Curves: Algebraic, Tropical, and Logarithmic

Yoav Len (University of St Andrews),
Hannah Markwig (University of Tübingen),
Dhruv Ranganathan (University of Cambridge)

August 6 to August 11, 2023

1 Overview of the Field

The last decade has seen significant progress in the geometry of algebraic curves, their Jacobians, and their moduli. Highlights include long-sought progress on Brill–Noether theory for special curves [13, 18], the determination of the Kodaira dimension of M_{22} and M_{23} [7], a resolution of the maximal rank conjecture [14], a definitive understanding of universal double ramification cycles, and the development of the logarithmic Picard group. These achievements have come from researchers in different fields: logarithmic geometry, tropical geometry, and limit linear series, whose overarching theme revolves around degenerations of algebraic varieties.

Limit linear series are a collection of techniques developed by Eisenbud and Harris [6] to study the manner in which linear series degenerate as algebraic curves degenerate. A number of fundamental results in algebraic curve theory have been proved in this fashion, chief among them the celebrated Brill–Noether theorem. Limit linear series continue to be a central tool in the study of curves. However, despite the success of limit linear series in studying general curves, there is significant interest in studying special loci in the moduli space of curves. For example, such a theory would be crucial in constructing Brill–Noether varieties over the compact moduli space of stable curves, which in turn constrains the topology and geometry of M_g .

Tropical geometry encodes limits of linear series by discrete data structures on the dual graph of a degeneration and is well-suited to study maximal degenerations of curves. The theory is powerful, and has led to new results in Brill–Noether theory for special curves [17], Prym–Brill–Noether theory [5], and the determination of the Kodaira dimension of M_g [12]. The theory predicts a number of consequences pertaining to dimensions of Prym– Brill–Noether loci and to semistable reduction for Brill–Noether varieties, providing new directions for the classical theory. Its precise relation to limit linear series is still poorly understood.

Logarithmic geometry is a powerful set of techniques that deal with degenerate objects in algebraic geometry, and has a rich interaction with tropical geometry [1]. Following the recent development of the logarithmic Picard group, it is expected that both classical and tropical limit linear series are specializations of a single object in logarithmic geometry [16]. Additional evidence is provided by recent connections between logarithmic enumerative geometry and mirror symmetry [9], and the role of logarithmic geometry in the study of the double ramification cycle [10]. In a different vein, Brill–Noether constructions give rise to geometrically significant cycles on the moduli space of curves M_g, parameterizing curves that admit certain special linear series. A study of the extension of these constructions to the boundary of M_g is a natural and important open problem and is likely to shed additional light on the cohomology of the moduli space of curves. As evidence, we note that a special case of these constructions includes the double ramification cycles discussed above, which determine nearly all that is currently known about the tautological ring of

$M_{g,n}$. These extension problems are typically well-served by logarithmic geometry methods, but it demands the development of a theory of logarithmic linear series.

These concrete questions, and others like it, provide fertile ground for cross-pollination of ideas, where experts from all three areas will have different perspectives. The tropical theory has been largely driven by applications to the geometry of curves, while logarithmic geometry has followed a ground-up conceptual view. By initiating a dialogue between these communities along with experts in the classical theory, we aimed to develop a shared understanding and appreciation of the recent breakthroughs and push for a new wave of developments in those areas.

2 Presentation Highlights

In order to give the participants a common ground for discussions, the first three talks gave an overarching view of the three main themes in the workshop: the classical, tropical, and logarithmic aspects of the geometry of curves. They were given by **Renzo Cavalieri**, **Nathan Pflueger**, and **Isabel Vogt**.

The introductory talks were pitched broadly, but included recent research breakthroughs as well. For example, Cavalieri explained how logarithmic intersection theory gave rise to a new and natural perspective on Hurwitz theory, as well as natural generalizations of it; Pflueger overviewed the basic techniques in the theory of tropical linear series, and also gave the first new construction of a family of Brill–Noether general tropical curves in a decade; Vogt introduced different classical perspectives on Brill–Noether theory, but presented a striking new proof of the Brill–Noether theorem based on stable map and deformation theoretic techniques, well-adapted to generalizations.

