10w5025 Optimal transportation and applications

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1 Summary

Our meeting took place during a week of many flight cancellations, due to the eruption of a volcano in Iceland. This caused a number of European participants to arrive late and prevented several more — 10 total, including one organizer — from attending at all. Nevertheless, by rearranging the planned activities, a lively and stimulating program was achieved involving the 32 odd participants who managed to attend. In particular, several summaries of striking progress were presented, directions for further research identified, introductions were made, collaborations advanced, and some new ones established.

2 Background

Optimal mass transportation can be traced back to Gaspard Monge's famous paper of 1781: 'Mémoire sur la théorie des déblais et des remblais'. The problem there is to minimize the cost of transporting a given distribution of mass from one location to another. Since then, it has become a classical subject in probability theory, economics and optimization. Following the seminal discoveries of Brenier in his 1987 paper 'Décomposition polaire et réarrangement monotone des champs de vecteurs' [6], optimal transportation has received much renewed attention in the last 20 years. It has become an increasingly common and powerful tool, at the interface between partial differential equations, fluid mechanics, geometry, probability theory, and functional analysis. At the same time, it has lead to significant developments in applied mathematics, including for instance in economics, biology, meteorology, design, and image processing.

2.1 Regularity of optimal transportation, fully nonlinear partial differential equations, and Riemannian geometry

The smoothness of optimal transport maps is an important issue in transportation theory since it gives information about qualitative behavior of the map, as well as simplifying computations and algorithms in numerical and theoretical implementations. Thanks to the results of Brenier [6, 7] and McCann [67], it is well known that the potential function of the map satisfies a Monge-Ampère type equation, an important fully nonlinear second order elliptic PDE arising in differential geometry. In the case of the quadratic cost function in Euclidean space, pioneering papers in this field are due to Delanoë [23], Caffarelli [8, 9, 10, 11], and Urbas [86].

More recently, Ma, Trudinger and Wang [69, 84] (see also [83]) discovered a mysterious analytical condition, now called the Ma-Trudinger-Wang condition (or simply MTW condition) to prove regularity estimates for general cost functions. Costs functions which satisfy such a condition are called regular. At this point, Loeper [58] gave a geometric description of this regularity condition, and he proved that the distance squared on the sphere is a uniformly regular cost, giving the first non-trivial example on curved manifolds. The Ma-Trudinger-Wang tensor is reinterpreted by Kim-McCann [52] in an intrinsic way, and they show that it can be identified as the sectional curvature tensor on the product manifold equipped with a pseudo-Riemannian metric with signature (n, n). Also, recent results of Loeper-Villani [61] and Figalli-Rifford [40] show that the regularity condition on the square distance of a Riemannian manifold implies geometric results, like the convexity of the cut-loci. Developments discussed at the workshop highlighted many fruitful interactions between analysis and geometry around optimal transportation.

2.2 Geometry of the space of probability measures, and it applications to geometric inequalities and nonlinear diffusion

Links from optimal transport to geometric analysis, including to the theory of Ricci curvature and Ricci flow, take their origin in the work of Otto and Villani [74], and have received even more attention after the recent works of Lott and Villani [61], Sturm [80, 81], McCann and Topping [68]. The possibility to define useful analogs of such concepts in a metric measure space setting has been a tantalizing goal, only partly realized so far. Still this progress, together with the original contribution due to Otto [73] on the formal Riemannian structure of the Wasserstein space and its application on PDE, is having a strong impact on the research community.

Indeed, on the one hand one exploits that geometric/functional inequalities are related to hidden convexity of appropriate entropy functionals, and thus governs the rate of convergence of the corresponding gradient flows. This framework yields a lot of interesting results, including extensions of (log-)Sobolev inequalities with completely new proofs of the original ones, as well as quantitative study of asymptotic structure of nonlinear diffusion processes, including the porous-medium equations and fast diffusion equations. There are a lot of literature in this direction: see, e.g., the references in [1, 87, 88].

