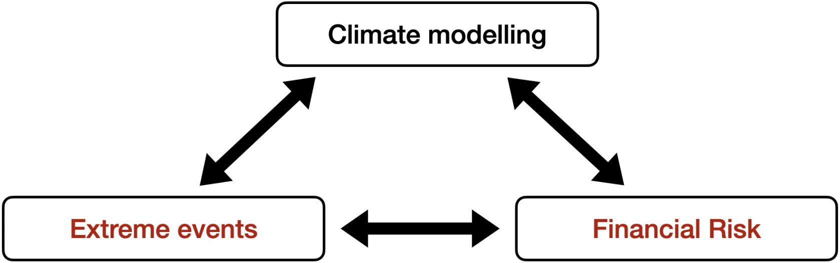
Extreme value analysis for financial risk management

Natalia Nolde

Department of Statistics
University of British Columbia
Vancouver, Canada

natalia@stat.ubc.ca

Joint work with Chen Zhou (Erasmus School of Economics, Erasmus University, and Economics and Research Division, Bank of the Netherlands)
and
Menglin Zhou (Department of Statistics, UBC)


Climate modelling
Extreme events
Financial Risk

Introduction

- **Risk in finance**: possibility of an adverse scenario that has potential to undermine financial stability of a financial institution or a market
- Hence, focus on extreme events corresponding to such adverse scenarios
- **Extreme value analysis** (EVA) offers a natural theoretical paradigm based on extreme value theory combined with a modern set of statistical tools and techniques to address a wide range of questions arising in the realm of financial risk assessment and management
- **Goal**: a review of advances in EVA that provide useful solutions to new challenges in financial risk management

Risk categories and EVA applicability

- Risk categories subject to quantitative methods and regulatory scrutiny:
  - Within a bank: credit, market and operational risk*
  - Systemic risk: when failure of a single financial institution could lead to a failure of the entire system (e.g., industry or economy)
- Different risk categories have different data availability and data characteristics
- EVA is a data intensive approach and hence:
  - EVA methods are well suited to measure market risk
  - For operational risk, data pooling across institutions is necessary
  - For credit risk, EVA can be used but usually as a modelling tool
  - For systemic risk, EVA can be applied to institutions with market indicators (e.g., stock prices, CDS spreads)

Talk overview

- Motivating example: Measuring risk of an investment portfolio
- Univariate extreme value analysis with application to market risk measurement
- Multivariate extreme value analysis with application to systemic risk and reverse stress testing
- Extreme value analysis for serially dependent data
- Open problems

Motivating example: Measuring risk of an investment portfolio

- Risk measurement is used
  - as a basis for setting regulatory capital requirements for financial institutions
  - as part of internal risk management, to constraint amount of risk traders at a bank may take
- While different methods exist, a statistically rigorous approach is based on the loss distribution
  - Consider an investment portfolio of financial assets (e.g., stocks, bonds)
  - X: loss on the portfolio over a set time horizon with fixed portfolio decomposition over this period
  - The distribution of X is referred to as the loss distribution (denoted F_X) and X is the loss random variable

Univariate risk measures

- A common way to quantify risk is via a real-valued risk functional or a risk measure defined on the space of loss random variables
- Popular risk measures:
  - ✤ Value-at-Risk (VaR): until 2016, a long time standard measure of market risk in banking regulation

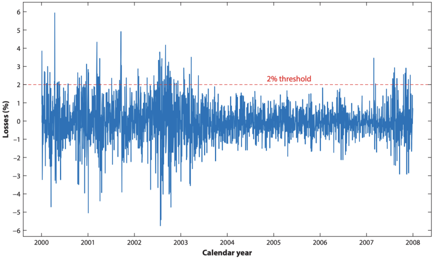
$$\mathrm{VaR}_p(X) = F_X^{\leftarrow}(p) = \inf_x \{F_X(x) \geq p\}$$

  - ✤ Expected shortfall (ES): the current standard measure of market risk

$$\mathrm{ES}_p(X) = \mathbb{E}(X \mid X \geq \mathrm{VaR}_p(X))$$

- $p \approx 1 \;\; \Rightarrow$ careful modelling of tail of the loss distribution is needed
  - ✤ for regulatory capital of banks' market risk: $p = 0.99$ for VaR and $p = 0.975$ for ES
  - ✤ for the banking book (assets on the bank's balance sheet to be held to maturity): $p = 0.999$

