpha}}||_{L^\infty(\mathbb{R}^n)} \leq 1.$$

For $n = 2$ and $\alpha = 1$, γ_1 is essentially the same as analytic capacity, and for $n \geq 2$ and $\alpha = n - 1$ it is essentially Lipschitz harmonic capacity.

Prat had showed in her thesis that if $0 < \alpha < 1$ then every set K with finite α-dimensional Hausdorff measure has $\gamma_\alpha(K) = 0$. The case $\alpha > 1$ is not so well understood, but here Prat also showed that $\gamma_\alpha(K) = 0$ if K is α Ahlfors-David regular.

The Riesz capacity is

$$C_{s,p}(K) = \inf\big\{||\varphi||_p^p : \varphi * \frac{1}{|x|^{n-s}} \geq 1 \text{ on} K,\ \varphi \in C_0^\infty\big\},$$

where $1 < p < \infty$ and $0 < s < \frac{p}{n}$.

Theorem: For every n and $0 < \alpha < 1$ there is a constant C depending only on n and α such that for all compact $K \subset \mathbb{R}^n$,

$$C^{-1} C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(K) \leq \gamma_\alpha(K) \leq C C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(K).$$

The Prat result about sets of finite α measure follows from the theorem and known estimates for $C_{s,p}$. It is also known that $C_{s,p}$ is subadditive, and hence the theorem implies that

$$\gamma_\alpha(K_1 \cup K_2) \leq C\gamma_\alpha(K_1) + \gamma_\alpha(K_2),$$

with constant C depending only on n and α. Moreover, since $C_{s,p}$ is bilipschitz invariant, the theorem also implies that

$$C^{-1}\gamma_\alpha(K) \leq \gamma_\alpha(T(K)) \leq C\gamma_\alpha(K)$$

for every bilipschitz homeomorphism T of $\mathbb{R}^n$, where the constant C depends only on n, α and the Lipschitz constants of T and T^{-1}.

The proof of the Theorem has two main steps. The first step is to compare γ_α with the corresponding "positive" capacity $\gamma_{\alpha,+}$. Here $\gamma_{\alpha,+}(K) = \sup \mu(K)$, where the supremum is over all positive measures μ supported on K such that for $1 \leq i \leq n$, $||\mu * \frac{x_i}{|x|^{1+\alpha}}||_{L^\infty(\mathbb{R}^n)} \leq 1$, and the first step of the proof is to show

$$C^{-1}\gamma_\alpha(K) \leq \gamma_{\alpha,+}(K) \leq C\gamma_\alpha(K).$$

The proof of this somewhat resembles Tolsa's proof of the corresponding result for analytic capacity, and uses Prat's earlier proof of the positivity of the symmetrization of the Riesz kernel k_α and a localization result for the kernel k_α. This localization result for $\alpha < n$ is non-trivial and constitutes the main technical difficulty of the proof. The second step of the proof is to use Wolff potentials to compare $\gamma_{\alpha,+}(K)$ to $C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(K)$.

6. Daniel Meyer: Quasisymmetric Embeddings of Self Similar Surfaces

A *quasiconformal* map $f : X \to Y$ is a homeomorphism of metric spaces (distance written as $|x-a|$) such that for all $x \in X$

$$\limsup_{\varepsilon\to 0} \frac{\max_{|z-x|<\varepsilon}|f(z)-f(x)|}{\min_{|z-x|<\varepsilon}|f(z)-f(x)|} \leq K$$

with K independent of x. If $K = 1$ f is *conformal*. The homemorphisms $f : X \to Y$ is η-*quasisymmetric* if there is an increasing homeomorphism $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all x, a, and $b \in X$,

$$\frac{|f(x)-f(a)|}{|f(x)-f(b)|} \leq \eta\Big(\frac{|x-a|}{|x-b|}\Big).$$

Every quasisymmetric map is quasiconformal and every quasiconformal self map of $\mathbb{R}^n$ is quasisymmetric.

In dimension 2, the images of the unit circle under a global quasiconformal mappings are characterized by the *Ahlfors three point condition*: if ζ lies on the smaller diameter arc with endpoints z and w, then

$$|z-\zeta| \leq C|z-w|.$$

The von Koch snowflake curve is an example. However, for $n \geq 3$ no characterization of the images of S^{n-1} under quasiconformal self maps of $\mathbb{R}^n$.

