RENEWAL EQUATIONS

a tribute to Kermack & McKendrick

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Building block

infectiousness as a function of time τ elapsed since exposure

τ aka "infection-age"

Outbreak

no demographic turnover

permanent immunity

second part

the simplest contact structure

in a

heterogeneous host population

but for the time being

every individual subject to
the same force-of-infection
and equally susceptible

$$\frac{dS}{dt} = -\overbrace{FS}^{\text{incidence}}$$

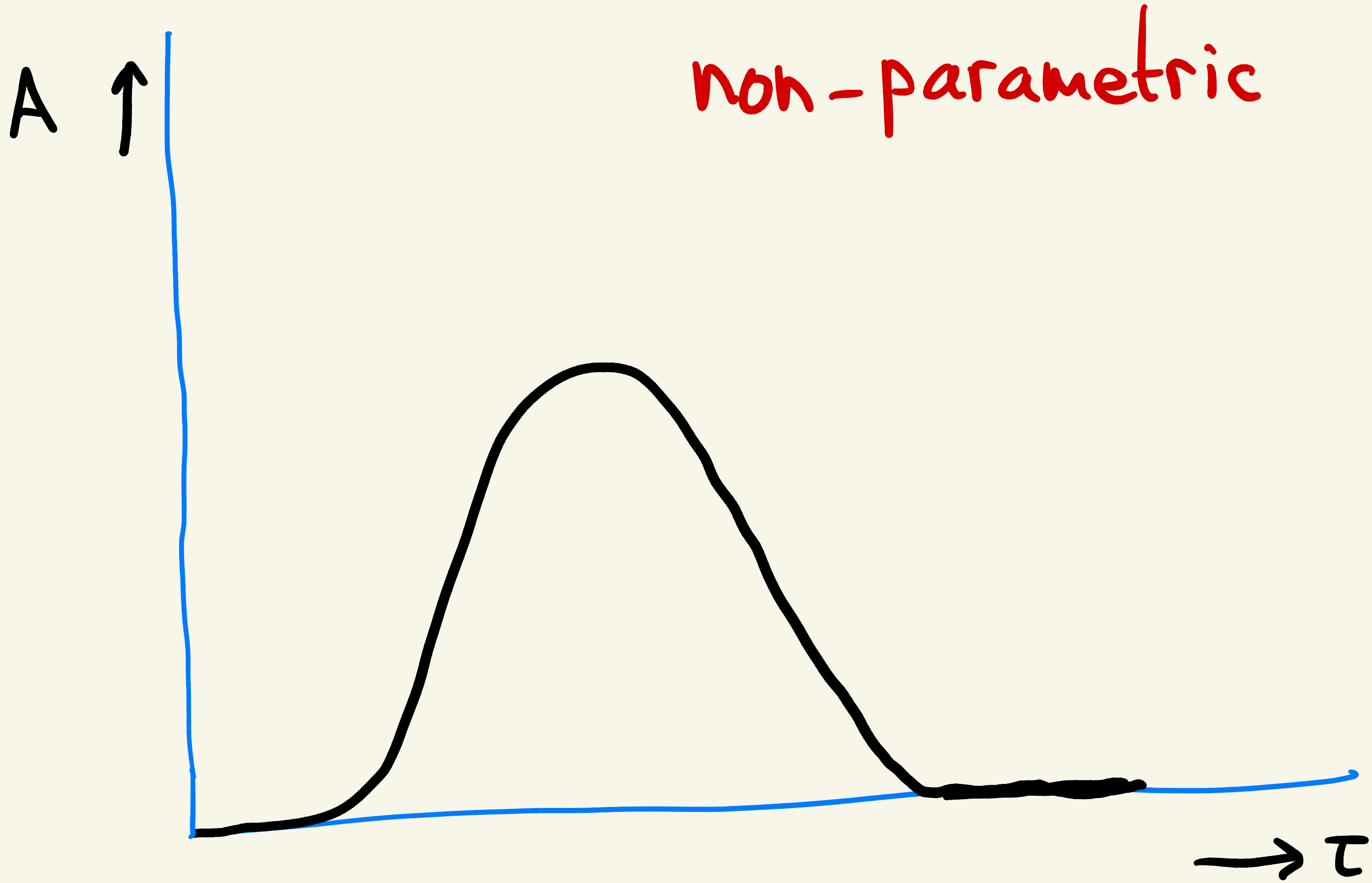
f-o-i

KM 27

$$F(t) = \int_0^\infty A(\tau)\,\text{incidence}(t-\tau)\,d\tau$$

$$A(\tau) = \text{expected contribution to the f-o-i at time } \tau \text{ after becoming infected}$$

input – output


non-parametric
A
τ

parametric example

$$A(\tau) = U e^{\tau \Sigma} V$$

$$U, V \in \mathbb{R}^n \qquad \Sigma \in \mathbb{R}^{n \times n}$$

individual level

V initial state

Σ state transitions

U output

i_state representation

fake

Analysis

Linearization:

$$\text{incidence} = F \cancel{S}$$
$$N$$

linear Renewal Equation $\leftarrow$ RE

$$F(t) = N \int_0^\infty A(\tau) F(t-\tau)\, d\tau$$

Lotka, Feller

Ansatz

$$F(t) = e^{\lambda t}$$

Euler Lotka

$$1 = N \int_0^\infty e^{-\lambda \tau} A(\tau)\, d\tau$$

real root

$$\lambda = r$$

Malthusian parameter

$$R_0 := N\int_0^{\infty} A(\tau)\,d\tau$$

Basic Reproduction Number