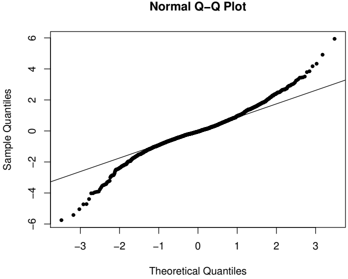
Motivating example (cont'd) - S&P 500 stock market index

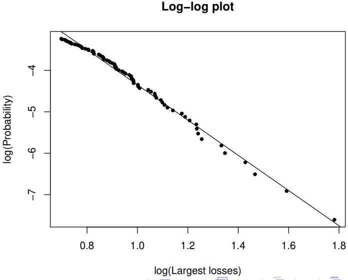


- Data are clearly heavy-tailed
- Tail behaviour is consistent with a Pareto-type model:

$$1 - F_X(x) \approx A x^{-\alpha}, \quad \alpha > 0, \ A > 0$$

- **Aim**: probabilistic models with focus on tail and only mild assumptions on F_X; inference using data in a tail region





Univariate Extreme Value Analysis

- Mathematically, heavy-tailed behaviour is usually characterized by the condition of regular variation
- A df F with infinite upper endpoint is said to have a **regularly varying** (upper) tail with tail index $\alpha > 0$, denoted as $1 - F \in \mathrm{RV}_{-\alpha}$, if

$$\lim_{x \to \infty} \frac{1 - F(tx)}{1 - F(x)} = t^{-\alpha}, \qquad t > 0$$

- Examples include Student's t, skew-t, Pareto and log-gamma distributions

Univariate Extreme Value Analysis (cont'd)

- Regular variation allows estimation of risk measures at some extreme probability level by extrapolating from a less extreme level
- For p, q close to one and $q < p$:

$$\frac{1-p}{1-q} = \frac{1 - F_X(\mathrm{VaR}_p(X))}{1 - F_X(\mathrm{VaR}_q(X))} \approx \left(\frac{\mathrm{VaR}_p(X)}{\mathrm{VaR}_q(X)} \right)^{-\alpha}$$

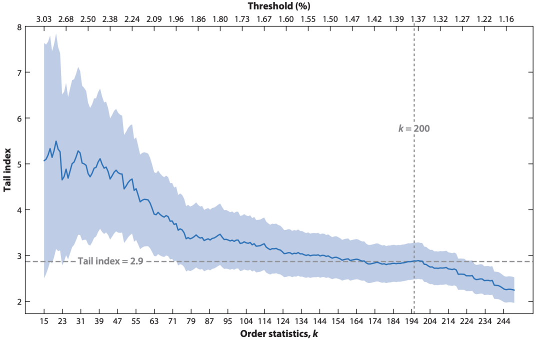
$$\Rightarrow \; \mathrm{VaR}_p(X) \approx \mathrm{VaR}_q(X) \left(\frac{1-q}{1-p} \right)^{1/\alpha}$$

- To use this asymptotic approximation, we need an estimate of tail index α

Tail index estimation

- Hill estimator (Hill (1975)) is a popular estimator of the (reciprocal of) tail index α
- $X_1, \ldots, X_n \overset{i.i.d.}{\sim} F$ with $1 - F \in \mathrm{RV}_{-\alpha}$
- With $\xi = 1/\alpha$ and $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$ order statistics

$$\hat{\xi}^H_{k,n} = \frac{1}{k} \sum_{i=1}^{k} \log X_{n-i+1,n} - \log X_{n-k,n}$$



Connections between limit results in extreme value theory

- $X_1, \ldots, X_n \overset{i.i.d.}{\sim} F$; let $M_n = \max\{X_1, X_2, \ldots, X_n\}$
- A df F belongs to the maximum domain of attraction of df G, $F \in \mathcal{D}(G)$, if G is non-degenerate and there exist $a_n > 0$, $b_n \in \mathbb{R}$ such that

$$\mathbb{P}\Big((M_n - b_n)/a_n \leq x\Big) = F^n(a_n x + b_n) \to G(x), \quad n \to \infty, \quad x \in \mathcal{C}(x)$$