On the other hand, optimal transportation has also provided a new and simpler way to establish sharp geometric inequalities like the isoperimetric theorem, optimal Sobolev inequalities and optimal Gagliardo-Nirenberg inequalities: see e.g., [74, 87, 88, 2, 3]. More recently, Figalli, Maggi and Pratelli [39] were able to exploit Gromov's proof of the anisotropic isoperimetric inequality via optimal transport to prove a sharp quantitative version of the Wulff inequality. This result shows that optimal transportation's proofs of functional inequalities, apart from being generally very short and elegant, are also very stable and can be used to get improved versions of the original inequality, see also for instance [38].

2.3 Application to economics, meteorology, design problems, image processing,

There are numerous applications of optimal transport, among which we concentrate on economics, meteorology, and design problems.

2.3.1 Economics

Mass transportation duality is useful in formulating the problem of existence, uniqueness and purity for equilibrium in hedonic models. Recent works of Ekeland [29, 30], Chiappori, McCann and Nesheim [16] have shown that optimal transportation techniques are powerful tools for the analysis of matching problems and hedonic equilibria. Work of Rochet and Choné [78] and Carlier [15] also exposed applications to the principal agent problem – a central paradigm in microeconomic theory, which models the optimal decision problem facing a monopolist whose must act based on statistical information about her clients. Although existence has generally been established in such models, characterization of the solutions, including uniqueness, smoothness, and comparative statics remain pressing open questions. Transportation theory has a wide range of further potential applications in econometrics, urban economics, adverse selection problems and nonlinear pricing. These were highlighted in a talk of Robert McCann concerning recent work with Figalli and Kim.

2.3.2 Geophysical dynamics

Geophysical dynamics seeks to understand the evolution of the atmosphere and oceans, which is fundamental to weather and climate prediction. It has been shown by Cullen and his collaborators that mass transportation theory can be applied to fluid dynamical problems, for instance those governing the large-scale behaviour of the atmosphere and oceans (see e.g., [21]). Here discontinuous solutions find important applications as models for atmospheric fronts, where the point is to analyze the geometry and dynamics of the discontinuity. The theory can also be given a geometrical interpretation, which has led to important ex- tensions in its applicability, and can be used to investigate the qualitative impact of geographical formations, such as mountain ranges. A related open problem to which mass transportation is relevant is the incorporation moisture and thermodynamics into the dry dynamics, to model, e.g., rainstorms. Since Cullen was one of the researchers who participation was prevented by the volcanic eruption in Iceland, his collaborator Mikhail Feldman gave beautiful survey of mathematical developments surrounding the semigeostrophic theory.

2.3.3 Engineering design

Mass transportation theory has a number of promising applications in engineering design – ranging from the construction of reflector antennas or shapes which minimize wind resistance, to problems in computer vision. Oliker [72] and X-J Wang [90] have pioneered the use of transportation theory in reflector design, while Plakhov has been exploring novel applications in aerodynamics, see e.g., [76, 77]. Image registration offers medical applications, in which the goal is establish a common geometric reference frame between two or more diagnostic images captured at different times. Based on the mass transportation theory, Tannenbaum and his group developed powerful algorithms for computing elastic registration and warping maps: see e.g., [4, 49, 28].

3 Open Problems and Progresses

When studying regularity of optimal maps, one of the major open problems is to find extensive classes of cost functions and domains where the MTW condition holds. In relation to geometry, the Riemannian distance squared cost on a manifold may be the most natural case to consider. In this direction, so far the list of manifolds where MTW is satisfied, includes tori, C^4-perturbations of the round sphere, and Riemannian products and submersions of the round sphere: see e.g., [58, 54, 61, 40, 25]. An important open problem is to relate a local geometric Riemannian curvature condition to MTW condition which is global in nature – but the relevant data are concentrated along geodesic paths. For example, can an estimate on the derivative of the sectional curvature imply MTW condition? Some partial results in this direction have been provided by [25, 42, 26]. Another angle is to understand what geometric restrictions the MTW condition gives. So far progress has been made regarding convexity of tangent cut loci in [61, 40, 41, 42, 43, 44].