The speaker gives explicit constructions of quasiconformal maps from S^2 to certain 2-dimensional topological surfaces, analogous to the snowflake curve, known as "snowball" surfaces. These maps are constructed by iterating specific rational maps. The speaker also proves that the maps constructed above have extensions to quasiconformal self maps of $\mathbb{R}^3$, again by explicit construction.

7. Cristian Rios: The L^p Dirichlet problem and nondivergence harmonic measure

For $k = 0, 1$ let $A_k(x) = \{a_k^{i,j}(x)\}$ be a symmetric $n \times n$ complex matrix function on $\mathbb{R}^n$ for which there exists $0 < \lambda < \Lambda < \infty$ such that for all $x, \xi \in \mathbb{R}^n$,

$$\lambda|\xi|^2 \leq \xi \cdot A_k(x)\xi \leq \Lambda|\xi|^2,$$

and let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Consider the Dirichlet problem

$$\mathcal{L}_k u = \sum_{i,j=1}^{n} a_k^{i,j}(x)\partial_{i,j}u(x) = 0, \; x \in D;$$

$$u = g, \; x \in \partial D.$$

Let σ be surface measure on ∂D and let $1 < p < \infty$. Say that $\mathcal{D}_p$ holds for $\mathcal{L} = \mathcal{L}_k$ if the solution u has nontangential maximal function $N(u)$ satisfying

$$||N(u)||_{L^p(\sigma)} \leq C_p||g||_{L^p(\sigma)}$$

for all $g \in C(\partial D)$. Define

$$a(x) = \max_{1\leq i,j\leq n} ||a_1^{i,j} - a_0^{i,j}||_{L^\infty\left(B(x,\frac{\mathrm{dist}(x,\partial D)}{2}\right)}.$$

Theorem: Assume there is $\rho > 0$ such that $A_k \in BMO_\rho$ where BMO_ρ is the Saracon class

$$\inf_c \frac{1}{(\sigma(\partial D \cap B(x,r))} \int_{\partial D \cap B(x,r)} |f(y) - c| d\sigma(y) \leq \rho\big(\sigma(\partial D \cap B(x,r)\big)$$

for all $x \in \partial D$, and assume that $\mathfrak{L}_0$ satisfies D_p. Then $\mathfrak{L}_1$ satisfies D_p if

$$\sup_{Q \in \partial D} \sup_{r>0} \frac{1}{\sigma(\partial D \cap B(Q,r))} \int_{D \cap B(Q,r)} \frac{a^2(x)}{\operatorname{dist}(x, \partial D)} dx < \infty.$$

A similar result for elliptic operators of divergence form was proved by Fefferman, Kenig and Pipher in 1991.

8. Nages Shanmugalingam; The Dirichlet Problem for Domains in Metric Measure Spaces

Let Ω be a domain in the Euclidean space $\mathbf{R}^n$. For $1 \leq p < \infty$ the classical $W^{1,p}(\Omega)$ is the collection of all $u \in L^p(\Omega)$ with distributional derivatives $\partial_i u$, $i = 1, ..., n$, in $L^p(\Omega)$, under the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \sum_{i=1}^n \|\partial_i u\|_{L^p}.$$

Some classical properties of Sobolev functions $u \in W^{1,p}(\mathbf{R}^n)$ include:

(1) Poincarè inequality: For each ball B, radius r,

$$\fint_B |u - u_B|\, dx \leq Cr \left(\fint_B |\nabla u|^p \, dx \right)^{1/p}.$$

where C depends only on n and p.

(2) If $p > n$, it is Hölder continuous with exponent $\alpha = 1 - n/p$.

(3) If $p < n$, $u \in L^{p^*}$, where $p^* = \frac{np}{n-p}$.

(4) *the weak upper gradient inequality:* there exists a zero p-modulus (defined below) curve family Γ so that if $\gamma \notin \Gamma$,

$$|u(x) - u(y)| \leq \int_\gamma |\nabla u| \, ds.$$

(5) *The Hajłasz Inequality:*

$$|u(x) - u(y)| \leq C|x-y| \left(M_p|\nabla u|(x) + M_p|\nabla u|(y) \right), a.e.$$

Let X be a metric space equipped with metric d and measure μ, and $1 \leq p < \infty$. We say a function u is in the *Hajłasz space* $H^{1,p}(X)$ if $u \in L^p(X)$ and there exists $g \geq 0$ $g \in L^p(X)$ such that

$$|u(x) - u(y)| \leq d(x,y) \left(g(x) + g(y) \right).$$

The definition of Hajłasz space has three advantages: when $p > 1$ it yields the same space as classical Sobolev space on $\mathbf{R}^n$; it is defined by a pointwise inequality; and the Poincaré inequality holds:

$$\fint_B |u - u_B| \leq Cr \left(\fint_B g^p \right)^{1/p}$$

whenever B a ball in X of radius r. It has has two disadvantages: for general domains in $\mathbf{R}^n$ it is not the same as the classical Sobolev space, and the Hajłasz gradient g may not have the truncation property.