  - Fisher-Tippett-Gnedenko theorem tells that G is the generalized extreme value distribution (up to type)

$$G_\xi(x) = \exp\{-(1+\xi x)^{-1/\xi}\}, \qquad 1 + \xi x > 0, \qquad \xi \in \mathbb{R}$$

  - If $\xi > 0$, $G_\xi(x)$ corresponds to the Fréchet distribution, and regular variation of $1 - F_X$ is a necessary and sufficient condition for $F \in \mathcal{D}(G_\xi)$

Connections between limit results in extreme value theory (cont'd)

- If $F \in \mathcal{D}(G_\xi)$, the conditional distribution of excesses $X - u$ given $X \geq u$, after proper scaling, converges to a generalized Pareto distribution with shape parameter ξ for $u \to x_F$

$$H(y; \sigma, \xi) = \begin{cases} 1 - (1 + \xi \frac{y}{\sigma})_+^{-1/\xi}, & \xi \neq 0; \\ 1 - \exp(-y/\sigma), & \xi = 0 \end{cases}$$

(Pickands-Balkema-de Haan theorem)

Peaks-over-threshold (POT) method

- The generalized Pareto distribution can be used to model losses exceeding a high threshold
- Write: $\overline{F}(x) = 1 - F(x)$ and $\overline{F}_u(y) = \mathbb{P}(X - u > y \mid X > u)$
- We have: $\overline{F}(x) = \overline{F}(u) \times \overline{F}_u(x - u)$ for $x > u$
- Then, for threshold u large:

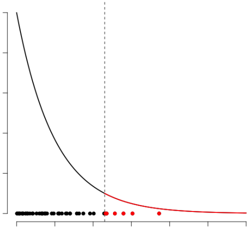
$$\begin{aligned} 1 - p = \overline{F}(\mathrm{VaR}_p(X)) &= \overline{F}(u) \times \overline{F}_u(\mathrm{VaR}_p(X) - u) \\ &\approx \overline{F}(u) \times \overline{H}(\mathrm{VaR}_p(X) - u; \sigma_u, \xi) \end{aligned}$$

$$\Rightarrow \ \mathrm{VaR}_p(X) \approx u + \frac{\sigma_u}{\xi}\left(\left(\frac{1-p}{\overline{F}(u)}\right)^{-\xi} - 1\right)$$

(POT high quantile estimator)

From univariate to multivariate EVA

- The concept of ordering plays an essential role in defining an extreme event
- For univariate data, there is a natural way to order sample points and hence a single direction for extrapolation
- Peaks-over-threshold method:



$$\mathbb{P}(X > x) = \mathbb{P}(X > u)\ \mathbb{P}(X > x \mid X > u), \quad x > u$$
$$\approx \mathbb{P}(X > u)\ \overline{H}(x - u; \sigma_u, \xi) \quad \text{for large } x,$$

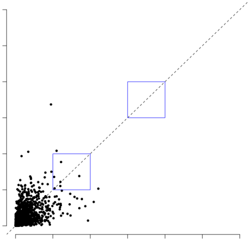
where $H(\cdot; \sigma_u, \xi)$ is the cdf of a generalized Pareto distribution

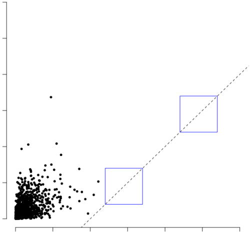
Multivariate extreme value analysis

- For multivariate data, there is no natural ordering
- Implication:
  - There exist various ways for multivariate ordering and different directions for extrapolation

⇓

  - Different approaches and representations for multivariate extremes
  - Tail dependence structure is often key to which representation is most useful





Background

- The original approach to the study of multivariate extremes was based on the coordinatewise maxima $\boldsymbol{M}_n = \left(\max_{1 \le i \le n} X_{1,i}, \ldots, \max_{1 \le i \le n} X_{d,i} \right)$
- Asymptotic behaviour is studied after applying a linear normalization:

$$\frac{\boldsymbol{M}_n - \boldsymbol{a}_n}{\boldsymbol{b}_n}$$

  with normalizing sequences $\boldsymbol{a}_n$ and $\boldsymbol{b}_n$ determined by marginal distributions
- The limiting distribution of normalized maxima exists when margins are in the domain of attraction and the dependence structure satisfies the condition of multivariate regular variation (MRV)