Establishing regularity of optimal maps for a cost satisfying MTW condition is a separate issue. Continuity of optimal maps with bounded transported mass distributions are now known on the manifolds satisfying the strong MTW condition due to the results in [58, 54, 63, 61, 40, 44], while the higher regularity of optimal maps for more regular mass distributions is still open for perturbation cases (though, there are some partial results [89, 61, 25]). There has been also progress in related problem, called reflector antenna in the works [90, 48, 91, 69, 13, 58, 50]: especially, in [50] it is identified the range of regions where regularity holds for general data. For a degenerate MTW condition, where the analysis is more subtle due to lack of strong local estimates, a global higher regularity is known [84] on domains in $\mathbf{R}^n$. For continuity with rough data, there has been a substantial progress in [34, 36] assuming a slightly stronger but still degenerate version of MTW condition, so-called nonnegative cross-curvature. The higher regularity in [34, 36] uses recently found interior a priori estimates of [65] (see also [64]). The work [36] on multiple products of spheres is the first time higher regularity is obtained on non-flat manifold satisfying only degenerate MTW condition. For general degenerate MTW case, continuity with rough data and interior regularity remain open.

A wide open problem of regularity theory is to understand the nature of discontinuity/singularity set of optimal maps when the MTW condition is not satisfied, e.g., the distance squared cost on negatively curved Riemannian manifolds. As Villani asked in his book [88], does such set have nice geometry or does it show fractal nature? Partial results in this direction have been recently proved in [31, 33], but a complete answer to this problem is still missing.

As (quite surplisingly) shown by the recent work in [35], a variant of the MTW condition naturally arises in the principal agent problem in multi-dimensional setting. More precisely, the authors identify a structural condition on the value $b(x, y)$ of product type y to agent type x – and on the principal's costs $c(y)$ – which is necessary and sufficient for reducing the profit maximization problem faced by the principal to a convex program. This is a key step toward making the principal's problem theoretically and computationally tractable; in particular, it allows us to derive uniqueness and stability of the principal's optimum strategy – and similarly of the strategy maximizing the expected welfare of the agents when the principal's profitability is constrained. This fact shows how the MTW condition plays a key role as a structural condition in principal agent problem, and it is likely that this fact could be useful in the future also in other problems coming from economics.

New applications for optimal transport are also appearing in statistical mechanics. The recent work described by Sei [79] concerned applications to directional statistics, and showed that a convex combination of optimal transport plans on a sphere gives a way to construct a family of probability distributions. Moreover, the the work in [36] allows Sei's result to be generalized to multiple products of spheres. Let us also observe that, since the state of a spin system is classically modeled as a point in the phase space obtained by taking many products of spheres, one may expect that optimal transportation will prove useful to estimate rates of decay for correlations in other models from statistical mechanics, with the goal of establishing phase transitions. The hope is that, in contrast with current methods that take advantage of the specific structure of models, convexity methods will be robust under small changes to the model.

Other important open problems arise in the contest of the theory of Lott-Villani [61] and Sturm [80, 81] of metric-measure spaces with Ricci curvature bounded from below. Indeed, still the theory presents many open problem, as the recent work of Bacher and Sturm shows (see also the discussion on the talk of Sturm below): indeed, the authors introduce a new curvature-dimension condition which is more flexible than the original one introduced by Sturm, and allows to obtain much more general result (at least at the moment) with the drawback of producing slightly worse constants in some inequalities. So, it becomes natural now to try to understand which is the right condition to consider on metric spaces (note that the two conditions coincide on Riemannian manifolds). Another important question concerns the study of gradient flows in these class of metric spaces. As shown by Gigli [45], the heat flow on a metric space with Ricci curvature bounded from below exists and is unique. However, questions concerning stability and long-time asymptotics are widely open.

