Computability, Reverse Mathematics and Combinatorics

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Sunday, December 7–Friday, December 12, 2008

1 Measuring the Strength of Theorems

Mathematicians all know what it means to prove a theorem from some set of axioms. In Reverse Mathematics we reverse the process and study what axioms are actually required to prove a theorem. If we can omit some of the axioms and assume instead the "theorem" and use this to prove the omitted axioms, then we know that the omitted axioms were really needed to prove the theorem. Thus Reverse Mathematics addresses the natural and philosophically important question of comparing the relative difficulty (or strength) of theorems by showing that some proofs need stronger axioms than others. Another approach to measuring relative complexity of mathematical notions is to use results from Computability Theory. Can we for example, go from an effective procedure in the finite case of a combinatorial problem to one in the countable case? If not, will a compactness argument suffice or do we perhaps need a transfinite induction?

This subject provides ways to compare the complexity of theorems by precisely calibrating the strength of each theorem along standard axiomatic (proof theoretic) and computational yardsticks. There are, moreover, intimate connections between the computational and proof-theoretic measures, and results from one approach often carry over to, or have consequences for, the other.

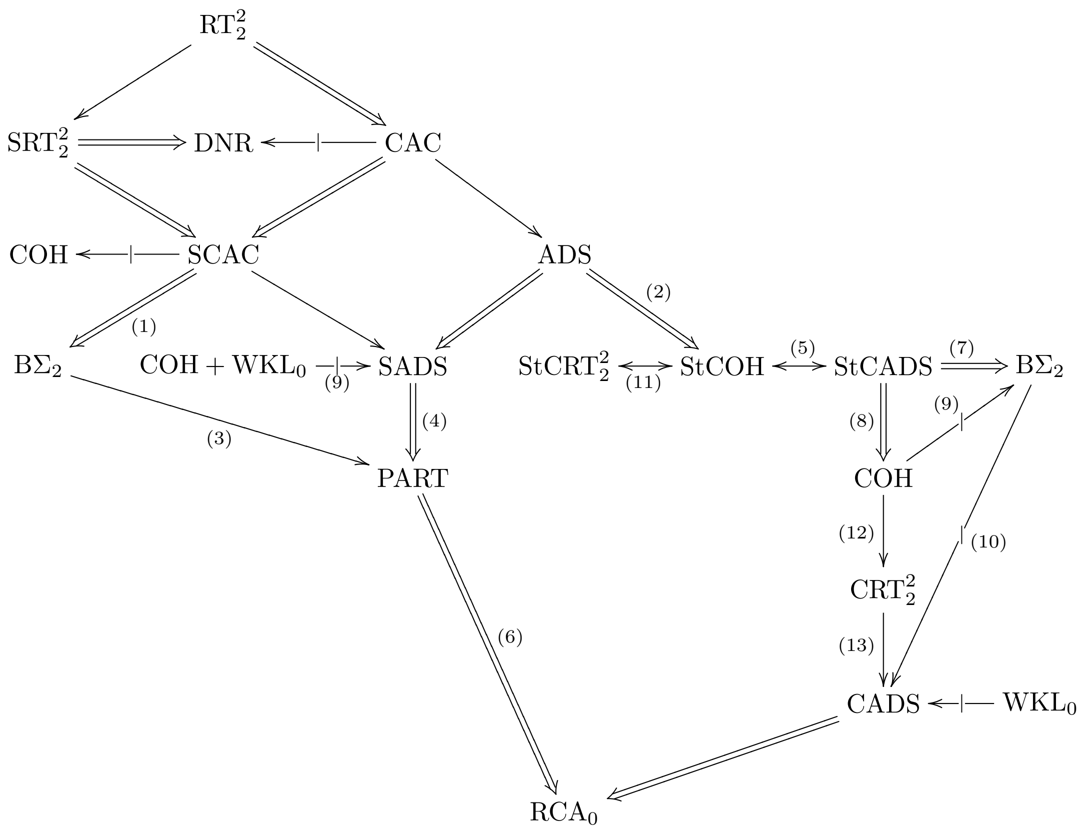
There are five standard levels on the proof-theoretic yardstick. Each represents an increasing level of set existence axioms and has a computational analog. The first (RCA_0) is a system of recursive (computable) mathematics. The existence of a proof in this system of a theorem of the form "for every A of some kind, there exists a B with some properties" implies there is a Turing machine M that, given access to information about an instance A, can compute a corresponding solution B. Generally, the converse holds as well. The second system (WKL_0) includes a weak version of König's Lemma: every infinite binary branching tree has an infinite path. It essentially adds the power of compactness arguments to the elementary effective procedures available in RCA_0. Computationally, it corresponds to the Jockusch-Soare low basis theorem that bounds the complexity of the path in terms of its halting problem. The third system (ACA_0) asserts that every set definable in (first order) arithmetic exists. It corresponds to being able to solve the halting problem (construct the Turing jump, X') relative to any set X. The last two systems (ATR_0 and Π^1_1-CA_0) are more powerful systems with second order existence axioms. The first of them corresponds to (effectively) iterating the Turing jump into the transfinite. The last adds the power to determine if a given ordering is a well ordering.

