Neostability theory

Alf Onshuus (Universidad de los Andes),
Anand Pillay (University of Leeds),
Caroline Terry (Ohio State University),
Frank Wagner (Université Lyon I),
Thomas Scanlon (University of California, Berkeley)

February 19-24, 2023

1 Overview of the Field

"Neostability theory" , also called "Generalized stability theory" (a subfield of pure model theory characterized by the use of ideas and methods from stability theory in more general "tame" and sometimes "untame" settings) has become one of the main areas of pure model theory. The development of neostability has been propelled by both problems internal to the subject and applications to other branches of mathematics. The theory has achieved a high degree of coherence and technical results and constructions reappear in various contexts.

During the last decade the field has taken shape with many of the most promising young model theorists working around and making very important contributions. Such developments, which we could only hope for when we proposed the first BIRS meeting in 2009, have been influenced and driven in a very significant manner by the previous BIRS meetings in 2009, 2012, 2015 and 2018. It is a rapidly growing field, with researchers spread out across the world. The BIRS meetings have clearly helped bring everybody together, and we would like to continue to forge this development.

Starting in the late Sixties and for all of the Seventies and Eighties, stability theory played a central role in model theory. Inaugurated with Morley's celebrated theorem on theories categorical in an uncountable cardinality, the theory reached a high degree of sophistication with Shelah's classification theory and was further developed by Shelah, Lascar, Poizat and others into what has arguably been the deepest and most applicable of the branches of model theory.

Stability theory reached an apex with the geometric stability theory of, especially, Hrushovski and Zilber. In the late 1970s Zilber introduced the group configuration in his work on totally categorical theories; this was then taken up by Hrushovski and generalized well beyond these logically perfect theories. In so doing, it was revealed that structures of algebraic or algebraic geometric origin explain the complexity of some very general theories in which there is no apparent geometry or algebraic structure. This approach of analyzing a stable structure according to the geometry of the types was incredibly fruitful with results such as the trichotomy theorem for Zariski geometries by Hrushovski and Zilber and its applications to diophantine number theory (including the Mordell-Lang Conjecture for fields of positive characteristic) by Hrushovski and later Scanlon.

Since the early 2000's, it has been clear that fundamental ideas and objects of study of stability theory, such as definable types and forking, may be used in other contexts. These developments prompted us to

organize the first BIRS meeting on the subject in 2009. Since then, various subfields have taken shape with significant development and interesting results in all of them.

2 Recent Developments and Open Problems

Infinite methods for finite settings.

The theory of non forking is a tool that can be applied to a broad class of infinite structures which has manifested itself in disparate areas of mathematics. A good example of how to use infinite-based methods to shed light in finitary settings is Hrushovski's breakthrough in additive combinatorics which is now known as the "non-abelian Freiman theorem". Hrushovski used the idea that non forking could be developed relative to any good notion of smallness (in the sense of ideals or measure zero sets) together with a generalization of the generically presented group theorem to complete the proof. Applying (infinite) model theoretic techniques to problems in combinatorics (which are by essence finite contexts), was also the tool used by Malliaris and Shelah to obtain a stronger version of Szemerédi's Regularity Lemma for stable graphs, and generalized to a regularity lemma with respect to arbitrary Keisler measures by Malliaris and Pillay. Terry also has some very interesting extremal graph results using 0-1 laws of both Fraïssé limits and ultraproducts. Work of Chernikov and Starchenko showed that in many regularity theorems one needed some form of non stability, and that the use of the orderability of the real numbers was needed in the proof of the Erdős-Szekeres Theorem, highlighting the concept of "distality" as an important tool for applications of model theory to combinatorics. Terry and Laskowski made progress on a (labeled) counting problem for hereditary properties and mutual algebraicity, and Laskowski and Braunfeld connected monadic stability to an unlabeled counting problem for hereditary properties.

This is a very active field with new research by, among others, Chernikov, Conant, Peterzil, Pillay, Starchenko and Terry, among others. There is also a body of recent and ongoing work on so-called arithmetic regularity theorems (the structure of subsets of finite groups) initiated by Tao's arithmetic regularity lemma, often in tame settings.

Pseudofinite model theory: Pseudofinite model theory (the theory of ultraproducts of finite structures) has seen a very strong development in recent years. This already appears above in the combinatorics environment, and some of results and approaches can be seen as a combination of methods from nonstandard analysis and generalized stability. Pseudofiniteness of the triangular-free random graph (or the Urysohn sphere) is a long-standing open question, and possibly the work of Conant and Terry on the Urysohn space (where they characterized dividing and forking in continuous logic) may be an important tool to attack this problem. Kruckman gave a criterion for pseudofiniteness of a countably categorical theory. Finally, foundational work of García, Steinhorn and Macpherson gives necessary condition for stability or simplicity, and describes the relation between forking and dimension in pseudofinite structures. Among the recent work, we mention results of Bukh, Hrushovski and Zimmerman on proper intersections and modularity, Wagner on the existence of big abelian subgroups in pseudofinite groups with almost chain condition on centralisers, and García and Wagner on unimodularity in the stable context.

Dependent Theories: Dependent theories (also known as NIP) is probably the most robust setting outside stability theories. Since very early on we knew that the notions of definable types, non forking, heirs and coheirs were very important in this setting, and this provided many more examples of structures amenable to these model theoretic techniques (algebraically closed valued fields, p-adically closed fields and any o-minimal expansion of the real field are all examples of dependent theories). Foundational study of dependent structures has continued to expand, and our understanding has grown substantially. Shelah now has a "(re-)counting of types" characterization of dependent theories, while efforts towards understanding the different properties of non forking in dependent theories have continued with work by Chernikov, Kaplan, Simon and Usvyatsov, with a highlight in Simon's proof of a decomposition theorem for types in dependent theories into a stable and a distal-like part. More recently, Onshuus and Simon worked towards characterizing homogeneous dependent theories of finite th-rank, generalizing analogue results in the stable contexts by Cherlin, Harrington, Hrushovski and Lascar.

