Smooth Functions on Rough Spaces and Fractals with Connections to Curvature Functional Inequalities

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1 Overview of the Field

The study of smooth functions on a metric measure space (X, d, μ) can be undertaken in two general settings. In both there are circumstances under which one obtains a Dirichlet form, which allows for the possibility of studying curvature via functional inequalities in the sense of Bakry and Émery. With regard to curvature, the theory is quite well-developed in one case and in its infancy in the other, so it makes sense to describe the situations separately even though the purpose of the workshop was to bring people from these fields together to learn from one another and consider problems of mutual interest.

First order calculus and Dirichlet forms on spaces with many rectifiable curves

Taking the perspective that a function $f : X \to \mathbb{R}$ can be thought of as smooth if it can be paired with some sort of "gradient object" g such that the pair satisfies some familiar estimates from calculus leads to function spaces analogous to classical Sobolev spaces. Various types of gradient are considered in the literature, involving limits of L^p norms of difference quotients (Korevaar-Schoen spaces), an analogue of the maximal function of the classical gradient (Hajłasz-Sobolev spaces), a notion of the size of the gradient as measured through the fundamental theorem of calculus on rectifiable curves (Newtonian Sobolev spaces), and relaxation of L^p norms of local Lipschitz constants (Cheeger Sobolev spaces). In each case one should think of the Sobolev space as having norm $\|f\|_p + \|g\|_p$.

In order to obtain a good first order calculus from the preceding construction, one requires that the underlying space contains suffciently many rectifiable curves. The standard condition for this, which has become central to the field, is the validity of a p-Poincaré inequality, which says that there is $\eta \geq 1$ so that for all balls B, integrable functions f and upper gradient g of f

$$\frac{1}{\mu(B)} \int_B |f - f_B| d\mu \leq C \mathrm{diam}(B) \left(\frac{1}{\mu(\eta B)} \int_{\eta B} g^p d\mu \right)^{1/p} \tag{1}$$

where f_B is the average over B and ηB is the ball concentric with B but having radius increased by the factor η. It is an important result of Keith and Zhong [23] that if X is complete, μ is doubling and (X, d, μ) satisfies a p-Poincaré inequality for some $p > 1$ then it also satisfies a q-Poincaré inequality for some $1 \leq q < p$. Using this it is possible to show that these hypotheses ensure all the preceding constructions of Sobolev spaces coincide. Moreover, it then follows from the work of Cheeger [13] that the spaces have a tangent structure and that Lipschitz functions are Cheeger differentiable μ-a.e. (an analogue of Rademacher's theorem). Just as an upper gradient can be considered an analogue of the size of the gradient, the Cheeger differential is an analogue of the gradient itself.

Since a Dirichlet form is an abstract version of the classical object $\int |\nabla f|^2$ it is natural to think that an L^2 Sobolev space constructed by the above procedure should provide a Dirichlet form simply by taking the seminorm $\|g\|_2$ with domain the Sobolev space. However, doing this does not ensure the space has a Hilbertian structure with respect to f, or even that the space is reflexive. The reason is that g was assumed to record some notion of the size of a gradient, but this size need not correspond to an L^2 norm. It is an open problem to determine optimal conditions under which Sobolev spaces arising from this construction are infinitesimally Hilbertian, for a positive result using a lower bound on Bakry-Émery curvature see [26]. A positive result is that when one has a 2-Poincaré inequality and a Cheeger differential then the L^2 norm of this differential is a strongly local regular Dirichlet form with domain the Sobolev space, see [30].

Calculus on spaces without many rectifiable curves

In cases where (X, d) does not have many rectifiable curves and there is no Poincaré inequality the preceding approach is not productive because the Sobolev spaces are degenerate, sometimes even coinciding with L^p. However, it is sometimes still possible to directly construct a local regular Dirichlet form $\mathcal{E}$ and its domain $\mathcal{F}$ without going via a notion of gradient. This approach has been particularly fruitful on highly symmetric self-similar fractal spaces. Initially, all such constructions were probabilistic: researchers constructed diffusions on fractal sets to investigate their anomolous properties [16, 10, 6]. This produced Dirichlet forms by well-established functional analytic methods, see [15]. Probabilistic constructions remain the primary way to produce Dirichlet forms on infinitely ramified self-similar sets, but on post-critically finite self-similar sets (which are typically gaskets like the Sierpinski gasket) there is a more elementary approach due to Kigami [24] that constructs a Dirichlet form as a suitably renormalized limit of Dirichlet forms on finite graphs.