How then do we show that one of these systems or types of extra computational power is actually necessary to prove a theorem or compute a solution. Proof-theoretically, this is done by showing that (relative to some weak base theory), the theorem in question actually implies the axioms of the stronger system used to prove it. (Thus the name, reverse mathematics.) Computationally, we can show, for example, that given any set X there is an instance of the problem computable from X such that any solution computes the halting problem relative to X or decides if a linear ordering computable from X is well founded. Such investigations show, in one direction, that certain problems cannot be solved without additional (perhaps unexpected) information. In the other direction, they often provide sharper proofs with more precise computational information than had been previously known. Nonimplications between various principles can also be established by both computational and proof-theoretic methods.

The flavor of the calibrations is suggested by some examples from the realms of countable algebra and combinatorics (with their equivalent systems): in general, algebraic and combinatorial existence theorems with algorithmic solutions are in RCA_0; every n-regular bipartite graph (any $n > 0$) has a perfect matching (WKL_0); every vector space has a basis, every Abelian group has a unique divisible closure, every finitely branching infinite tree has an infinite path, Ramsey's theorem giving homogeneous sets for m-colorings of n-tuples for any $n > 2, m > 1$, various marriage/matching theorems such as Hall's (ACA_0); Hindman's theorem is at or slightly above ACA_0; Ulm's theorem on reduced Abelian p-groups, König's duality theorem (ATR_0); every tree has a largest perfect subtree, every Abelian group is the direct sum of a divisible and a reduced group (Π^1_1-CA_0).

Several theorems about or using Nash-Williams's well (or better) quasiordering theory by Kruskal, Laver, Robertson-Seymour, etc., are known to be at levels around Π^1_1-CA_0 but their precise strengths have not been determined. Other new examples at this level have recently come to light and need to be studied. In addition, many questions remain about combinatorial principles that are known to inhabit the area strictly below ACA_0 such as RT^2_2 (Ramsey's theorem for pairs and two colors), CAC (Dilworth's theorem that every infinite partial order has an infinite chain or antichain), and ADS (the Erdős-Szekeres theorem that every infinite linear order contains an infinite sequence which is strictly ascending or descending). The last two, which are easy consequences of Ramsey's theorem for pairs, are also known to be incomparable with WKL_0. Whether Ramsey's theorem for pairs implies WKL_0 is, however, an open problem.

Though the five standard levels on the proof-theoretic yardstick are linearly ordered, the situation is much more complicated in general. Here is a diagram of some combinatorial principals weaker than Ramsey's Theorem for pairs, as taken from Hirschfeldt and Shore [4]. Double arrows indicate a strict implication and single ones an implication that is not known to be strict. Negated arrows indicate known nonimplications.



2 Outcome of the Meeting

In Stephen Simpson's introductory talk "The Gödel Hierarchy and Reverse Mathematics", he roughly separated the Gödel Hierarchy (a linear order of foundational theories) into three levels: weak (eg. bounded arithmetic), medium (eg. 2nd order arithmetic) and strong (eg. ZFC). At this meeting, we had the rare opportunity of interacting with researchers whose combined studies represent the full realm of the Hierarchy. Through the talks and the open problem session, participants were able to share their approaches to major known problems, and also to invite others to consider questions that they had encountered through their work. In some cases questions posed at the meeting have already been answered; please see the "Scientific Progress Made" section of this report.