Now let X have metric d and measure μ. The p-modulus of a path family Γ is

$$\mathrm{Mod}_p\Gamma = \inf_\rho \|\rho\|_{L^p}^p,$$

where the infimum is taken over all non-negative Borel-measurable functions ρ such that for each rectifiable γ in Γ

$$\int_\gamma \rho ds \geq 1.$$

Fuglede proved that modulus is an outer measure on the collection of all curves in X, and Fuglede and Koskela-MacManus proved that a curve family Γ has zero p-modulus if and only if there exists $L^p(X) \ni \rho \geq 0$ such that for each rectifiable $\gamma \in \Gamma$,

$$\int_\gamma \rho \, ds = \infty.$$

Any property holding on all compact curves except for a zero modulus family of curves is said to hold on *p-almost every curve*, or p-a.e.

Definition: A Borel function $\rho \geq 0$ on X is an *upper gradient* if $u : X \to [-\infty, \infty]$ if on all curves γ,

$$|u(x) - u(y)| \leq \int_\gamma \rho. \tag{5}$$

If ρ satisfies (1) only p-a.e., it is called a *p-weak upper gradient* of u.

We say (X, μ) has the *truncation property* if whenever u is constant on a closed (or open) set E, and $\rho \in L^p(X)$ is a p-weak upper gradient of u, then

$$\rho_{\mathrm{new}}(x) = \begin{cases} \rho(x) & \text{if } x \notin E \\ 0 & \text{if } x \in E \end{cases}$$

is a p-weak upper gradient of u.

We say a function u is in the *Newtonian space* $N^{1,p}(X)$ if $u \in L^p(X)$ and if u has an upper gradient $\rho \in L^p(X)$, and we define its ”norm” by:

$$\|u\|_{N^{1,p}} = \|u\|_{L^p} + \inf_\rho \|\rho\|_{L^p},$$

where we identify u and v if $\|u - v\|_{N^{1,p}} = 0$.

Koskela and MacManus proved that every p-weak upper gradient in $L^p(X)$ can be approximated to desired accuracy by upper gradients in $L^p(X)$. We use p-weak upper gradients rather than upper gradients because minimal p-weak upper gradients exist and are unique, and if $||f_n - f||p \to 0$, if f_n has upper gradient g_n and if and $||g_n - g||_p \to 0$, then g is a weak upper gradient of f.

Properties of Newtonian Spaces:

- If u is in $N^{1,p}(X)$, it is absolutely continuous on p-a.e. curve.
- $N^{1,p}(X)$ is Banach.
- Every Cauchy sequence in $N^{1,p}(X)$ has a subsequence that converges uniformly outside arbitrarily small p-capacity sets, where $\mathrm{Cap}_p(E) = \inf_u \|u\|_{N^{1,p}}^p$, the infimum being over all $u \in N^{1,p}(X)$ s.t. $u\big|_E = 1$.
- If X is any domain in $\mathbf{R}^n$ and $1 \leq p < \infty$, then $N^{1,p}(X)$ is isometrically the classical Sobolev space.

We say μ is *doubling* if there exists $C > 0$ such that for all $x \in X$ and all $r > 0$

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)),$$

and we say X supports (q, p)-*Poincaré inequality* if there is $C > 0$ such that for all balls $B \subset X$ and for all $u :\in N^{1,p}(X)$ and all weak upper gradients ρ of u,

$$\left(\fint_B |u - u_B|^q \, d\mu\right)^{1/q} \leq C\mathrm{rad}(B) \left(\fint_B \rho^p \, d\mu\right)^{1/p}.$$

Semmes has proved that if μ is doubling and X supports $(1, p)$-Poincaré inequality, then Lipschitz functions are dense in $N^{1,p}(X)$, and X is a quasiconvex space. Hajłasz–Koskela and Shanmugalingam proved that if μ is Ahlfors regular and if X supports a $(1, q)$-Poincaré inequality for some $q < p$, then $N^{1,p}(X)$ satisfies the Sobolev embedding theorems. Cheeger proved that if μ is doubling and if X supports a $(1, p)$-Poincaré inequality, then $N^{1,p}(X)$ is reflexive and admits a natural derivation.