Other generalizations: The class of NTP_2 theories has established itself as a very adequate generalization of both simple and dependent theories. Recent work has shown that in many cases one can generalize common

results on simple and dependent theories to this context. Chernikov and Kaplan's proof of Kim's Lemma about the behavior of forking, Hempel's work on definable abelian "envelopes" of infinite abelian subgroups are good examples.

NTP_2 theories combine quite well when one assumes the existence of finitely additive measures on definable sets, is amenability, which has been particularly useful in studying the topological dynamics of both definable groups and automorphism groups. Definable amenability of a definable group G, meaning the existence of a translation invariant finitely additive probability measure on the definable subsets of G, played a decisive role in Hrushovski, Peterzil, and Pillay's proof of Pillay's Conjecture for definably compact groups in o-minimal expansions of the real field. The use of definable amenability was recently extended to NTP_2 theories where a stabilizer theorem was proved and used to show that amenable groups in geometric fields were isogenous to algebraic groups.

In a different direction, work of Chernikov and Ramsey give a criterion for NSOP_1 based on the existence of an independence relation satisfying independent amalgamation, a result which was improved recently by Ramsey showing that any NSOP_1 theory admits a symmetric notion of independence satisfying the independence theorem.

Mutchnink's equivalence.

Stable forking conjecture and related problems: One area in which we have not made any progress for more than a decade, is settling some of the open problems in simplicity theory. The stable forking conjecture (even in the supersimple case), elimination of hyperimaginaries in simple theories, the equivalence between forking and thorn forking are all questions that seem as far away from being solved as we were ten years ago.

3 Presentation Highlights

The talks were very well appreciated by the audience. We made a survey among participants to which 24 people replied. The summary of the answers is as follows:

"Which talks do you think presented the highlights of the meeting?", most replies included "too many" together with "talks were excelent" or "all talks were extremely good".

Out of the 32 speakers, 25 were mentioned as having presented highlights of the meeting, 14 of which were voted as highlights by at least 2 people.

The talks that were most mentioned by responders (together with what we as organizers thought were the highlights) were the results presented by Nick Ramsey, Scott Mutchnik, Ben Castle, and Artem Chernikov:

Scott Muchnik presented new results in understanding the NSOP_n-category. After proving the equivalence of NSOP_1 and NSOP_2, Mutchnik showed how one can construct generic expansions of theories with a ternary relation R associated with a Hrushovski group configuration using the geometry of the group configuration to find examples of strictly NSOP_4 theories. The equivalence of NSOP_3 and NSOP_1 is unknown.

Artem Chernikov spoke of higher classification theory, presenting his results (joint with Hempel) on n-stability, $-$-distality and n-dependence.

Nick Ramsey spoke of Kim-independence, a notion that has seen incresing amounts of studies and work since proveing its relevance in NSOP_1 theories.

Finally, Ben Castle spoke about his new approach to find group configurations in strongly minimal reducts of topological fields. His approximation already proved useful in showing Zilber's Trichotomy in an algebraically closed field K of characteristic zero when the universe of the strongly minimal reduct was not a curve in K, a case that had been open since Zilber proved the result for curves and stated his conjecture.

4 Scientific Progress Made

Many people reported starting collaborations, collecting interesting questions, or getting fresh insights on problems they were working on.

Some specific progress mentioned by attendees were:

- Starchenko, Peterzil and Chernikov reported progress on explicit Elekes-Szabo in arbitrary dimension.
- Kaplan Ramsey and Chernikov reported some progress on variants of treeless and C-less theories.

5 Outcome of the Meeting

As mentioned in the testimonials, mathematically we find this meeting to be one of the most stimulating, fruitful and productive meetings we have organized, or that we have been a part of. The excellent level of the talks, together with the interactions that happened in the meeting made it a very successful and enjoyable mathematical meeting for our field.

Beyond mathematics, we wanted to foster diversity and inclusion in our community. Diversity wise we were not so successful. Despite our efforts, late cancelations and travel issues and sometimes refusal to speak meant that only seven out of the 32 speakers were female. Also, the representation from outside Europe and North America was also less than what we had hoped for.

On the inclusion side, however, we found that the idea of the mentorship was very good and consequential. We assigned to every Ph.D. student and postdoc a mentor, and the interactions were great. In the survey we had replies from seven of the mentees and they were all extremely positive about the mentorship idea, one of them saying his/her conversations with the mentor was one of the highlights of the meeting. Some were suggesting assigning specific meeting times (we handled this very informally) something that we will consider doing in future meetings.

We also made big efforts in having the speakers feel they were in a safe environment. We actively worked on this through the moderation, which was done exclusively by the organizers. When we asked the participants in the survey about this, "Did you feel that during the talks speakers had an active interaction with the audience, and that they felt comfortable with this interaction?" we found that the opinions were split. The answers were numerical, graded 1-5 with with 5 being completely agree. 23 out of the 25 respondents graded with either 4 or 5, but one replied with a 2 and another with a 1. This of course means we still have to work on this. One of the participants suggested to have the speaker decide beforehand whether or not the speaker was willing to have questions from the audience during the talk, or only after the talk. We think this is a very good idea, (possibly with an in-between solution of allowing questions during specific pauses in the talk. One more thing we plan to implement in future meetings we organize.

So there are things to improve, as always, that we will take into account in future meetings. It was a very successful and well received conference, which brought together many different researchers. Both the testimonials and the very high acceptance rate we had in the invitations are both evidence of the importance of this meeting for the community.