Higher order and fractional smoothness

In either setting, once one has a local regular Dirichlet form it is standard to consider not only the associated Markov process but the corresponding (heat) semigroup P_t and its generator $\mathcal{L}$, which is non-positive and self-adjoint. These satisfy $P_t = e^{t\mathcal{L}}$ and $\mathcal{E}(f, g) = -\int f\mathcal{L}g$ for suitable classes of functions. On Euclidean spaces $\mathcal{L}$ is the (negative) Laplacian, and it is well known that a complete description of the usual Sobolev and Besov spaces in this context can be defined from the Laplacian via spectral theory and interpolation [33]. A parallel approach based on $\mathcal{L}$ can be used to define spaces of smooth functions on metric measure spaces with a local regular Dirichlet form.

Differences between the settings

It should be emphasized at this point that although one has a local regular Dirichlet form in both the rectifiable and the fractal settings we have described, the properties of these forms and their domains are rather different. One difference is seen in the walk dimension, which is an exponent β measuring the rate at which the diffusion P_t spreads through the space. If P_t has an integral kernel $p_t(x, y)$ then under fairly mild conditions (see [19] Theorem 4.1) one has bounds of the form

$$C_1 t^{-\alpha/\beta}\Phi\Big(c_1 \frac{d(x,y)}{t^{1/\beta}}\Big) \leq p_t(x,y) \leq C_2 t^{-\alpha/\beta}\Phi\Big(c_2 \frac{d(x,y)}{t^{1/\beta}}\Big)$$

with $\Phi(s) = \exp\big(s^{\beta/(\beta-1)}\big)$. In the rectifiable analysis setting one always has $\beta = 2$, while on fractals $\beta > 2$ is typical; α is the Hausdorff dimension and it is known both that $2 \leq \beta \leq \alpha + 1$ and all pairs (α, β) satisfying this inequality can occur [5]. Another difference is seen in the structure of energy measures. According to the seminal work of Beurling and Deny, corresponding to bounded $f, g \in \mathcal{F}$ one can introduce a measure $d\Gamma(f, f)$ via

$$\int g d\Gamma(f,f) = \mathcal{E}(fg, f) - \frac{1}{2}\mathcal{E}(f, g^2)$$

by virtue of the regularity assumption that $\mathcal{F}$ is dense in $C(X)$ in the uniform norm. A recent result of Kajino and Murugan [21] establishes that having walk dimension $\beta = 2$ is equivalent to absolute continuity of all $d\Gamma(f,f)$ with respect to μ. Earlier results of Kusuoka [27] and Hino [20] established that $d\Gamma(f,f) \perp d\mu$ was the case in many fractal examples. In turn, this property is closely connected to a significant property of smooth functions, an elementary proof of which was given by Ben-Bassat, Strichartz and Teplyaev for the case of the Sierpinski gasket [11]: functions in the domain of the generator $\mathcal{L}$ do not form an algebra. The connection to the singularity of energy measures is that (modulo some technicalities) the Beurling-Deny-LeJan theorem may be applied to compute $\mathcal{L}(fg) = f\mathcal{L}g + g\mathcal{L}f + 2d\Gamma(f,g)$, which is a measure singular to μ. Strichartz [34] has identified a connection to the fact that $\mathcal{L}$ does not act as a 2nd order operator on the Hölder-Lipschitz scale of spaces. We close this section by noting that the replacement for the $p = 2$ Poincaré inequality (1) in such fractal settings is

$$\int_B |f - f_B|^2 \le C(\operatorname{diam}(B))^\beta \int_{\eta B} d\Gamma(f,f).$$