Ohtsuka showed that for every $L^p(\mathbf{R}^n) \ni \rho \geq 0$ there is a set G_ρ such that $|\mathbf{R}^n \backslash G_\rho| = 0$, and for all $x \neq y \in G_\rho$ there is a rectifiable curve γ connecting x to y with

$$\int_\gamma \rho \, ds < \infty.$$

Suppose $L^p(\mathbf{R}^n) \ni \rho \geq 0$. Then ρ partitions X into equivalence classes via the following equivalence $x \sim y$ if and only if $x = y$ or there exists a rectifiable $\gamma_{x,y}$ such that $\int_{\gamma_{x,y}} \rho \, ds < \infty$. X is said to have the MEC_p property if for each such ρ there exists an equivalence class G_ρ with $\mu(X \backslash G_\rho) = 0$. For example $\mathbf{R}^n$ is MEC_p for all $1 \leq p < \infty$, $\mathbf{R}^n$ with the snowflake metric ($d(x, y) = |x - y|^\epsilon$, $0 < \epsilon < 1$ fixed) is not MEC_p for any p, and if X supports local $(1, p)$-Poincaré then X is MEC_p. **The Dirichlet Problem:** Given a domain $V \subset \mathbf{R}^n$ and $f \in W^{1,p}(\mathbf{R}^n)$, we seek $u \in W^{1,p}(\mathbf{R}^n)$ so that:

(i) $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$ on V, and

(ii) $f - u \in W_0^{1,p}(\mathbf{R}^n)$.

Condition (i) is equivalent to:

$$\int_V |\nabla(u + h)|^p \geq \int_V |\nabla u|^p,$$

for all $h \in W_0^{1,p}(\mathbf{R}^n)$. Known results on the Dirichlet problem include:

- If $1 < p < \infty$ μ doubling, and if X is proper and supports the $(1, p)$-Poincaré, and $E \subset X$ is open with $\mathrm{Cap}_p(X \backslash E) > 0$, then minimizing u exists for boundary data $f \in N^{1,p}(X)$ and satisfies Harnack.
- If $1 < p < \infty$ $E \subset X$ is a bounded open set, and $f \in N^{1,p}(X)$ is bounded, then minimizing u exists.
- Cheeger (1998): If X is MEC_p, then given the "boundary value" f, the solution u is unique.
- If X is MEC_p, such solutions satisfy the maximum principle: If u, v are solutions on E to two problems involving (possibly) different boundary functions, and $u \geq v$ p-q.e. on $X \backslash E$, then $u \geq v$ p-q.e. on E.

Definitions: $u : E \to (-\infty, \infty]$ is *p-superharmonic* if u is lower semicontinuous, $u \not\equiv \infty$, and if for all $\Omega \subset\subset E$ and all $v \in N^{1,p}(X)$ p-harmonic in Ω: $v \leq u$ in $E \sim \Omega \Longrightarrow v \leq u$ on Ω.

$u : E \to \mathbf{R}$ is a *p-superminimizer* if $u \in N^{1,p}_{loc}(E)$ and for all $\Omega \subset\subset E$, and all $\phi \in N^{1,p}_0(\Omega)^+$,

$$\int_\Omega g^p_{u+\phi}\, d\mu \geq \int_\Omega g^p_u\, d\mu.$$

Given Borel $f : \partial E \to \mathbf{R}$, define the Perron families

$$\begin{aligned} U_f := \{u : E \to (-\infty, \infty] \,:\, & u \text{ } p\text{-superharm. on } E, \\ & u \text{ bdd below on } E, \liminf_{E \ni x \to y \in \partial E} u(x) \geq u(y)\}, \end{aligned}$$

and

$$L_f := -U_{-f},$$

the *Upper Perron solution*

$$\overline{P}f(x) := \inf_{u \in U_f} \tilde{u}(x),$$

and the *Lower Perron solution*

$$\underline{P}f(x) := \sup_{u \in L_f} \widehat{u}(x) = -\overline{P}(-f)(x).$$

Say f is *resolutive* if $\overline{P}f = \underline{P}f$.