3 Presentation Highlights

Many of the talks included slides that are available at the BIRS page for the workshop, under the tab "Workshop Files".
Speaker: **Sam Buss** (Univ. of Calif., San Diego)
Title: *Polynomial Local Search higher in the Polynomial Hierarchy and Bounded Arithmetic* (slides available)
Summary: Buss presented joint work with Arnold Beckmann, discussing provably total functions of bounded arithmetic, and recent characterization of the Σ^b_i definable functions of S^{k+1}_2 (or T^k_2), for all $i \leq k$. The main tool is extensions of polynomial local search problems to higher levels of the polynomial time hierarchy, where the feasible set is defined by a Π^b_k predicate but the cost and neighborhood functions are definable by polynomial time terms. These higher level PLS problems can be used to determine the truth of Π^b_k properties and also allow "witness doubling". These results can be formalized and then Skolemized with a weak base theory such as S^1_2 — the stronger theory

S_2^{k+1} (or T_2^k) is needed only to prove the existence of a solution. The Skolemization allows us to define sets of clauses that are refutable in a depth m propositional refutation system (a Tait style system for propositional logic), but are conjectured not to be provable in a depth $m - 1/2$ system. Related open problems and future directions for research were discussed.

Speaker: **Timothy J. Carlson** (Ohio State)
Title: *Combinatorics of Words* (slides available)
Summary: Carlson discussed some results on the infinite combinatorics of words and their connection to idempotent ultrafilters.

Speaker: **Douglas Cenzer** (University of Florida)
Title: *Space Complexity of Abelian Groups* (slides available)
Summary: Cenzer presented joint work with Rodney G. Downey, Jeffrey B. Remmel, and Zia Uddin in which they develop a theory of $LOGSPACE$ structures and apply it to construct a number of examples of Abelian Groups which have $LOGSPACE$ presentations. They show that all computable torsion Abelian groups have $LOGSPACE$ presentations and we show that the groups $\mathbb{Z}$, $Z(p^\infty)$, and the additive group of the rationals have $LOGSPACE$ presentations over a standard universe such as the tally representation and the binary representation of the natural numbers. They also study the effective categoricity of such groups. For example, they give conditions are given under which two isomorphic $LOGSPACE$ structures will have a linear space isomorphism.

Speaker: **Chi Tat Chong** (National University of Singapore)
Title: Π^1_1 *conservation of the COH Principle over models of* $B\Sigma_2$ (slides available)
Summary: Chong reported on joint work with Ted Slaman and Yue Yang. Given a model M of RCA_0 + $\mathrm{B}\Sigma_2$ and $R \subset M$, a set G is R-cohesive if for all s, G is either contained in R_s, modulo M-finite sets, or disjoint from R_s modulo M-finite sets, where R_s is the sth co-ordinate set of R. The principle COH states that for all R in M, there is an R-cohesive set for R in M. Chong gave a sketch of the proof that RCA_0 + COH + $\mathrm{B}\Sigma_2$ is Π^1_1 conservative over RCA_0 + $\mathrm{B}\Sigma_2$. He also discussed some open problems related to the existence of solutions for Δ_2 sets A (i.e. those contained in or disjoint from A) that preserve RCA_0 + $\mathrm{B}\Sigma_2$. The existence of such solutions will point a way towards solving the problem of resolving the complexity of Ramsey's theorem for pairs (whether it implies Σ_2 induction) and separating it from Stable Ramsey's Theorem for pairs.

Speaker: **Valentina Harizanov** (George Washington University)
Title: *Computability and orders on structures* (slides available)
Summary: A magma is left-orderable if there is a linear ordering of its domain, which is left invariant with respect to the magma operation. If the ordering is also right invariant, then the magma is bi-orderable. For arbitrary magmas (not necessarily associative), there is a natural topology on the set of all left orders, and this space is compact. Harizanov presented results on computable orderable groups, in particular, free groups, and computability theoretic complexity of their orders.