The first full-fledged research talk was given by **Diane Maclagan**, who spoke on the topic of toric and tropical vector bundles. The topic turned out to have a number of topics touched upon in talks and discussions later in the conference.

The Tuesday began with the talk of **Eric Larson** who discussed joint work with Isabel Vogt solving the famous interpolation problem for Brill–Noether curves. The solution brings to an end a long line of inquiry, which Larson and Vogt have both been involved in. The techniques involved degeneration and the study of vector bundles on singular curves, and was a strong fit for the themes of the conference.

In the second Tuesday talk was given by **Jonathan Wise**, who gave an introduction to his forthcoming work with Battistella–Carocci on moduli of linear series, beyond the compact type case, using logarithmic geometry. Again, vector bundles on degenerate objects (in this case, Artin fans of curves) played a crucial role. The talk would turn out to be closely related to the one given by Payne later in the conference.

Later on Tuesday **Navid Nabijou** spoke on the relationship between logarithmic and orbifold Gromov–Witten theory, reporting on his recent joint work with Battistella–Ranganathan. The work tied into Cavalieri's introductory lecture on the first day. The final full talk on Tuesday was given by Gavril Farkas, who reported on his recent work with Jensen–Payne, solving several problems concerning the Kodaira dimensions of moduli spaces, using tropical geometry. Farkas raised non-abelian Brill–Noether theory as an avenue for exploration through tropical geometry, where many fundamental open problems remain.

In addition to these, there were four talk by junior participants on Tuesday: **Shiyue Li, Siddarth Kannan, Patrick Kennedy-Hunt, and Alheydis Geiger**. These covered a range of topics, spanning tropical geometry, logarithmic geometry, matroids, and the moduli space of curves. It gave these early participants an opportunity to introduce themselves to the audience and served as a starting point for conversations later in the week.

The first talk on Wednesday was given by **Felix Röhrle** on the topology of tropical moduli spaces of p-cyclic covers. The line of inquiry is a natural and rich continuation of work of Chan–Galatius–Payne on the topology of spaces of tropical curves, in turn leading to several new results on the cohomology of the moduli space of curves.

Later on Wednesday, **Samouil Molcho** discussed logarithmic intersection theory in the context of the universal Picard variety, and especially his joint work with Abreu and Pagani, who were both in attendance. Molcho's talk highlighted the new input of logarithmic geometry in the study of compactified Picard schemes. The latter has been a rich line of inquiry in the study of curves in the 20th century.

The final two talks on Wednesday concerned the arithmetic of curves and their Jacobians, and were given by **Padmavathi Srinivasan and Farbod Shokrieh**. Srinivasan's talk concerned recent progress in non-abelian Chabauty-Kim methods concerning rational points on curves, where the work of Litt–Katz on iterated p-adic integrals plays an interesting role. Shokrieh's talk concerned the theory of heighs on abelian varieties, and his work with R. de Jong using electrical network theory and tropical geometry to give a new explanation of the difference between different heights.

Wednesday also hosted a problem session, chaired by **Dan Abramovich**. Several interesting problems were put forward by experts, and these stimulated significant further discussion. A number of researchers put forward problem, ranging from concrete questions to inspiring speculations. For example, **Isabel Vogt** proposed interesting and fundamental questions about the classical curve theory, while **Dhruv Ranganathan** suggested a systematic study of the cohomology of the space of parameterized curves in projective space. Several smaller groups of researchers informally continued discussions after the session, late into the evening.

There were two talks each on Thursday and Friday. On Thursday, **Alex Abreu** discussed additional new results on the intersection theory on universal Picard varieties. Specifically, he reported on his recent work with Pagani, giving significant new insight into the structure of Brill–Noether classes on the universal Jacobian. The talk tied in with those of Cavalieri and Molcho earlier in the week. The second talk on Thursday was given by **Sam Payne**. He reported further aspects of his work with Farkas–Jensen, which Farkas had already spoken about. While Farkas's talk explained the context for the results, Payne's talk gave a detailed account of how calculations are actually done in their work, how tropical geometry enters the picture, and how the resulting tropical problems can be solved. The remainder of Thursday was taken up by the hike and evening discussions.