Background

- The original approach to the study of multivariate extremes was based on the coordinatewise maxima $\boldsymbol{M}_n = \left(\max_{1 \le i \le n} X_{1,i}, \ldots, \max_{1 \le i \le n} X_{d,i} \right)$
- Asymptotic behaviour is studied after applying a linear normalization:

$$\frac{\boldsymbol{M}_n - \boldsymbol{a}_n}{\boldsymbol{b}_n}$$

  with normalizing sequences $\boldsymbol{a}_n$ and $\boldsymbol{b}_n$ determined by marginal distributions
- The limiting distribution of normalized maxima exists when margins are in the domain of attraction and the dependence structure satisfies the condition of multivariate regular variation (MRV)
- While block maxima approach is nowadays less common, MRV assumption is widely used in applications

Background (cont'd)

- A measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is regularly varying at infinity with index ρ (written $f \in \mathrm{RV}_\rho^\infty$) if

$$f(tx)/f(t) \to x^\rho, \qquad x > 0, \qquad t \to \infty$$

- Random vector $\mathbf{X}$ is said to be multivariate regularly varying on cone $\mathbb{E} = [0, \infty]^d \setminus \{\mathbf{0}\}$, with index $\alpha > 0$, if for any relatively compact $B \subset \mathbb{E}$,

$$t\mathbb{P}(\mathbf{X}/b(t) \in B) \to \nu(B), \qquad t \to \infty,$$

with $\nu(\partial B) = 0$, $b(t) \in \mathrm{RV}_{1/\alpha}^\infty$, and the limit measure ν homogeneous of order $-\alpha$

Background (cont'd)

- Note that the limiting distribution of normalized maxima (when it exists) characterizes tail dependence when all marginal components are simultaneously extreme
- If the limiting distribution is a product measure, the components are asymptotically independent, in which case the limit measure ν is degenerate and MRV cannot be utilized in statistical modelling
- Intuitively, asymptotic independence refers to situations in which marginal components cannot be extreme at the same time
- This situation calls for alternative multivariate tail characterizations
  - hidden regular variation, conditional extreme value models, ...

Application to systemic risk

- Modelling of extremes of multivariate random vectors goes back to 1980's
- In financial risk managements, application of multivariate EVA is fairly recent and is related to modelling of systemic risk
- Systemic risk arises in situations when a financial distress experienced by an individual firm causes instability of the entire financial system
- The global financial crisis of 2008-2009 revealed the far-reaching impact of systemic risk on the global economy, and identified inadequacy of the existing risk management framework for financial institutions

CoVaR

- CoVaR is a popular measure of systemic risk, introduced by Adrian and Brunnermeier (2011)
- Define
  - X: loss for a financial institution
  - Y: loss for a system proxy such as a market index
- CoVaR at level $1-p$ is defined as the $(1-p)$-quantile of the conditional loss distribution

$$\mathbb{P}\{Y \geq \mathrm{CoVaR}_p \mid X \geq \mathrm{VaR}_p(X)\} = 1-p$$

i.e., CoVaR is the value-at-risk (quantile) of a market index conditional on an institution being in financial distress

CoVaR estimation - literature review

- Quantile regression techniques have been proposed for CoVaR estimation in the Adrian and Brunnermeier (2011) formulation

- Girardi and Ergün (2013) adopt a fully parametric approach via a bivariate AR(1)-GARCH(1,1) model with the Engle (2002) DCC specification, and a bivariate skew-t distribution for innovations

- Nolde and Zhang (2018) propose an EVT-based semi-parametric approach assuming multivariate regular variation and a parametric model for the spectral density motivated by the class of skew-elliptical distributions

  - **Remark:** While this approach alleviates some of the model risk in Girardi and Ergün (2013), it is restrictive in modelling different tail decays for losses of an institution and a system proxy, and the tail dependence structure

CoVaR estimation - literature review

- Quantile regression techniques have been proposed for CoVaR estimation in the Adrian and Brunnermeier (2011) formulation
- Girardi and Ergün (2013) adopt a fully parametric approach via a bivariate AR(1)-GARCH(1,1) model with the Engle (2002) DCC specification, and a bivariate skew-t distribution for innovations
- Nolde and Zhang (2018) propose an EVT-based semi-parametric approach assuming multivariate regular variation and a parametric model for the spectral density motivated by the class of skew-elliptical distributions
    - **Remark:** While this approach alleviates some of the model risk in Girardi and Ergün (2013), it is restrictive in modelling different tail decays for losses of an institution and a system proxy, and the tail dependence structure