Speaker: **Denis Hirschfeldt** (University of Chicago)
Title: *The Atomic Model Theorem and Related Model Theoretic Principles* (slides available)
Summary: Hirschfeldt reported on the complexity of several classical model theoretic theorems about prime and atomic models and omitting types. Some are provable in RCA_0, others are equivalent to ACA_0. One, that every atomic theory has an atomic model, is not provable in RCA_0 but is incomparable with WKL_0, more than Π^1_1 conservative over RCA_0 and strictly weaker than all the combinatorial principles of Hirschfeldt and Shore [2007] that are not Π^1_1 conservative over RCA_0. A priority argument with Shore blocking shows that it is also Π^1_1-conservative over $\mathrm{B}\Sigma_2$. We also provide a theorem provable by a finite injury priority argument that is conservative over $\mathrm{I}\Sigma_1$ but implies $\mathrm{I}\Sigma_2$ over $\mathrm{B}\Sigma_2$, and a type omitting theorem that is equivalent to the principle that for every X there is a set that is hyperimmune relative to X. Finally, we give a version of the atomic model theorem that is equivalent to the principle that for every X there is a set that is not recursive in X, and is thus in a sense the weakest possible natural principle not true in the ω-model consisting of the recursive sets.

Speaker: **Jeffry L. Hirst** (Appalachian State University)
Title: *Two variants of Ramsey's theorem* (slides available)
Summary: This talk explored the computability theory and reverse mathematics of some versions of Ramsey's theorem, including Ramsey's theorem on trees (TT^n_k) and the polarized Ramsey's theorem (PT^n_k). Here are statements of those theorems:

TT^n_k: Let $2^{<\mathbb{N}}$ denote the full binary tree and $[2^{<\mathbb{N}}]^n$ denote all n-tuples of comparable nodes in $2^{<\mathbb{N}}$. If $f : [2^{<\mathbb{N}}]^n \to k$, then we can find a $c < k$ and a subtree S such that S is order isomorphic to $2^{<\mathbb{N}}$, and $f(\sigma) = c$ for every n-tuple σ of comparable nodes in S.

PT^n_k: If $f : [\mathbb{N}]^n \to k$, then we can find a $c < k$ and a sequence $H_1, H_2, \dots H_n$ of infinite sets such that $f(\{x_1, x_2, \dots, x_n\}) = c$ for every nonrepeating n-tuple $(x_1, x_2, \dots, x_n) \in H_1 \times \cdots \times H_n$.

Speaker: **Carl Jockusch** (UIUC)
Title: *Bounded diagonalization and Ramseyan results on edge-labeled ternary trees* (slides available)
Summary: Jockusch discussed recent joint work with Rod Downey, Noam Greenberg, and Kevin Milans. The class of weakly 1-random sets is not strongly (or Medvedev) reducible to DNR_3, the class of diagonally noncomputable functions taking values in 0,1,2. The key element of the proof is a new Ramseyan result on rooted ternary trees with certain edges having labels in 0,1. In fact a family of related results on this topic is obtained.

Speaker: **H. Jerome Keisler** (UW–Madison)
Title: *Nonstandard Arithmetic, Reverse Mathematics, and Recursive Comprehension* (slides available)
Summary: In the paper "Nonstandard Arithmetic and Reverse Mathematics" (Bull. Symb. Logic 2006), it was shown that each of the five basic theories of second order arithmetic that play a central role in reverse mathematics has a natural counterpart in the language of nonstandard arithmetic. This lecture surveyed the results in that paper, and then gave an even more natural counterpart of the weakest the basic theories, the theory RCA_0 of Recursive Comprehension.

The language L_2 of second order arithmetic has a sort for the natural numbers and a sort for sets of natural numbers, while the language ${}^*L_1$ of nonstandard arithmetic has a sort for the natural numbers and a sort for the hyperintegers. In nonstandard analysis one often uses first order properties of hyperintegers to prove second order properties of integers. An advantage of this method is that the hyperintegers have more structure than the sets of integers. The method is captured by the Standard Part Principle (STP), a statement in the combined language $L_2 \cup {}^*L_1$ which says that a set of integers exists if and only if it is coded by a hyperinteger. We say that a theory T' in $L_2 \cup {}^*L_1$ is conservative with respect to a theory T in L_2 if every sentence of L_2 provable from T' is provable from T.

For each of the basic theories $T = \mathsf{WKL}_0, \mathsf{ACA}_0, \mathsf{ATR}_0, \Pi^1_1\text{-}\mathsf{CA}_0$ in the language L_2 of second order arithmetic, the 2006 paper gave a theory U of nonstandard arithmetic in the language ${}^*L_1$ such that:

(1) $U + \mathsf{STP}$ implies T and is conservative with respect to T.