Theorem (Björn-Björn-Shanmugalingam): If μ is doubling and X supports the $(1,p)$-Poincaré, then the following are resolutive:

- $f \in N^{1,p}(X)$ $(Pf \in N^{1,p}(X))$.
- continuous functions.
- If E is p-regular, then bounded semicontinuous functions.
- If $K \subset \partial E$ is compact and F zero p-capacity set containing all p-irregular boundary points, then $\chi_{K\cup F}$ $(P\chi_{K\cup F} = \overline{P}\chi_K)$.

3. The Mumford-Shah Problem and Minimal Surfaces (written by G. David, T. De Pauw and B. Hardt)

1. The Mumford-Shah functional in dimension 3

Guy David’s lecture focused mainly on open problems connected too the Mumford-Shah functional. This functional was introduced in image processing, and is a reference tool in image segmentation, but the main concern here is the study of its minimizers. It is given by

$$J(u, K) = \int_{\Omega\sim K} |\nabla u|^2 + \int_{\Omega\sim K} |\nabla u - g|^2 + H^{n-1}(K),$$

where Ω is a simple bounded domain in $\mathbf{R}^n$, g is a given bounded function on Ω, and the competitors are pairs (u, K) such that K is closed in Ω, with finite Hausdorff measure $H^{n-1}(K)$ of codimension 1, and u is, say, locally of class C^1 away from K.

A rapid account of recent results of regularity for K was given (C^1 regularity in many places by Ambrosio, Fusco, Pallara, Rigot; blow-up techniques and regularity for the isolated components of K, recent work by Léger and David), but the main point of the lecture was open questions on the functional itself and on its global version on $\mathbf{R}^n$ obtained by blow-up.

The most the well-known problem is the conjecture of Mumford and Shah, which concerns minimizers in dimension 2, and says that if (u, K) is a reduced minimizer for J, then the singular set K is a finite union of curves of class C^1, which may only meet by sets of 3 and with 120 degrees angles. But David mostly wanted to convince the audience that there are other, equally interesting and perhaps easier questions, mainly in dimension 3.

Of course many of the known theorem in dimension 2 become questions in higher dimensions, because Bonnet's monotonicity argument and Léger's magic formula do not seem to have counterparts, but let us name a few.

First, is the function u essentially determined by K when we know that (u, K) is a global minimizer in space?

Also, a perturbation result of Ambrosio, Fusco, and Pallara says that of in a small ball B, K is very flat (i.e., close to a hyperplane) and $\int_{B\sim K} |\nabla u|^2$ is very small, then K is a nice C^1 surface in half the ball. It would be interesting to know whether in this result, planes can be replaced with the other minimal sets in $\mathbf{R}^3$, like the product of a Y and a line.

Finally, we are lacking a precise analogue of the Mumford-Shah conjecture in 3-space: we know precisely a few global minimizers, but we are probably missing a last basic one.

There are connections between this and other lectures of the conference (such as Thierry De Pauw and Robert Hardt's), not only because the techniques mostly belong to Geometric Measure Theory, but also because a good understanding of the minimal sets in 3-space, for instance, will almost surely help with the perturbation results. Conversely, one can hope that Mumford-Shah techniques will be used in other parts of Geometric Measure Theory.

Some references:

L. Ambrosio, N. Fusco and D. Pallara, Functions of bounded variation and free discontinuity problems, Oxford Mathematical Monographs, Clarendon Press, Oxford 2000.

A. Bonnet, On the regularity of edges in image segmentation, Ann. Inst. H. Poincaré, Analyse non linéaire, Vol 13, 4 (1996), 485-528.

G. David and J.-C. Léger, Monotonicity and separation for the Mumford-Shah functional, Annales de l'I.H.P., Analyse non linéaire 19, 5, 2002, 631-682.

G. David, Singular sets of minimizers for the Mumford-Shah functional, book in preparation, some parts can be found at http://www.math.u-psud.fr/ gdavid/

2. On minimizing Scans

For two integers $1 \leq m \leq n$ the problem of Plateau can be stated as follows. Given an $m-1$ dimensional *boundary* $B \subset \mathbf{R}^n$, we seek an m dimensional *surface* $S \subset \mathbf{R}^n$, *spanning* B, having least *area* among all such surfaces. Solving the problem consists partly in making sense of the italicized words.