Aim: develop an EVT-based methodology for CoVaR estimation that relaxes assumption of multivariate regular variation

Probabilistic Framework

Assumptions:

Suppose (X, Y) has df F with continuous margins F_X and F_Y

(i) F has upper tail dependence function $R \not\equiv 0$

$$\lim_{u \to 0} \frac{\mathbb{P}\{F_X(X) \geq 1 - ux, F_Y(Y) \geq 1 - uy\}}{u} =: R(x, y)$$

(ii) $1 - F_Y \in RV_{-1/\gamma}$ for some $\gamma > 0$

Probabilistic Framework (cont'd)

- Define a constant η_p:

$$\eta_p := \frac{\mathbb{P}\big(Y \geq \mathrm{CoVaR}_p\big)}{\mathbb{P}\big(Y \geq \mathrm{CoVaR}_p \mid X \geq \mathrm{VaR}_p(X)\big)}$$

- It follows that $\mathbb{P}\big(Y \geq \mathrm{CoVaR}_p\big) = (1-p)\eta_p$

  i.e., CoVaR is related to quantile at level $(1-p)\eta_p$ of the unconditional distribution of Y via

$$\mathrm{CoVaR}_p = \mathrm{VaR}_{1-(1-p)\eta_p}(Y)$$

Empirical Analysis

- Data description
  - 14 financial institutions with data between Jan.1, 2000 to Dec.30, 2021, consisting of 5535 daily closing price records for each time series
  - The S&P 500 Index is used as a proxy for the aggregate financial system
  - Daily losses (%) are calculated as negative log-returns

Data description - List of financial institutions

- AFLAC INC (AFL)
- AMERICAN INTERNATIONAL GROUP INC (AIG)
- ALLSTATE CORP (ALL)
- BANK OF AMERICA CORP (BAC)
- HUMANA INC (HUM)
- J P MORGAN CHASE & CO (JPM)
- LINCOLN NATIONAL CORP (LNC), M B I A INC (MBI)
- PROGRESSIVE CORP OH (PGR)
- U S A EDUCATION INC (SLM)
- TRAVELERS COMPANIES INC (TRV)
- UNUMPROVIDENT CORP (UNM)
- WELLS FARGO & CO NEW (WFC)
- WASHINGTON MUTUAL INC (WM)

Comparative backtests[1]

(a) MBI (b) Combined

[1] See, e.g., Nolde and Ziegel (2017) for details

Traditional vs. Reverse Stress Testing

- Traditional stress testing: given extreme scenarios of risk factors, what are potential consequences for a given institution?
  - Choice of stress scenarios is arbitrary based on expert opinion or historical data
    - may not meet the plausibility requirement of stress scenarios as set by the Basel Committee on Banking Supervision (2005)
    - can exclude scenarios leading to highly adverse outcomes
  - The global financial crisis of 2007-2009 revealed limitations of traditional stress testing
- Reverse stress testing: given an adverse outcome (a loss of given magnitude), what scenarios of risk factors would lead to that outcome?
  - emphasized by supervisory authorities
    (Basel Committee on Banking Supervision (2009), Committee of European Banking Supervision (2009) and Financial Services Authority (2009)
  - used for internal risk management decisions (e.g., limits on trading)

Reverse Stress Testing (RST)

- Aim: to identify most probable scenarios for risk factors that lead to a specified adverse portfolio outcome (a large portfolio loss)
- Let
  - $\mathbf{X} \in \mathbb{R}^d$: changes in risk factors
  - $L = g(\mathbf{X})$: portfolio loss
  - $f(\cdot \mid L \geq \ell)$: conditional density of $\mathbf{X}$ given $L \geq \ell$
- Define a stress scenario as

$$\mathbf{m}^*(\ell) = \text{argmax}_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x} | L \geq \ell)$$