The nonstandard counterpart for RCA_0 in that paper does not have property (1), but instead has a weakened form of the STP. In this lecture we give a new nonstandard counterpart of RCA_0 which does have property (1). That is, we give a theory U of nonstandard arithmetic in ${}^*L_1$ such that $U + \mathsf{STP}$ implies and is conservative with respect to RCA_0.

Speaker: **Hal Kierstead** (Arizona State University)
Title: *Recursive and On-line Graph Coloring* (slides available)
Summary: A survey of results on recursive and on-line graph coloring. A recursive graph is a graph whose vertex and edge sets are both recursive. The basic question is whether for a class $\mathcal{C}$ of graphs there exists a function f such that every recursive k-colorable graph $G \in \mathcal{C}$ has a recursive $f(k)$-coloring. In order to prove positive results, early researchers strengthened the requirements for the effective presentation of graphs under consideration. One method was to restrict to *highly recursive* graphs. A graph is highly recursive if it is recursive, every vertex has finite degree, and its degree function is recursive. Another method was to consider digraphs, and in particular posets. Here the additional structure of the orientation of an edge provides useful information. Later, more sophisticated methods from graph theory led to positive results for the original problem.

At the same time, computer scientists began considering the problem of on-line coloring. An on-line graph coloring algorithm receives the vertices of a graph one at a time. When a vertex is received, the algorithm also learns the edges between it and the previous vertices. At this time it must irrevocably color it. There is an obvious, although not exact, correspondence between on-line and recursive coloring algorithms. However, there are many interesting results concerning on-line coloring finite graphs that have no recursive version, because the performance is measured not only in terms of chromatic number, but also in terms of the number of vertices. Moreover, there are some real world motivations for considering on-line coloring.

Speaker: **Hal Kierstead** (Arizona State University)
Title: *The Survival Game* (slides available)
Summary: The following (p, s, t)-*survival game* plays a critical role in my analysis with Konjevod of on-line Ramsey theory. The game is played by two players Presenter and Chooser. It begins with presenter choosing a positive integer n and fixing a hypergraph $H_0 = (V_0, E_0)$ with n vertices and no edges. The game now proceeds in rounds. Let $H_{i-1} = (V_{i-1}, E_{i-1})$ be the hypergraph constructed in the first $i - 1$ rounds. On the i-th round Presenter plays by presenting a p-subset $P_i \subseteq V_{i-1}$ and Chooser responds by choosing an s-subset $X_i \subseteq P_i$. The vertices in $P_i - X_i$ are discarded and the edge X_i is added to E_{i-1} to form E_i. So $V_i = V_{i-1} - (P_i - X_i)$ and $E_i = E_{i-1} \cup \{X_i\}$. Presenter wins the survival game if for some i, the hypergraph H_i contains a copy of the complete s-uniform hypergraph K_s^t on t vertices. Kierstead discuss a proof that Presenter has a winning strategy for all positive integers p, s, t with $s \leq p$. The case $s = 2$ is an entertaining puzzle, but for larger s the only known strategy uses finite model theoretic techniques and requires more than $n = A(2^s - 1, t)$ starting vertices, where A is the Ackermann function.

Speaker: **Bjørn Kjos-Hanssen** (University of Hawai‘i at Mānoa)
Title: *Birth-death processes, bushy trees, and a law of weak subsets* (slides available)
Summary: The proof that every set of integers that is Martin-Löf random relative to $0'$ has an infinite subset that computes no Martin-Löf random set was presented. The relation between this result and the still-open question whether Stable Ramsey's Theorem for Pairs implies Weak Weak König's Lemma was discussed.

Speaker: **Ulrich Kohlenbach** (Technische Universität Darmstadt)
Title: *Tao's correspondence principle, a finitary mean ergodic theorem and conservation results for Ramsey's theorem for pairs.* (slides available)
Summary: In the first part of the talk focussed on the proof theory of a correspondence principle implicit in recent work of T. Tao and apply this principle to study the strength of different finitary versions (one by Tao and another one – inspired by monotone functional interpretation – due to ourselves) of the infinite pigeonhole principle (joint work with J. Gaspar). In the second part we show how recent proof theoretic metatheorems can be used to provide quantitative finitizations even in the absence of compactness. As an example we give a new quantitative form of the mean ergodic theorem for uniformly convex Banach spaces (joint work with L. Leustean). In the third part we calibrate the provable recursive function(al)s of systems that may use fixed sequences of instances of Ramsey's theorem for pairs (joint work with A. Kreuzer).