Curvature

In the situation where $d\Gamma(f,f)$ is absolutely continuous with respect to μ we abuse notation to simply write $\Gamma(f,f)$ for the Radon-Nikodym derivative and use polarization to define $\Gamma(f,g)$ (which could have also been defined by polarizing the definition from $\mathcal{E}$). Notice that classically, $\Gamma(f,g)$ is the scalar product of ∇f and ∇g, and that on a manifold the scalar products encode the Ricci curvature. For this reason, Bakry and Émery introduced the form

$$\Gamma_2(f,f) = \frac{1}{2}\mathcal{L}\Gamma(f,f) - \Gamma(f,\mathcal{L}f) \tag{2}$$

and defined curvature dimension spaces $\mathrm{CD}(K,d)$ with curvature K and dimension d to be those for which

$$\Gamma_2(f,f) \ge K\Gamma(f,f) + \frac{1}{d}(\mathcal{L}f)^2. \tag{3}$$

It is a result of Ambrosio, Gigli and Savaré [2] that this condition is equivalent to the definition of synthetic curvature given in terms of displacement convexity of the entropy functional in the 2-Wasserstein space due to Sturm [36, 35] and Lott and Villani [29], thereby giving access to results proved by methods of optimal transport. The achievements in describing the geometry of metric measure spaces using synthetic curvature are too many to consider here; we refer to [37] for further information and references.

Having said that this approach to curvature is very successful, it should immediately be noted that in fractal settings where $d\Gamma(f,f)$ is singular to μ one cannot define Γ_2 by (2). It is not known whether there is a version of Γ_2 or the curvature condition (3) in these settings but, given the apparent difficulties of devising such, it is useful to know that that there are weaker curvature notions that do not involve Γ_2. Some examples of conditions equivalent to the $\mathrm{CD}(K,\infty)$ condition that only involve Γ are described in Proposition 3.3 of [3]. An even weaker condition of this type was introduced in [1] as a Hölder smoothing property of the heat kernel; such estimates are available in the setting of fractal spaces with fractal diffusion established by Barlow [4].

The workshop was targeted at the nexus of the preceding areas of study, providing experts the opportunity to share techniques from recent work and early-career faculty and students the chance both to learn about the foundations of these disciplines, discover open problems, and talk about their own work to senior researchers in the fields.

2 Recent Developments and Open Problems

As the workshop lay at the meeting point of several fields we summarize some results and open problems that involve at least pairwise intersections of them. Many of these were presented and discussed at the workshop.

Linear structure of $W^{1,p}$ in the rectifiable analysis setting

It was mentioned above that a notable obstruction to obtaining a Dirichlet form in the setting where one has rectifiable analysis is that the space $W^{1,2}$ may fail to be infinitesimally Hilbertian. More generally, one is interested in knowing when there is a tangent linear space such that the upper gradient g is given by a norm of a gradient ∇f. Significant recent progress on this problem has been made by Eriksson-Bique and Soultanis, who provide a notion of a p-weak chart that exists without the assumption of a Poincaré inequality and coincide with Cheeger's Lipschitz charts when a Poincaré inequality is valid. This is achieved by taking a tangent space defined in terms of directions of a.e. rectifiable curve through the base point. Significant open problems include generalizing to the case $p = \infty$, determining how this structure behaves under tensorization, and understanding the properties of tangent space directions in general spaces.

2.0.1 Correspondence between metric and Dirichlet form

We have described above how one might go about constructing a Dirichlet form from the metric measure structure of (X, d, μ). This is successful in a complete separable metric space on which the measure is semilocally finite and satisfies some growth condition, provided that the resulting form is infinitesimally Hilbertian. Problems include determining when the semilocally finite assumption is necessary and providing a geometric characterization for the infinitesimal Hilbertian condition. Conversely, if energy measures are absolutely continuous with respect to μ then one can define the intrinsic metric from the Dirichlet form $\mathcal{E}$ by setting $d_{\mathcal{E}}(x,y) = \sup\{f(x) - f(y) : f \in \mathcal{F} \cap C(X),\, |\Gamma(f,f)| \leq 1\}$. It is then natural to ask: if we begin with (X, d, μ), construct $\mathcal{E}$ and then $d_{\mathcal{E}}$, when does $d = d_{\mathcal{E}}$?, and if we begin with $(X, \mathcal{E}, \mu)$, construct $d_{\mathcal{E}}$, and then use $(X, d_{\mathcal{E}}, \mu)$ to build a new form $\tilde{\mathcal{E}}$, when does $\mathcal{E} = \tilde{\mathcal{E}}$?