$$\operatorname{sign} r = \operatorname{sign}(R_0 - 1)$$

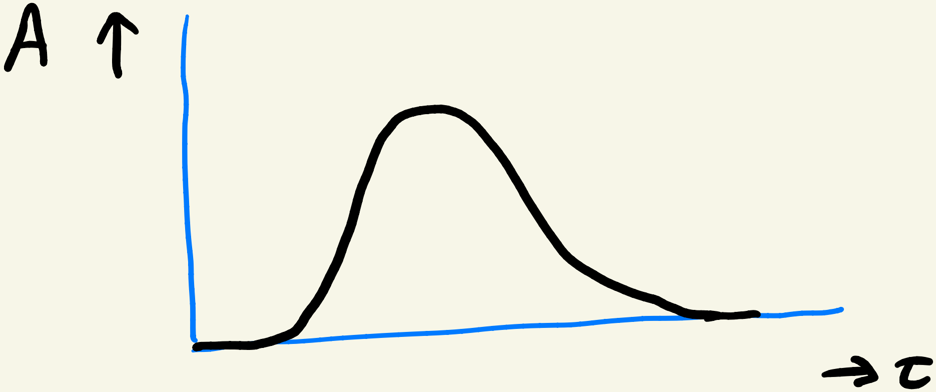
emerging disease

data $\Rightarrow$ estimate of r

Wanted

estimate of R_0

in view of control



shape matters

generation interval

Nonlinear

$$F(t) = \int_0^\infty A(\tau)\left\{-S'(t-\tau)\right\} d\tau$$

↑ integrate over $(-\infty, t]$

$$s(t) := \frac{S(t)}{N}$$

$$\boxed{s = e^{-w}}$$

w = cumulative f-o-i

$$w(t) = \int_0^\infty A(\tau)\,\Psi\big(w(t-\tau)\big)\,d\tau$$

scalar nonlinear RE

$$\boxed{\Psi(w) = N\left(1 - e^{-w}\right)}$$

Interlude

RE is a

Delay Equation

A delay equation is a rule for extending a function of time towards the future on the basis of the (assumed to be) known past

$$DE = DDE \cup RE$$

A dynamical system is defined by translation along the extended function

$$s_t(\theta) = s(t+\theta) \quad , \quad \theta \leq 0$$

end of interlude

Final Size Equation

$$s(\infty) = e^{-R_0(1 - s(\infty))}$$

- probabilistic consistency condition
- "shape" of A is irrelevant

Compartmental Models

J.A.J. Metz
M. Gyllenberg

$$A(\tau) = U e^{\tau \Sigma} V$$

$$U, V \in \mathbb{R}^n \qquad \Sigma \in \mathbb{R}^{n \times n}$$

$$w = U \cdot Q$$

$$\boxed{\frac{dQ}{dt} = \Sigma Q + \Psi(U \cdot Q) V}$$

recall: $\Psi(w) = N(1 - e^{-w})$

integrated form

Y, V, Σ, U

may incorporate

asymptomatic
quarantined
hospitalized
....