Speaker: **Alberto Marcone** (Universitá di Udine, Italy)
Title: *An interaction between reverse mathematics and computable analysis* (slides available)
Summary: There is a natural correspondence between subsystems of second order arithmetic and some functions studied in computable analysis, although results cannot be translated automatically in either direction. In recent work with Guido Gherardi we use this correspondence to solve an open problem in computable analysis la Weihrauch. We introduce the computable analysis version of $\mathbf{WKL}_0$, and prove its equivalence with the computable analysis version of the Hahn-Banach theorem.

Speaker: **Antonio Montalbań** (Univerity of Chicago)
Title: *On the Strength of Fraïssé's conjecture.* (slides available)

Summary: Fraïssé's conjecture, which is now Laver's theorem, says that the class of countable linear orderings is well-quasi-ordered by the relation of embeddability. A well-quasi-ordering is a partial ordering with no infinite descending sequences and no infinite antichains. The question of what is the proof theoretic strength of Fraïssé's conjecture has been open for twenty year. Some progress has been made but the question is still open.

In Montalbán's Ph.D. thesis he proved that Fraïssé's conjecture is equivalent to many statements about embeddability of linear orderings. This shows the statement is robust in the sense that is equivalent to many theorems in a certain area of math, and equivalent to all small variations of these theorems. Only the big five systems are known to be robust.

Montalbán also talked about the plan of attack that he and Alberto Marcone have, which involves ordinal notations up to the ordinal Γ_0.

Speaker: **Jim Schmerl** (UCONN)
Title: *Grundy colorings of graphs* (slides available)
Summary: A Grundy coloring of a graph is a special kind of proper coloring. Chromatic numbers of graphs are defined in terms of proper colorings, and Grundy numbers are defined using Grundy colorings. These concepts were discussed in the context of Reverse Mathematics.

Speaker: **Stephen G. Simpson** (Pennsylvania State University)
Title: *The Gödel Hierarchy and Reverse Mathematics* (slides available)
Summary: The Gödel Hierarchy is an array of foundationally significant theories in the predicate calculus. The theories range from weak (bounded arithmetic, elementary function arithmetic) through intermediate (subsystems of second-order arithmetic), through strong (Zermelo/Fraenkel set theory, large cardinals). The theories are ordered by inclusion, interpretability, and consistency strength. Reverse Mathematics is a program which seeks to classify mathematical theorems by calibrating their places within the Gödel Hierarchy. The theorems are drawn from core mathematical areas such as analysis, algebra, functional analysis, topology, and combinatorics. Remarkably, the Reverse Mathematics classification scheme exhibits a considerable amount of regularity and structure. In particular, a large number of core mathematical theorems fall into a small number of foundationally significant equivalence classes (the so-called "big five"). There are close connections with other foundational programs and hierarchies. In particular, concepts and methods from degrees of unsolvability play an important role.

Speaker: **Reed Solomon** (UCONN)
Title: *Classically equivalent definitions of well quasi-orders*
Summary: There are several classically equivalent ways to define a well quasi-order. The equivalence of these definitions follows from Ramsey's Theorem for Pairs and we explore the reverse mathematical strength of these equivalences. This work is joint with Alberto Marcone and Peter Cholak.

Speaker: **Frank Stephan (NUS)**
Title: *Implementing Fragments of ZFC within an r.e. Universe*
Summary: Rabin showed that there is no r.e. model of the axioms of Zermelo and Fraenkel of set theory. In the present work, it is investigated to which extent one can have natural models of a sufficiently rich fragment of set theory. These models are generated by considering the relation $x \in A_y$ to be generated from a Friedberg numbering $A_0, A_1, A_2, \ldots$ of all r.e. sets and then a member A_x of this numbering is called a set in the given model iff the downward closure of the induced ordering from x is well-founded. It is shown which axioms and basic properties of set theory can be obtained and which cannot be obtained in such a model. The major short-coming is that the axioms of comprehension and replacement do not hold in full generality, as, for example, the r.e. sets are not closed under complement and therefore differences of sets might not exist. Furthemrmore, only partial-recursive functions can be used in the axiom of replacement. The existence of transfinite ordinals depends on the model; some Friedberg models do not contain any of them while other Friedberg models contain all recursive ordinals. The axioms of pair, union and similar set constructions are satisfied, but not in a uniform way. That is, the union of two sets is a

set but one cannot find the index of it effectively. For the power set axiom, one can find a Friedberg model where for every set there is a further set consisting of all the (r.e.) subsets of the first set. This is the best possible result which one can obtain and the operation is again not effective. Besides these constructions, it has been determined which sets exist in all Friedberg models. Some sets like the set V_ω of all hereditarily finite sets does not exist in any Friedberg model.