- RST involves estimation of stress scenarios $\mathbf{m}^*(\ell)$ for given ℓ

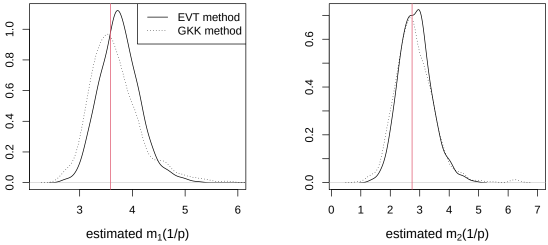
Starting point

- Glasserman, Kang and Kang (2015) propose the following method (referred to as the GKK method) for estimating $\mathbf{m}^*(\ell)$:
  - $(\mathbf{X}, L)$ is assumed to be elliptically distributed (with a regularly varying density generator)
  - A scaling factor is established between $\mathbf{m}^*(\ell)$ and conditional mean $\boldsymbol{\mu}(\ell) = E(\mathbf{X}|L \geq \ell)$ using elliptical symmetry
  - The conditional mean $\boldsymbol{\mu}(\ell)$ is estimated empirically with observations satisfying $L \geq \ell$
- The use of the empirical estimator of $\boldsymbol{\mu}(\ell)$ is warranted only when threshold ℓ is not too large, while the scaling factor between $\mathbf{m}^*(\ell)$ and $\boldsymbol{\mu}(\ell)$ is justified only under above model assumptions
- We propose an EVT-based estimator which uses direct extrapolation of the conditional mode into extreme regions under assumption of multivariate regular variation

Simulation Studies (1): bivariate t distribution

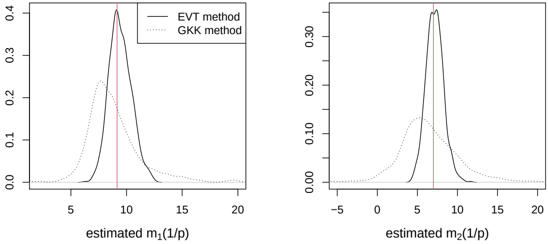
Top panel:

- $p = 1/100$
- $n = 1000$



Bottom panel:

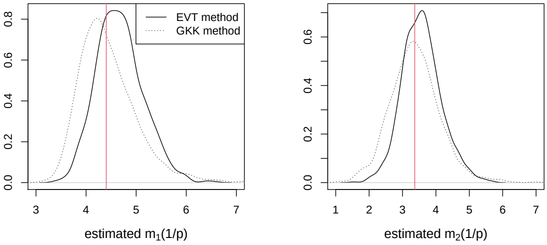
- $p = 1/3000$
- $n = 3000$



Simulation Studies (2): bivariate skew-t distribution

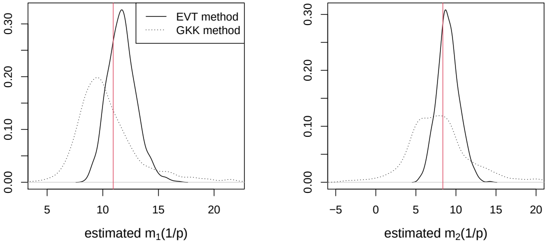
Top panel:

- $p = 1/100$
- $n = 1000$

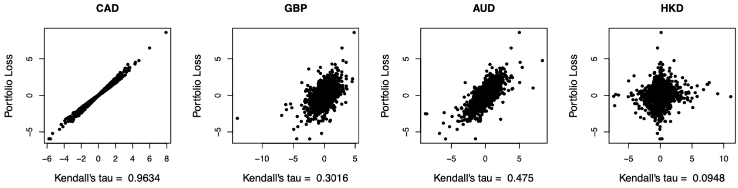


Bottom panel:

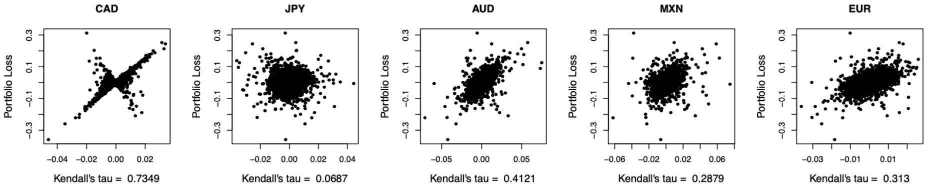
- $p = 1/3000$
- $n = 3000$



Application: Data

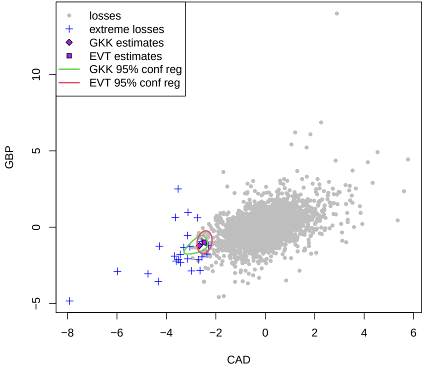


(a) Portfolio A

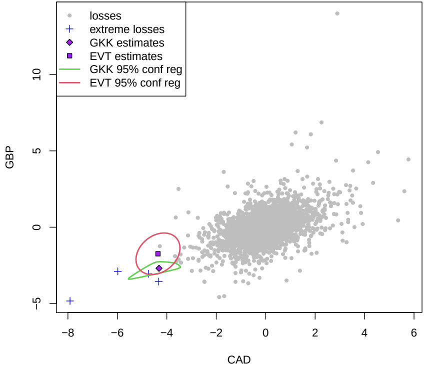


(b) Portfolio B

Application: Results - Portfolio A

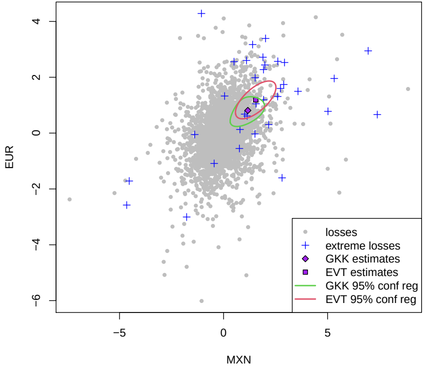


(c) $p = 0.01$

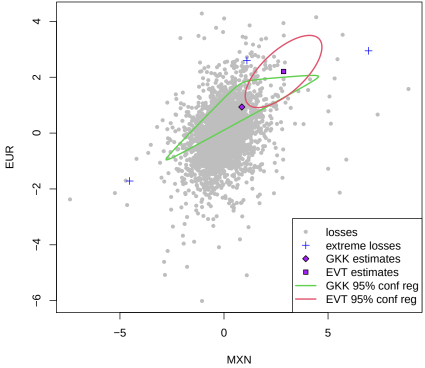


(d) $p = 0.001$

Application: Results - Portfolio B



(e) $p = 0.01$



(f) $p = 0.001$

Remarks on RST

- The proposed EVT-based estimator of stress scenarios provides a clear improvement over existing approaches when considered adverse outcomes are extreme, and it is applicable under weaker model assumptions

- In large aggregate portfolios, risk factors appear to be weakly associated with large portfolio losses.

  How to construct stress scenarios in this setting?

  Is it possible that a combination of (non-extreme) values in several risk factors leads to large portfolio losses?

EVA for serially dependent data

- Serial dependence in financial data
  - Filtering through GARCH-type models
  - Unconditional risk versus dynamic risk
- Extreme value theory for stationary serially dependent data

Open problems

- As financial risks are likely to change over time, there is a need for non-stationary modelling of extremes both univariate and multivariate

- Mixed dependence structures that can bridge asymptotic dependence and asymptotic independence

- Curse of dimensionality: dimension reduction techniques, sparse dependence structures, inference in high dimensions, ...

References

Main reference: N. Nolde and C. Zhou. Extreme value analysis for financial risk management. *Annual Review of Statistics and Its Application* 8: 217-240, 2021.

T. Adrian and M. Brunnermeier. CoVaR. Technical report, National Bureau of Economic Research, 2011.

R. Engle. Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics*, 20:339–350, 2002.

G. Girardi and A. Ergün. Systemic risk measurement: Multivariate GARCH estimation of CoVaR. *Journal of Banking & Finance*, 37:3169–3180, 2013.

N. Nolde and J. Zhang. Conditional extremes in asymmetric financial markets. *Journal of Business & Economic Statistics*, pages 1–29, 2018.

N. Nolde and J. Ziegel. Elicitability and backtesting: Perspectives for banking regulation. *The Annals of Applied Statistics*, 11:1833–1874, 2017.