The first question has a fairly satisfactory answer: provided that the form $\mathcal{E}$ is infinitesimally Hilbertian and given some mild condition on the measure one has always $d_{\mathcal{E}} \geq d$ and equality holds if there is an embedding $f \in \mathcal{F} \cap C(X)$ and $|\Gamma(f,f)|$ bounded implies f is Lipschitz. For the second condition there is a sufficient condition from the work of Ambrosio Gigli and Savaré, which in particular guarantees that it is sufficient that one has the preceding Sobolev to Lipschitz embedding without the a-priori assumption of continuity and a Bakry-Émery curvature lower bound of the form (3). An optimal condition is not known.

Conformal structure and quasisymmetric deformation on fractal spaces

One of the most natural problems in the circumstances we have described is the question of when the theory in one framework is equivalent to that in another. A concrete realization of this is to consider how a notion of smoothness transforms under suitable maps between metric spaces, and to ask whether or when it is possible to transform, for example, from the fractal analysis setting to that where a rectifiable analysis applies. Kigami showed this was possible for the Sierpinski gasket using a harmonic parametrization [25]. Significant recent progress on problems of this type has been achieved by Kajino and Murugan [22]. They use the walk dimension β, which always satisfies $\beta \geq 2$ and has $\beta = 2$ in the case where the space admits a rectifiable analysis, as a measure of how different the notion of smoothness is on a fractal to that in the rectifiable setting. They then show that a space on which initially $0 < \beta < 2$ can be quasisymetrically deformed to have β arbitrarily close to 2. They also show examples in which one cannot achieve $\beta = 2$ by this kind of deformation. The same authors have investigated the connection between the walk dimension and the singularity or absolute continuity of energy measures to the reference measure μ [21]; this work is connected to investigations of the geometric nature and stability of the Harnack inequality [9] and to the general question of finding the "correct" conformal structure on these types of spaces. Among the open problems in this area is the need for a more detailed understanding of boundary behavior of harmonic functions, especially in infinitely ramified fractals, as this is implicated in the the failure of the deformation to achieve $\beta = 2$.

Geometry of Harnack inequalities and boundary Harnack principle

A core estimate in Dirichlet spaces is the elliptic Harnack inequality (EHI), which says that if h is harmonic on a ball $B(x, 2R)$ then $\operatorname{ess\,sup}_{B(x,R)} h \leq C \operatorname{ess\,inf}_{B(x,R)} h$ for some constant C depending only on the Dirichlet space, not the function h. For divergence form elliptic PDE this is due to Moser, who also proved the closely related parabolic Harnack inequality (PHI) for solutions of the associated heat equation. Other well-known approaches to this result are due to DeGiorgi and to Nash. The PHI for a Laplace-Beltrami operator on a Riemannian manifold was given a metric measure characterization as the conjunction of volume doubling and Poincaré inequality by Grigory'an [17] and by Saloff-Coste [32]; subsequently this was extended to more general settings in several ways [7, 8, 18] that involve a third condition on the existence of suitable cutoff functions. More recently, Barlow and Murugan [9] recognized that the possibility of changing the measure, and thus altering the PHI without altering the harmonic functions, can be used to give a deeper understanding of spaces that have EHI but not PHI. They developed these ideas to prove stability of the EHI under rough isometries and give a geometric characterization of the EHI under certain local regularity assumptions.

The boundary Harnack principle (BHP) states that positive harmonic functions u, v in a domain Ω that vanish on the portion $\partial\Omega \cap B(z, 2R)$ of the boundary must satisfy $\frac{u(x)}{v(x)} \leq C\frac{u(y)}{v(y)}$ for x, y in $\Omega \cap B(z, R)$. This inequality is crucial for understanding boundary values of elliptic PDE, and has a long history. Recent work of Barlow, Chen and Murugan shows that the EHI implies a BHP on inner uniform domains, but also showed that the EHI is not necessary for the BHP, leaving open the question of which diffusions admit a BHP. BHPs for processes involving both diffusions and non-local operators such as Levy processes are also an active area of study.