4 Presentation Highlights

4.1 Theoretical aspects of optimal transportation

4.1.1 Regularity of optimal transportation, nonlinear PDE, and related geometry.

A number of speakers discussed topics concerning the Ma, Trudinger and Wang conditions for regularity of optimal transportations and fully nonlinear elliptic Hessian equations.

Jiakun Liu presented his joint work [65] with Xu-Jia Wang and Neil Trudinger on the continuity of second derivatives of solutions to the Monge-Ampère type equations arising in optimal transportation. His result includes Hölder and more general continuity estimates for second derivatives, when the inhomogeneous term in the equation is Hölder and Dini continuous, together with corresponding regularity results for the potentials in optimal transportation.

Ludovic Rifford discussed joint work [44] with Alessio Figalli and Cédric Villani on the regularity of optimal transport maps associated with quadratic costs on Riemannian manifolds. He gave necessary and sufficient conditions related to the so-called Ma-Trudinger-Wang and extended Ma-Trudinger-Wang conditions, with examples and counterexamples.

Micah Warren presented a joint work [55] with Young-Heon Kim and Robert McCann. He introduced a new pseudo-Riemannian metric on the products space of source and target domains of optimal transportation. This metric involves both the mass densities and the cost; it is a conformal perturbation of the metric previously defined by Kim and McCann [52], which gives a geometrization of the condition of Ma, Trudinger and Wang as a curvature condition of the metric. It leads to a new geometrical extremization property of optimal maps. Namely, in the conformally perturbed metric, the graph of the optimal map between two given smooth densities becomes a calibrated maximal Lagrangian rectifiable n-current, thus special Lagrangian in the sense of Hitchin; it has zero mean curvature as an embedded submanifold. This gives an unexpected link between optimal transportation and the more classical problem of finding mass minimizing currents in geometric measure theory. The calibrations which detect these special Lagrangians are pseudo-Riemannian analogues of the special Lagrangian calibrations for Calabi-Yau manifolds.

Simon Brendle talked about his joint work [5] with Micah Warren concerning existence results for minimal Lagrangian graphs. Given two uniformly convex domains in $\mathbf{R}^n$, he showed existence of a diffeomorphism between them, whose graph is a minimal Lagrangian submanifold. This question comes down to a boundary value problem for a fully nonlinear PDE, and is in the similar spirit as the second boundary value problems of Monge-Ampère type equations arising in optimal transportation, and is also related in broader sense to the talk of Micah Warren. Brendle also discussed a similar question for domains in the hyperbolic plane.

Paul Lee discussed transportation costs arising from natural mechanical actions. He reported a joint work with R. McCann [57], where they found a class of costs based on Lagrangian actions, which satisfy the Ma, Trudinger and Wang conditions for regularity of optimal transportation.

4.1.2 Geometry of the space of probability measures

Nicola Gigli discussed heat flow in metric measure spaces [45]. Under the assumption of Ricci curvature bounded from below, he obtained well-posedness of the definition of of the heat flow as the gradient flow of the Entropy with respect to the quadratic Wasserstein distance. In particular, uniqueness is proved.

Karl-Theodor Sturm presented some partial results (joint work with K. Bacher) to open questions concerning the curvature-dimension condition $CD(K;N)$ for metric measure spaces. They introduced a new version $CD^*(K;N)$ (called reduced curvature-dimension condition) of $CD(K;N)$. Like the original condition, on Riemannian manifolds this new condition it is equivalent (roughly speaking) to the conditions "Ricci bounded from below by K" and "dimension bounded from above by N". However, it provides worse constants in functional inequalities than the ones that can be deduced from $CD(K,N)$. On the other hand, it has properties which make it more suitable for many applications. Indeed, it satisfies a local-to-global property: that is, if it holds locally then it is also true globally (a fact not known for $CD(K,N)$). Moreover, it satisfies a tensorization property, which allows to deduce $CD(K;N)$ for metric cones and suspensions, under suitable assumptions on the basis. As an application of these results, they can prove finiteness of the fundamental group of any metric measure space $(M;d;m)$ which satisfies $CD(K;N)$ locally with positive K and finite N.