Speaker: **Henry Towsner** (UCLA)
Title: *How Constructive is Furstenberg's Multiple Recurrence Theorem?* (slides available)
Summary: On its face, Furstenberg's method for proving Szemeredi's Theorem seems to be as non-constructive as possible, with multiple applications of compactness and a transfinite induction. Yet other proofs, such as those by Szemeredi and Gowers, show that these theorems can be proven by explicit combinatorial means. The use of a "partial Dialectica translation" to eliminate the transfinite aspects of the argument was discussed, as was how conventional unwinding methods suffice to eliminate the other non-constructive aspects.

Speaker: **Andreas Weiermann** (Ghent University)
Title: *Well partial orderings and their strengths in terms of maximal order type* (slides available)
Abstract: A well partial ordering (wpo) is a partial ordering which is well-founded and which does not admit infinite anti-chains. Famous examples for wpo's are provided by results of Higman, Kruskal, Friedman and Kriz. Every wpo can be extended to a well-ordering on the same domain such that the resulting order type is maximal possible and we may call this order type the maximal order type of the wpo under consideration. The reverse mathematics strengths of assertions about a wpo can typically be measured in terms of its maximal order type. It might therefore be of some interest to get a "formula" providing in natural situations the maximal order type of a wpo. In our talk we will suggest a general principle which yields appropriate maximal order types for the standard trees classes. The conjecture is that it also applies to the tree classes (equipped with an ordering fulfilling a certain gap condition) studied by Harvey Friedman. For classes of trees labeled with two labels some first results have been obtained and we believe that a general result will soon be available. Parts of the talk are based on joint work with A. Montalban, H. Friedman, and M. Rathjen.

Speaker: **Keita Yokoyama** (Tohoku University)
Title: *Non-standard analysis within second order arithmetic* (slides available)
Summary: Some systems of non-standard second order arithmetic and their interpretation to second order arithmetic were introduced. Then, he showed some Reverse Mathematics for non-standard analysis. For example, a non-standard version of the Weierstraßapproximation theorem is equivalent to ns-WKL_0, which is a conservative extension of WKL_0, and Σ^0_1-transfer principle for real numbers is equivalent to ns-ACA_0, which is a conservative extension of ACA_0. We can apply non-standard arguments in non-standard systems to standard Reverse mathematics. For example, we can show that the Riemann mapping theorem for Jordan region is provable within ns-WKL_0, thus, it is provable within WKL_0.

4 Open Problems

Probably the most famous open problem in the field is to determine the strength of Ramsey's Theorem for Pairs. In particular, does Ramsey's theorem for pairs imply Weak König's Lemma over RCA_0? There are many related open problems who's aim is to get a handle on this question.

During the meeting there was a Problem Session where many open problems were presented, far too many to include with context into this report. We are compiling these problems into a paper that we will make available to participants and also perhaps publish.

5 Scientific Progress Made

Stephen Simpson solved a problem that was presented by Keita Yokoyama in the problem session. Simpson showed that if (M, S) is a countable model of WWKL_0, then we can find $\overline{S} \supseteq S$ such that

$(M, \overline{S})$ is a countable model of WKL_0 and every closed set of positive measure which is coded in $\overline{S}$ contains points in S. Yokoyama needed this result in order to prove that his formal system for nonstandard analysis with Loeb measures, ns-BASIC + LMP, is conservative over WWKL_0.

At the meeting, Denis Hirschfeldt raised the question of whether the principle PART, which arose in his work with Richard Shore, is not just implied by $\mathrm{B}\Sigma_2$ but actually equivalent. Since the meeting, Chi Tat Chong, Steffen Lempp and Yue Yang have shown these to be equivalent, thus showing that $\mathrm{B}\Sigma_2$ is very robust over $\mathrm{I}\Sigma_1$ in both first-order and second-order arithmetic.

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