Nonlinear potential theory

The linear potential theory of Dirichlet forms is a long-studied area with a rich and detailed theory, particularly as it interacts with probabilistic notions [15, 14]. In Euclidean spaces the same is true of the non-linear potential theory corresponding to the p-energy $\int |\nabla f|^p$, and much of this has been extended to the setting of rectifiable analysis on metric measure spaces [12]. For example, p-harmonic functions are known to be Hölder continuous, satisfy comparison principles, have a Harnack inequality and a Liouville type theorem. They can be constructed by a Perron method from boundary data, though determining the "weakest" possible assumptions on the boundary data is an open problem. There are also open problems around removability of sets for these function spaces, many of which remain open even in the Euclidean setting.

In fractal spaces, there are at least two recent approaches to p-energy and non-linear potential theory. Kigami has generalized both his successful earlier approach to 2-energy (Dirichlet form) via graph approximations, as well as the local Poincaré type conditions from the work of Kusuoka and Zhou [28] to a construction of p-energy under two hypotheses: p is greater than the Ahlfors regular conformal dimension (which is conjectured to be the critical exponent for Sobolev embedding of the p-energy space into the continuous functions) and X is p-conductive homogeneous. The latter condition can be verified in a wide variety of fractal examples using curve modulus techniques. There are many open questions, including what should be done when p is less than the Ahlfors regular conformal dimension, whether the p-energy space admits a core and whether it can be characterized as a Korevaar-Schoen space, whether there are p-energy measures, what structural information (such as Harnack inequalities) can be deduced about p-harmonic functions, and to what extent the construction might depend on how the approximating graphs are constructed from a partition of the space. Workshop participants are also investigating when it is possible to (weakly) define a p-Laplacian operator

At the same time, Baudoin and co-authors have developed an approach in which the p-energy space is a Korevaar-Schoen space at the critical exponent for which such spaces contain non-constant functions. A key ingredient in this approach is the notion of an L^p mean value inequality, which provides that the Korevaar-Schoen norm controls L^p norms of difference quotients at all scales. Under this assumption there are Sobolev embedding theorems [31] and a Gagliardo-Nirenberg theorem (Baudoin). Again there are many open questions. They include whether the critical Korevaar-Schoen exponent is independent of the Hausdorff and walk dimensions or has a connection to the Ahlfors regular conformal dimension, whether the L^p mean value inequality can be localized to a Poincaré-type inequality, and how these notions behave under standard methods to construct new spaces from old spaces (e.g. gluing, products, cones, Gromov-Hausdorff limits). We also note that in the case $p = 1$ the L^1 mean value inequality was deduced from a weak variant of a Bakry-Émery curvature-type inequality in [1], indicating that curvature functional inequalities might provide structural information about p-energy spaces; the extent to which these concepts are related more generally remains to be explored.

3 Presentations

Fabrice Baudoin: Korevaar-Schoen-Sobolev spaces and critical exponents on metric measure spaces. We review some of the recent developments in the theory of Korevaar-Schoen-Sobolev spaces. While this theory is equivalent to that of Cheeger and Shanmugalingam if the space supports a Poincare inequality, it offers new perspectives in situations, like fractals, where such inequalities are not available.

Jana Björn: Potential theory, p-harmonic and Green functions on metric spaces.
A review of some achievements of nonlinear potential theory in the setting of metric measure spaces with Poincaré inequality.

Shiping Cao: Dirichlet forms on unconstrained Sierpinski carpets.
We construct symmetric self-similar Dirichlet forms on unconstrained Sierpinski carpets, which are natural extension of planar Sierpinski carpets by allowing the small cells to live off the $1/k$ grids. The intersection of two cells can be a line segment of irrational length, and we also drop the non-diagonal assumption in this recurrent setting. A uniqueness theorem is also provided. Moreover, the additional freedom of unconstrained Sierpinski carpets allows us to slide the cells around. In this situation, we view unconstrained Sierpinski carpets as moving fractals. We prove that the self-similar Dirichlet forms will vary continuously in a Γ-convergence sense, and the generated diffusion processes, viewed as processes in $\mathbb{R}^2$, will converge in distribution. This is a joint work with Hua Qiu.