individuals

put

$$Q(t) = \int_{-\infty}^{t} y(\sigma)\, d\sigma$$

then

$$\frac{dS}{dt} = -FS$$

$$\frac{dy}{dt} = \Sigma y + (FS)V$$

$$F = u \cdot y$$

standard form

SEIR example

$$\frac{d}{dt}\underbrace{\begin{pmatrix} E \\ I \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} -\gamma & 0 \\ \gamma & -\alpha \end{pmatrix}}_{\Sigma}\begin{pmatrix} E \\ I \end{pmatrix} + FS\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{V}$$

$$F = \underbrace{\begin{pmatrix} 0 & \beta \end{pmatrix}}_{u}\begin{pmatrix} E \\ I \end{pmatrix} = \beta I$$

why are compartmental models

omnipresent

?

i) omnipresence is self reinforcing

ii) RE are unfamiliar

iii) lack of tools for RE

Discrete time ← M. Kreck

$$s(t+1) = e^{-\hat{F}(t)}\, s(t)$$

time unit one day

$$\hat{F}(t) = \sum_{k=\cancel{1}\ m_1}^{\cancel{\infty}\ m_2} A_k \left(1 - e^{-\hat{F}(t-k)}\right) s(t-k)$$

essential parameters

M. Bootsma
H. Othmer
R. Planqué

PNAS (2021) 118 39

$$\text{peak} \begin{cases} \text{when} \\ \text{how high} \end{cases} ?$$

Herd Immunity Threshold

$$s_{crit} = \frac{1}{R_0}$$

$\left.\begin{array}{l}\text{vaccin induced}\\ \text{disease induced}\end{array}\right\}$ immunity disease induced

HIT

in the presence of

heterogeneity

(some more susceptible than others)

G. Gomes

T. Britton

F. Jülicher

N. Goldenfeld

M. Bootsma

D. Chan

H. Inaba

(in progress)

Trait $x \in \Omega$ ← static

$$S(t,x) = e^{-N\int_0^\infty \int_\Omega A(\tau,x,\xi)\left[1 - S(t-\tau,\xi)\right]\Phi(d\xi)\,d\tau}$$

trait distribution

abstract RE

For general theory see book by H. Inaba and recent paper by E. Franco e.a.

arXiv: 2201.05323

Separable Mixing:

$$A(\tau, x, \xi) = a(x)\, b(\tau)\, c(\xi)$$

independence makes "life" easy

$$\boxed{s(t,x) = e^{-a(x)w(t)}}$$

w = cumulative f-o-i

Side remark (for later use)

$$s(t, x_1) = \left(s(t, x_2)\right)^{\frac{a(x_1)}{a(x_2)}}$$

$$w(t) = \int_0^\infty b(\tau)\,\Psi\big(w(t-\tau)\big)\,d\tau$$

Scalar RE

$$\boxed{\Psi(w) := N \int_{\Omega} c(\xi)\left(1 - e^{-a(\xi)w}\right)\Phi(d\xi)}$$