Asuka Takatsu reported a joint work with Shin-ichi Ohta [71] on an extension of the notion of displacement convexity to a more general entropy functional, called the Tsallis entropy. This convexity induces new examples of measure concentration.

Gershon Wolansky presented his investigation [92] of a link between optimal transformations obtained by different Lagrangian actions on Riemannian manifolds. In particular, he explained how the 1-Wasserstein metric arises as a limit of the p-Wasserstein distance W_p between two small perturbations of a suitably chosen reference measure μ: for any non-negative measures λ^+, λ^- of equal mass

$$W_1(\lambda^-, \lambda^+) = \lim_{\epsilon \to 0} \epsilon^{-1} \inf_{\mu} W_p(\mu + \epsilon\lambda^-, \mu + \epsilon\lambda^+),$$

where the infimum is over the set of probability measures μ on the ambient space M. He used a version of this limit theorem as the foundation for a series of developments concerning optimal network theory.

4.1.3 Extensions of the notion of optimal transportation

A couple of speakers discussed how to extend the notion of optimal transportation.

First, Brendan Pass talked about his result [75] on multi-marginal optimal transportation where the goal is to minimize matching of multiple (more than two) mass distributions. To study the local structure of the optimal plan in this multi-marginal setting, he used a family of semi-Riemannian metrics derived from the mixed, second order partials derivatives of the cost function, which is a reminiscence of the metric of [52], to provide upper bounds on the dimension of the support of the optimal measure.

Qinglan Xia discussed the ramified optimal transportation which he has been developing in a series of papers since 2003: see e.g., [93, 95] . Here, he allows more flexible geometry for each path of the transportation plan. Namely, each such path is allowed to branch out, while each branching point contributes some weight in the transportation cost. This theory has many nice applications. In particular, he successfully modeled the formation of a tree leaf for various tree types [94]. In his talk, he connected his theory to another variational problem: p-harmonic maps on graphs.

Application of optimal transportation:

There were talks concerning various applications of optimal transportation.

Chris Budd showed how optimal transport ideas can be used to generate a mesh suitable for discretize time-dependent partial differential equations. The point is that the mesh should also evolve to capture the change of the underlying structure so that the computations are efficient, accurate and reliable. The meshes are constructed by optimally transporting a reference mesh according to a metric dictated by the solution of the underlying PDE. He showed how this procedure is applied to a series of problems, including the formation of tropical storms in meteorological systems.

Mikhail Feldman discussed a model in geophysical dynamics (e.g. meteorology) called semi-geostrophic system. He presented his recent work with A. Tudorascu showing rigorously energy conservation property for weak Lagrangian solutions to this system. Though formally well-known, finding solutions which respect conservation of energy is both important and delicate.

Wilfrid Gangbo reported a joint work with Alessio Figalli and Turkay Yolcu [32], where they extend De Giorgi's interpolation method to a class of parabolic equations which are not gradient flows but possess an entropy functional and an underlying Lagrangian. The new fact in the study is that not only the Lagrangian may depend on spatial variables, but it does not induce a metric. Assuming the initial condition to be a density function, not necessarily smooth, but solely of bounded first moments and finite "entropy", they could use a variational scheme to discretize the equation in time and construct approximate solutions (this scheme is analogous to the minimizing movement scheme introduced by De Giorgi to construct gradient flows in metric spaces). Then, De Giorgi's interpolation method turns out to be a powerful tool for proving convergence of the algorithm. Finally, they show uniqueness and stability in L^1 of their solutions.