Li Chen: Poincaré inequalities on the Vicsek set.
The Vicsek set is a tree-like fractal on which neither analog of curvature nor differential structure exists, whereas the heat kernel satisfies sub-Gaussian estimates. I will talk about Sobolev spaces and scale invariant L^p Poincaré inequalities on the Vicsek set. Several approaches will be discussed, including the metric approach of Korevaar-Schoen and the approach by limit approximation of discrete p-energies.

Zhen-Qing Chen: Boundary Harnack principle for non-local operators on metric measure spaces.
It is well known that scale invariant boundary Harnack inequality holds for Laplacian Δ on uniform domains and holds for fractional Laplacians Δ^s on any open set. It has been an open problem whether the scale-invariant boundary Harnack inequality holds on bounded Lipschitz domains for Levy processes with Gaussian components such as the independent sum of a Brownian motion and an isotropic stable process (which corresponds to $\Delta + \Delta^s$). In this talk, I will present a necessary and sufficient condition for the scale-invariant boundary Harnack inequality to hold for a class of non-local operators on metric measure spaces. This result will then be applied to give a sufficient geometric condition for the scale-invariant boundary Harnack inequality to hold for subordinate Brownian motion on bounded Lipschitz domains in Euclidean spaces. A counter-example will be given showing that the scale-invariant BHP may fail on some bounded Lipschitz domains with small Lipschitz constants. Joint work with Jie-Ming Wang.

Stathis Chrontsios-Garitsis: Fractals under quasiconformal maps.
There are various dimension notions that are used to distinguish different fractals. Some depend on measures (e.g. Hausdorff and packing dimension) and others depend only on the metric of the space (e.g. box-counting and Assouad dimension). Even when considering all these notions, however, they might not be enough to distinguish or classify certain fractals. In such situations, it is useful to consider a collection of dimensions instead, known as a dimension spectrum. In this talk, we will present how the Assouad spectrum of a given set changes under quasiconformal maps and use this result to quasiconformally classify polynomial spirals, which would not be possible considering only the Hausdorff, box-counting and Assouad dimension notions. This talk is based on joint work with Jeremy Tyson.

Simone Di Marino: Sobolev and BV spaces on metric measure spaces: a review.
A review of foundational results regarding Sobolev and BV functions in the metric measure setting.

Estibalitz Durand-Cartagena: Basics of Lipschitz analysis in metric spaces.
We review some Lipschitz-type results in connection with geometric properties and differentiable structures of metric measure spaces.

Sylvester Eriksson-Bique (Part 1) and Elefterios Soultanis (Part 2): p-weak differentiable structure on metric spaces.

A fundamental question in analysis on metric measure spaces is when $W^{1,p}$ has a linear structure. Cheeger resolved this under the assumption of measure doubling and p-Poincaré inequality for some $p > 1$. We report on recent progress in removing the assumption of a Poincaré inequality.

Behnam Esmayli: Coarea Inequality for Sobolev functions on Metric Spaces.
By substituting the modulus of gradient with the notion of upper gradient, one can ask if there is a universal inequality on metric spaces that mimics the classical coarea formula. It is reasonable to assume for example that the metric space is (locally) homeomorphic to $\mathbb{R}^n$ and with locally finite Hausdorff-n measure. Under stronger further geometric assumptions on a metric space, the affirmative answer follows from a localization of the Eilenberg's inequality. Only under the former assumptions, we prove for the case of $n = 2$, such an equality for monotone Sobolev functions. I will also discuss counterexamples showing difficulties of generalizing further.

Tuomas Hytönen: Dyadic cubes on metric spaces.
Dyadic cubes are ubiquitous in analysis in Euclidean spaces. First constructions preserving some of their key features in much more general spaces have been given by David and Christ. I have explored further elaborations in my works with Martikainen, Kairema, Auscher, and Tapiola; in particular, metric versions of random dyadic cubes (inspired by several works of Nazarov, Treil and Volberg on Euclidean spaces), the "1/3 trick" of adjacent/shifted dyadic cubes, and constructions of Hölder-regular "splines" and "wavelets" adapted to these dyadic structures.