The final two talks were given by **Margarida Melo** and by **David Holmes**. Melo spoke about joint work with Molcho, Ulirsch, Viviani, and Wise on tropical universal Jacobians. This gave a complete account of the combinatorics lurking behind the universal logarithmic Jacobian, the compactified Jacobians, and the relationship between them. The talk nicely complemented those of Abreu and Molcho. Holmes discussed gluing maps for curves in tropical geometry, especially in the context surrounding logarithmic enhancements of CohFTs and punctured logarithmic curves due to Abramovich–Chen–Gross–Siebert.

3 Recent Developments and Open Problems

In the last five years, tropical geometry has matured significantly as a research area. The period from 2010–2015 saw the subject produce elegant new combinatorial proofs of results in classical geometry, including the Brill–Noether and Gieseker–Petri theorems [4, 11] and the enumerative geometry of plane curves [8]. However, recent contributions have gone far beyond what was previous known. These include developments in Hurwitz–Brill–Noether theory [13, 3], the determination of the Kodaira dimension of M_{22} and M_{23} [7], and advances in the cycle theory of Prym varieties [5]. Each of these developments have been anchored by theoretical advances from logarithmic geometry and Berkovich geometry. There has been remarkable contemporaneous progress in the classical side. For example, H. Larson developed a systematic study of degeneracy loci on $\mathbb{P}^1$-bundles [15] in order to address questions raised in Jensen and Ranganathan's work in Hurwitz–Brill–Noether theory, and E. Larson used Hilbert scheme and stable map techniques to resolve the maximal rank conjecture [14]. In another example, the search for a robust formalism for tropical Jacobians has led to the development of the logarithmic Picard group and related developments on the double ramification cycle [16, 10]

There are many overlapping themes and analogies among the breakthroughs on the two sides, and what appears to be missing is a bridge linking them together. For instance, there is no understanding of the relationship between the algebraic and tropical Riemann–Roch theorem [2], and the situation is similar for the tropical notion of the rank of a divisor. In light of recent developments in the tropical, logarithmic, and classical study of curves, it seems particularly timely to combine the expertise across these research communities.

4 Outcome of the Meeting and Scientific Progress Made

As mentioned earlier in this document, the conference brought together experts from three distinct communites, linked by their common interest in the theory of curves – logarithmic, classical, and tropical geometry. These communities each have expertise that promised to benefit the others, and we believe this promise was fulfilled. For example, the formalism and framework of logarithmic geometry is well-developed and gives a robust connection between tropical and algebraic geometry. At the same time, the expertise of tropical geometers in combinatorial aspects of the subject have helped clarify and direct developments in logarithmic geometry. Both these subjects have in turn, given new input to the classical theory of curves, as for example advanced in the talks of Farkas and Payne.

The lectures described in the previous section were well-received, and the introductory talks on the first day created a common language for the rest of the conference. The time between and after the lectures gave time for genuine discussions, and several researchers mentioned important conversations. There were also intangible benefits – community building and community linking, between these three groups. For example, strong connections appear to have been discovered between work of Battistella–Carocci–Wise and Jensen–Payne that will be explored at a future workshop, that may not have been discovered for a long time as the relevant papers are not yet public. Similarly, Kennedy-Hunt, one of the early career participants, had the chance to discuss the results of his PhD, on logarithmic moduli theory, with Jonathan Wise, who is one of world experts in this area.

The conference also included a mentorship activity. The participants were broken into groups of 4-5, with each group including a range in levels of seniority. The groups were encouraged to discuss some aspects of the profession. The groups were given a list of potential topics to discuss, such as what makes good research problems, how to deal with rejection, and how to transition between different phases of one's career. The organizers received positive feedback on the activity, with several participants noting the freedom to talk more freely in smaller groups than they might have in a larger activity.

Testimonials

Sam Payne said "I found the combination of different perspectives on algebraic curves represented at the conference especially useful. It was illuminating to see the commonalities between the logarithmic Picard theory that Jonathan Wise et al. are developing and the notions of tropical linear series that I have been considering with Farkas and Jensen. The most likely possibility for a new collaboration would be something with Dhruv, using tropical linear series and connections to log geometry to study the Castelnuovo Problem on the existence of smooth projective space curves of a given degree and genus."