Note $\Psi'(0) = N \int_{\Omega} c(\xi)a(\xi)\,\Phi(d\xi)$

side remark

if $a(x) = 1 \quad \forall x \in \Omega$

then

$$\Psi(w) = \bar{c} N \left(1 - e^{-w}\right)$$

$$\bar{c} := \int_\Omega c(\xi) \, \Phi(d\xi)$$

if

heterogeneity only affects infectiousness

and

numbers are large

simply take the

average infectiousness

end of side remark

$$R_0 = \Psi'(0) \int_0^\infty b(\tau)\, d\tau$$

$$R_{eff} = \Psi'(\omega) \int_0^\infty b(\tau)\, d\tau$$

HIT

$$\Psi'(w) = \frac{\Psi'(0)}{R_0} \Rightarrow w = \tilde{w}$$

$$\tilde{S} = \int_{\Omega} e^{-a(\xi)\tilde{w}} \, \Phi(d\xi)$$

Special Case

$$\Omega = (0, \infty)$$

$$a(x) = x$$

$$\int_\Omega x \, \Phi(dx) = 1$$

Gamma Distribution

Novozhilov

Φ has density

$$x \longmapsto \frac{p^p}{\Gamma(p)} x^{p-1} e^{-px}$$

$$\text{Variance} = \frac{1}{\rho}$$

$$\hat{\Phi}(\lambda) = \left(\frac{\lambda}{\rho} + 1\right)^{-1}$$

$$\Psi(w) = 1 - \left(\frac{w}{p} + 1\right)^{-q}$$

$$c(\xi) = 1 \quad \Longrightarrow \quad q = p$$

$$c(\xi) = \xi \quad \Longrightarrow \quad q = p+1$$

HIT does not depend on $b(\tau)$

$$c(\xi) = 1 \qquad \tilde{s} = R_0^{-1+\frac{1}{p+1}}$$

$$c(\xi) = \xi \qquad \tilde{s} = R_0^{-1+\frac{1}{\frac{1}{2}p+1}}$$

When $\quad b(\tau) = U e^{\tau \Sigma} V$

then $\quad w = U \cdot Q$

with $\quad \boxed{\dfrac{dQ}{dt} = \Sigma Q + \Psi(U \cdot Q) V}$

normalisation

$$a(\bar{x}) = 1$$

$$c(\bar{x}) = 1$$

for "representative" $\bar{x} \in \Omega$

standard form

$$\bar{s}(t) = s(t, \bar{x})$$

$$\frac{d\bar{s}}{dt} = -F\bar{s}$$

$$\frac{dY}{dt} = \Sigma Y + F \boxed{\Psi'(-\ln \bar{s})} V$$

$$F = u \cdot Y$$

$$\Psi'(-\ln \bar{s}) = N \int_{\Omega} c(\xi)\, a(\xi)\, \bar{s}^{\,a(\xi)}\, \Phi(d\xi)$$

$$s_{tot.} = \int_{\Omega} \bar{s}^{\,a(\xi)} \, \Phi(d\xi)$$

Example: two types

A	not-vaccinated	$a=1$	$c=1$
B	vaccinated	$a=\varepsilon_1$	$c=\varepsilon_2$

$$\Psi'(-\ln \bar{s}) = N_A \bar{s} + N_B \varepsilon_1 \varepsilon_2 \bar{s}^{\varepsilon_1}$$

$$s_{tot.} = \frac{N_A}{N} \bar{s} + \frac{N_B}{N} \bar{s}^{\varepsilon_1}$$

Gamma Distribution

$$\frac{ds_{tot.}}{dt} = -F\, s_{tot.}^{1+\frac{1}{p}}$$

$$\frac{dY}{dt} = \Sigma Y + F H(s_{tot.}) V$$

$$F = u . Y$$

$$H(s) = \begin{cases} N s^{1+\frac{1}{p}} & \longleftarrow \ c(\xi) = 1 \\[2ex] N\left(1+\frac{1}{p}\right) s^{1+\frac{2}{p}} & \longleftarrow \ c(\xi) = \xi \end{cases}$$

Summary

the KM 1927 model takes the form of a

RE

the kernel

$A(\tau)$ resp. $\{A_k\}$ discrete time

is the

essential ingredient

compartmental models

miss the mark

sojourn times are not

exponentially distributed

RE

exhibit remarkable

bookkeeping efficiency

also when dealing with

heterogeneity

Appeal

next time you formulate
an epidemic model,
please try a

RE