Naotaka Kajino: Conformal walk dimension: its universal value and the non-attainment for the Sierpiński carpet.
It is an established result in the field of analysis of diffusion processes on fractals, that the transition density of the diffusion typically satisfies analogs of Gaussian bounds which involve a space-time scaling exponent β greater than two and thereby are called SUB-Gaussian bounds. The exponent β, called the walk dimension of the diffusion, could be considered as representing "how close the geometry of the fractal is to being smooth". It has been observed by Kigami in [Math. Ann. **340** (2008), 781–804] that, in the case of the standard two-dimensional Sierpiński gasket, one can decrease this exponent to two (so that Gaussian bounds hold) by suitable changes of the metric and the measure while keeping the associated Dirichlet form (the quadratic energy functional) the same. Then it is natural to ask how general this phenomenon is for diffusions.

This talk is aimed at presenting (partial) answers to this question. More specifically, the talk will present the following results: (1) For any symmetric diffusion on a metric measure space in which any bounded closed set is compact, the infimum over all possible values of the exponent β after "suitable" changes of the metric and the measure is ALWAYS two unless it is infinite. (We call this infimum the conformal walk dimension of the diffusion.) (2) The infimum as in (1) above is NOT attained, in the case of the Brownian motion on the standard (two-dimensional) Sierpiński carpet, as well as on the standard three- and higher-dimensional Sierpiński gaskets. Some related open problems will also be discussed. For (1), it is not known whether the changes of the metric can be provided by geodesic metrics, or in other words, whether we can require the sub-Gaussian bounds to hold in the full off-diagonal regime. For (2), some (slight) new knowledge about local and global behavior of harmonic functions on the fractal is the key, and for better understanding of related phenomena it would be very important to analyze behavior of harmonic functions on Sierpiński carpets more deeply. This talk is based on joint works with Mathav Murugan (University of British Columbia). The results are given in https://link.springer.com/article/10.1007/s00222-022-01148-3 (Invent. math., in press), except for the non-attainment result for the Sierpiński carpet in (2) above, which is in progress.

Jun Kigami: Yet another construction of "Sobolev spaces" on metric spaces.
The counterpart of "Sobolev spaces" on metric spaces has been intensively studied for the last 20 years after the pioneering works by Cheeger, Hajlasz, and Shanmugalingam. The mainstream of the ideas is to use the local Lipschitz constant of a function as a suitable substitute for its gradient. However, a recent study by Kajino and Murugan on the conformal walk dimension revealed that the Dirichlet form associated with the Brownian motion on the Sierpinski carpet can not be a Sobolev space in this sense. In this talk, we will propose a new way of constructing "Sobolev spaces" on compact metric spaces including the Sierpinski carpet.

Mathav Murugan: Conformal Assouad dimension as the critical exponent for combinatorial modulus.

The conformal Assouad dimension is the infimum of all possible values of Assouad dimension after a quasisymmetric change of metric. We show that the conformal Assouad dimension equals a critical exponent associated to the combinatorial modulus for any compact doubling metric space. This generalizes a similar result obtained by Carrasco Piaggio for the Ahlfors regular conformal dimension to a larger family of spaces. We also show that the value of conformal Assouad dimension is unaffected if we replace quasisymmetry with power quasisymmetry in its definition.

Enrico Pasqualetto: Isoperimetric Problem on nonsmooth spaces with Ricci curvature bounded from below.
In the setting of nonsmooth spaces verifying synthetic lower Ricci curvature bounds (the so-called RCD metric measure spaces), a very refined differential calculus is available by now. By combining these calculus tools with the compactness and stability properties of the class of RCD spaces, it was possible to obtain several results on the isoperimetric problem that are new even in the case of non-compact Riemannian manifolds. Among other things, I will discuss the second-order differential behaviour of the isoperimetric profile, as well as some of its consequences, such as the existence of isoperimetric sets for large volumes and the sharp Lévy-Gromov isoperimetric inequality with the rigidity case.