Navid Nabijou said "The conference was one of the best I've been to. The quality of the talks was very high, with a great mix of topics. For me personally it was very stimulating, and I've come away with a couple of nascent collaborations. The mentoring activity worked extremely well. "

Alheydis Geiger said "I really enjoyed the conference! Here a few comments from me: The reflection groups were a really good idea, they provided an excellent opportunity to discuss questions within different career stages and get new perspectives and experience reports from others. The way you had set up the groups was also very good - to me it felt that you took pains to mingle the people, so that one would speak with people one probably hadn't approached with similar topics before. That was really great. The organization was good and smooth- only the "get-together part" on the first evening could maybe have happened in a more "mingle-friendly" environment, but I understand that the evening had been planned differently originally."

Siddarth Kannan said "As an early career mathematician, the BIRS meeting on curves provided a valuable opportunity to discuss a broad range of topics in algebraic geometry with potential collaborators at various career stages. These discussions led to two concrete research directions that I hope to explore further during my postdoctoral years: the weight filtration on the cohomology of moduli spaces of maps from genus one curves to projective space, and the topology of logarithmic Hilbert schemes. I was also able to present my past research on moduli spaces of weighted stable curves, and this led to several additional interesting conversations with workshop participants."

References

[1] Dan Abramovich, Qile Chen, Steffen Marcus, Martin Ulirsch, and Jonathan Wise. Skeletons and fans of logarithmic structures. In *Nonarchimedean and tropical geometry*, pages 287–336. Springer, 2016.

[2] M. Baker and S. Norine. Riemann–Roch and Abel–Jacobi theory on a finite graph. *Adv. Math.*, 215(2), 2007.

[3] K. Cook-Powell and D. Jensen. Tropical methods in Hurwitz–Brill–Noether theory. *arXiv:2007.13877*, 07 2020.

[4] F. Cools, J. Draisma, S. Payne, and E. Robeva. A tropical proof of the Brill-Noether theorem. *Adv. Math.*, 230(2), 2012.

[5] S. Creech, Y. Len, C. Ritter, and D. Wu. Prym–Brill–Noether loci of special curves. *Int. Math. Res. Not. IMRN*, 08 2020. rnaa207.

[6] David Eisenbud and Joe Harris. Limit linear series: basic theory. *Invent. Math.*, 85(2), 1986.

[7] G. Farkas, D. Jensen, and S. Payne. The Kodaira dimensions of $\overline{M}_{22}$ and $\overline{M}_{23}$. *arXiv:2005.00622*, 2020.

[8] A. Gathmann and H. Markwig. The numbers of tropical plane curves through points in general position. *J. Reine Angew. Math.*, 602:155–177, 2007.

[9] Mark Gross and Bernd Siebert. Intrinsic mirror symmetry. *arXiv preprint arXiv:1909.07649*, 2019.

[10] David Holmes, Samouil Molcho, Rahul Pandharipande, Aaron Pixton, and Johannes Schmitt. Logarithmic double ramification cycles. *arXiv:2207.06778*, 2022.

[11] D. Jensen and S. Payne. Tropical independence I: Shapes of divisors and a proof of the Gieseker-Petri theorem. *Algebra Number Theory*, 8(9), 2014.

[12] D. Jensen and S. Payne. Tropical independence ii: The maximal rank conjecture for quadrics. *Algebra Number Theory*, 10(8):1601–1640, 2016.

[13] D. Jensen and D. Ranganathan. Brill-Noether theory for curves of a fixed gonality. *Forum Math. Pi*, 9, 2021.

[14] E. Larson. The maximal rank conjecture. *arXiv:1711.04906*, 2018.

[15] Hannah Larson. A refined Brill–Noether theory over Hurwitz spaces. *Invent. Math.*, 11 2020.

[16] Samouil Molcho and Jonathan Wise. The logarithmic Picard group and its tropicalization. *arXiv:1807.11364*, July 2018.

[17] Nathan Pflueger. Brill-Noether varieties of k-gonal curves. *Adv. Math.*, 312, 2017.

[18] Nathan Pflueger. Special divisors on marked chains of cycles. *Journal of Combinatorial Theory, Series A*, 150, 2017.