H. Federer and W. Fleming introduced the *integral currents* in $\mathbf{R}^n$ (the surfaces), their *mass* (the area) and their boundary. We now briefly review their theory. An m dimensional rectifiable current consists in the following data:

(1) a Borel $\mathcal{H}^m$ rectifiable set $M \subset \mathbf{R}^n$;
(2) a Borel map $\xi : M \to \wedge_m \mathbf{R}^n$ such that for $\mathcal{H}^m$ almost every $x \in M$ a simple m vector $\xi(x)$ associated with the approximate tangent space to M at x, of length $|\xi(x)| = 1$;

(3) a Borel function $\theta : M \to \{1, 2, 3, \ldots\}$.

We moreover assume that this triple (M, ξ, θ) is such that its *mass*

$$\int_M \theta d\mathcal{H}^m < \infty \,.$$

Therefore we can associate with this data an m current T (in the sense of de Rham) in the following way:

$$T : \mathcal{D}^m(\mathbf{R}^n) \to \mathbf{R} : \phi \mapsto \int_M \langle \phi, \xi \rangle \theta d\mathcal{H}^m \,.$$

Now the *boundary* of a current T of degree $m \geq 1$ is the $m-1$ dimensional current ∂T defined by $\langle \partial T, \zeta \rangle := \langle T, d\zeta \rangle$ whenever ζ is a compactly supported differential form of degree $m-1$ with smooth coefficients. An m dimensional integral current T is an m dimensional rectifiable current such that also ∂T is rectifiable.

The Theorem of Federer and Fleming proves the existence of a mass minimizing current T among all those having boundary $\partial T = B$ for some $m-1$ dimensional compactly supported rectifiable current B with $\partial B = 0$. These mass minimizers model some but not all soap films when $n = 3$ and $m = 2$.

Given $0 < q < 1$ we let the q *mass* of a triple (M, ξ, θ) as above be

$$\int_M \theta^q d\mathcal{H}^m$$

Requiring that the q mass be finite does not imply that the mass is finite. Therefore one cannot interpret anymore the triple (M, ξ, θ) as a current, and we simply call it a *scan*. Nevertheless it is still possible to define an appropriate notion of boundary for these objects. We prove that given B as before there exists a q mass minimizing scan (M, ξ, θ) whose boundary is B. In case B is associated with a smooth embedded submanifold of $\mathbf{R}^n$ (without boundary) then the minimizing scan we obtain is in fact a current (that is it has finite mass). In general we prove that its underlying set M enjoys the following regularity: there exists an m dimensional properly embedded $C^{1,\alpha}$ submanifold $W \subset \mathbf{R}^n$ such that the Hausdorff dimension of the symmetric difference $M \triangle W$ is at most $m-1$.

4. Geometric Measure Theory in Singular Metric Spaces

There are several totally different approaches of the notion of rectifiability in singular metric spaces, in particular Carnot groups (for instance, Heisenberg groups). Three talks were about possible definitions:

- by B. Kirchheim (joint work with L. Ambrosio) in the setting of general metric spaces. For them, a Borel subset S of a metric space E is d-rectifiable if there exists a (countable) sequence of Lipschitz mappings $f_j : A_j \subset \mathbf{R}^d \to E$ such that $H^d(S / \cup_j f_j(\mathbf{R}^d)) = 0$. From this, they develop a rather complete theory of rectifiable sets. As applications, they get a version of the Rademacher theorem (differentiability of Lipschitz functions), area and co-area formulas, ... They also developed a theory of currents supported on rectifiable sets in metric spaces.

- by R. Serapioni (joint work with B. Franchi and F.Serra-Cassano) in the case of Heisenberg groups (and some special Carnot groups). For them, rectifiable sets in the Heinsenberg group are defined modulo a set of zero measure as subsets of the union of C^1-manifolds (with respect to the Carnot-Caratheodory structure of the group). As application, they get a version of the famous theorem of E. De Giorgi about sets of finite perimeter.

- by S. Pauls in the case of Carnot groups. In his definition, he replaces Lipschitz images of subsets of Euclidean spaces by Lipschitz images of some fixed subgroup of the original Carnot group.

In their talks, V. Magnani, P. Mattila J. Tyson discussed classical tools in (euclidean) geometric measure theory (as weak tangent measures, area and co-area formulas, ...) in the setting of Carnot groups.

It should be mentioned that there was a lot of discussions about this subject between the talks. This area of research is quite new and most of the definitions are not totally satisfactory.

Other classical problems of geometric measure theory (in Euclidean spaces) have been discussed by F. Germinet (comparison of dimensions), T. O'Neil (Visible sets), I. Laba (The Kakeya problem and related topics), N. Zobin (Whitney-type extension theorems for functions).