Katarzyna Pietruska-Paluba: The existence of the integrated density of states on fractals.
For an operator of the form $\Delta + V$, where Δ is a "Laplacian" and V is a potential, the integrated density of states is obtained by confining to a finite volume with Dirichlet boundary conditions so that the operator has N eigenvalues λ_j, forming the sum $\frac{1}{N}\sum_j \delta_{\lambda_j}$ of Dirac measures and taking the limit over volumes filling the space. We report on results in fractal settings with potentials obtained from a Poisson cloud model.

Ryosuke Shimizu: Construction of a canonical p-energy on the Sierpinski carpet.
We provide a review of construction of p-energy and $(1,p)$-Sobolev space on the Sierpinski carpet when p is strictly greater than its Ahlfors regular conformal dimension. For $p = 2$, our 2-energy and $(1,2)$-Sobolev space correspond to the canonical Dirichlet form on the Sierpinski carpet given by Barlow–Bass and Kusuoka–Zhou. We will see that the condition related to the Ahlfors regular conformal dimension plays the role of "strongly recurrent", which implies very good regularity of functions in our Sobolev space.

Giacomo Sodini: Density of subalgebras of Lipschitz functions in metric Sobolev spaces and applications to Sobolev-Wasserstein spaces.
We present a general criterium for the density in energy of suitable subalgebras of Lipschitz functions in the p-metric-Sobolev space associated with a Polish metric-measure space. We then apply our result to the case of the algebra of cylindrical functions in the 2-Sobolev-Wasserstein space arising from a positive Borel measure on the 2-Kantorivich-Rubinstein-Wasserstein space of probability measures on the Euclidean space. We show that such a Sobolev space is always Hilbertian, independently of the choice of the reference measure and we briefly mention how the density result can be extended to more general Sobolev-Wasserstien spaces. This talk is based on a joint work with Massimo Fornasier (TU München, Germany) and Giuseppe Savaré (Bocconi University, Milano, Italy).

Karl Theodor Sturm: Dirichlet forms and metric measure spaces.
We provide a comprehensive survey on Dirichlet forms on metric measure spaces. In particular, we discuss how to pass from metric-measure spaces to Dirichlet forms and vice versa, and under which conditions these transitions commute. Moreover, we outline the basic transformations of the respective data: measure change, metric change, time change, conformal change.

Alexander Teplyaev: Fine structure of BV functions on fractals (preliminary report).
We present some recent results and work in progress regarding the structure of"gradients" of BV functions on certain fractal sets.

Jing Wang: Spectral bounds and exit times of diffusions on metric measure spaces.
It is widely known that the exit time of a diffusion process from a domain reflects geometric and spectral information of the domain. In this talk we consider a diffusion on a metric measure space equipped with a local regular Dirichlet form. With suitable assumptions such as volume doubling property and heat kernel sub-Gaussian upper bound we obtain estimates on the survival probability $P(\tau_D > t)$ of the diffusion, where τ_D is its first exit time from domain D. The applications of this estimate include a uniform upper bound for the product $\lambda(D) \sup_{x \in D} \mathbb{E}_x(\tau_D)$ and a partial answer to a conjecture of Grigor'yan, Hu and Lau. These

results apply to many examples in sub-Riemannian manifolds, fractals, as well as fractal-like manifolds. This is a joint work with Phanuel Mariano.

4 Scientific Progress and Outcomes of the Meeting

The workshop involved many high-quality presentations, and almost as many discussion sessions, at which the audience participated enthusiastically. The small discussion rooms provided by BIRS were used extensively by groups of mathematicians participating in the program to brainstorm and work on projects; and we expect that many future papers and research programs have their beginnings rooted in this BIRS workshop. Judging from these observations, and the informal feedback received by the organizers, it seems that the workshop was very successful at fostering interaction between people from these distinct but related fields of mathematics and making new connections within and between their research areas. Participants also raised many questions during and after the presentations that suggested new directions of research and different approaches to established topics, some of which are mentioned in this report, and we anticipate that these research questions will be influential in the ongoing development of these areas.

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