Robert McCann described recent work with A. Figalli and Y.-H. Kim [35] on the principal agent problem in multi-dimensional setting. He showed that some Ma-Truginder-Wang type conditions give a general structural conditions on the value $b(X, Y)$ of product X to buyer Y, and on monopolists cost $c(X)$ of producing X, which reduce the relevant minimizing problem to a convex program in a Banach space– leading to uniqueness and stability results for its solution, confirming robustness of certain economic phenomena observed in previous researches. This results extends the special cases when either in 1 dimension or $b(X, Y)$ is linear, to a much general class of situations.

Quentin Merigot [70] explained topological inference problem, which addresses the question of recovering the topology (eg. homotopy type) of a compact subset K of $\mathbf{R}^d$ using approximation by a discrete set. In his work, he defined a notion of distance function to a probability measure on $\mathbf{R}^d$ whose sub-level sets can be used to associate a geometry and topology to the measure. This function has a Lipschitz-dependence on the measure with respect to the quadratic Wasserstein distance, and can be used to obtain topological reconstruction results in the presence of noise.

Garry Newsam reviewed two practical issues in transport problems: The first is the issue of appropriate function space settings for objects such as images and whether the penalty functions used to distinguish specific transportations / registrations from the class of all possible maps are consistent with these settings. The second is the opportunity to match theory to data opened up by new massive data records on real transportation flows, such as those on maritime cargo traffic provided by signals from the Automatic Identification System (AIS) transmitters now carried by all ships.

Tomanari Sei explained two topics on statistics which relate to the optimal transportation theory [79]. One is directional statistics, where convex combination of optimal transport plans on a manifold gives a way to construct a family of probability distributions on the manifold. The mathematical method behind is closely related to the talk of McCann [35]. Another topic is on information geometry, where he discussed the dual structure of a cost function, called divergence in statistics.

Results in other related areas

Maria Gualdani reported on a mean field model arising in price formation. Her result on this free boundary problem uses tools from non-interacting stochastic particle systems and multiscale analysis.

Amir Moradifam discussed the regularity issue of biharmonic equation of the form $\Delta^2 u = \frac{\lambda}{(1-u)^2}$, which is originated from the theory of Micro-Electro-Mechanical Systems (MEMS). He demonstrated that the regularity depends on the dimension, with a proof is based on certain improved Hardy-Rellich inequalities.

Shawn Xianfu Wang addressed an issue in convex analysis, describing how the proximal average method can be used to find the least norm minimizers of a convex function and its Fenchel conjugate.

Ramon Zaraté presented results from his thesis, on how to apply variational method to finding the unknown nonlinearity of given partial differential equations, ranging from equations of Euler-Lagrange type to parabolic equations.

5 Outcome of the Meeting

It was unfortunate that airport closures due to volcanic eruptions in Iceland prevented several European participants — many of them distinguished scientists — from attending. These included: Stefano Bianchini, Guillaume Carlier, Mike Cullen, Luigi de Pascale, Yuxin Ge, Peter Topping, Alexander Plakhov, Max von Renesse, and Cédric Villani.

However, those who were in attendance rose to the occasion and filled the breach admirably. Indeed, the workshop succeeded in addressing many of the most important research directions in optimal transportation theory in spite of these absences, through talks that included those of Gigli, Feldman, Liu, McCann, Rifford, Sturm and Warren. Other such as Brendle and Gualdani gave lucid and stimulating lectures on complementary topics. There was a high level of participation by young people, including graduate students, recent postdoctoral fellows, minorities and women, who benefitted from a relaxed schedule allowing additional time for interactions with researchers both senior and junior.

This workshop brought together researchers from a range of different fields with common interests in subjects related to the mathematics of optimal transportation, both theoretical and practical aspects. It showcased recent progress and set the stage for future developments, while stimulating new collaborations, new questions, and new lines of research. By making these connections, we believe that the meeting has accelerated the rate of progress within mathematics and in the transfer and application of mathematical techniques between mathematics and adjacent areas of science, including economics and statistics. Its lasting impact will be reflected in research directions which grow from the interactions which were catalyzed here.

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