Area-Depth Symmetric Catalan Polynomial

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joint work with:
Digjoy Paul

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Table of Contents

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q, t-Catalan polynomial

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Theorem (Garsia, Haglund 2002; Haiman 2002)

$C_n(q,t)$ is symmetric in q and t.

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Theorem (Garsia, Haglund 2002; Haiman 2002)

$C_n(q,t)$ is symmetric in q and t.

Open Problem: Find a combinatorial proof that shows $C_n(q,t)$ is symmetric.

New Symmetric Polynomials

Definition (P., Paul, S. 2021)

Let the area-depth polynomial $F_n(q,t)$ and dinv-ddinv polynomial $G_n(q,t)$ be defined as follows:

- $F_n(q,t) = \sum_{\pi \in D_n} q^{\text{area}(\pi)} t^{\text{depth}(\pi)}$
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- $G_4(q,t) =$
  $q^6 + q^5 t^2 + q^4 t^3 + q^4 t^2 + q^2 t + 2q^3 t + 2qt^3 + qt^2 + q^2 t^4 + q^3 t^4 + q^2 t^5 + t^6$

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Theorem (P., Paul, S. 2021)

$C_n(q,t) = \sum_{\pi \in D_n} q^{\mathrm{depth}(\pi)} t^{\mathrm{ddinv}(\pi)}$

Statistics on Dyck Paths

Definition

Let $(a_1(\pi), a_2(\pi), \ldots, a_n(\pi))$ be the area sequence of π where $a_i(\pi)$ is the number of full cells between π and the diagonal in the ith row.
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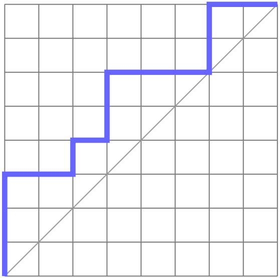

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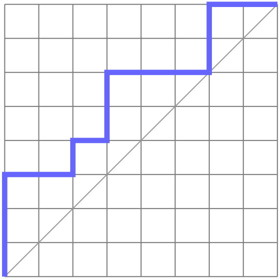

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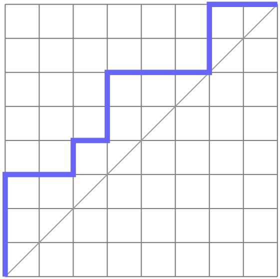

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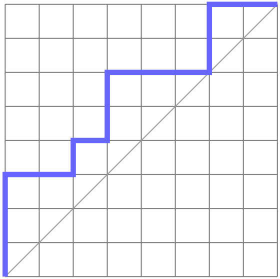

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- $(a_1(\pi), \ldots, a_n(\pi)) = (0, 1, 2, 1, 1, 2, 0, 1)$
- $\text{area}(\pi) = 8$
- Remark: Dyck paths are uniquely characterized by their area sequences.

Statistics on Dyck Paths

Definition

A diagonal inversion of π is a pair (i, j) such that

- $i < j$
- $a_i(\pi) = a_j(\pi)$ or $a_i(\pi) = a_j(\pi) + 1$

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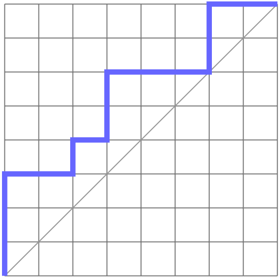

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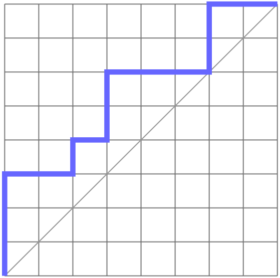

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- $(a_1(\pi), \ldots, a_n(\pi)) = (0,1,2,1,1,2,0,1)$
- Diagonal inversions of π:
  $(1,7),(2,4),(2,5),(2,8),(4,5),(4,8),(5,8),(3,6),$
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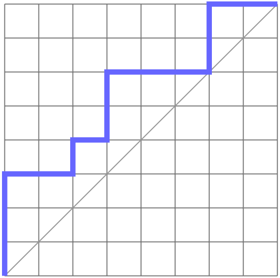

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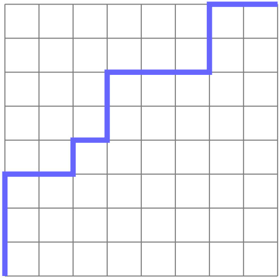
- $\mathrm{dinv}(\pi) = 15$

New Statistics on Dyck Paths

Definition (P., Paul, S., 2021)

The depth labelling of π is a labelling of the cells directly right of the North steps in π by:

- labelling all relevant cells in the first column with a 0



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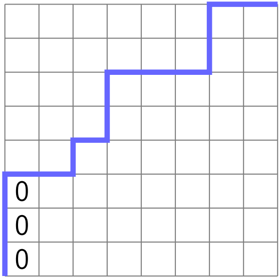

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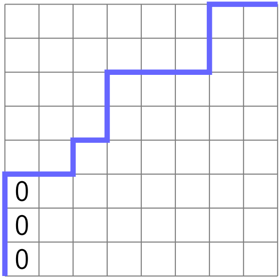

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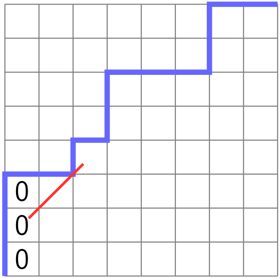

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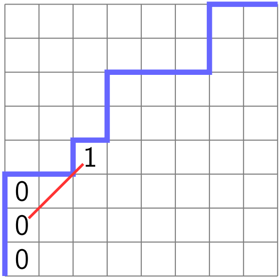

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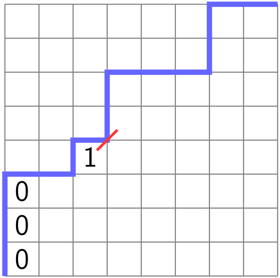

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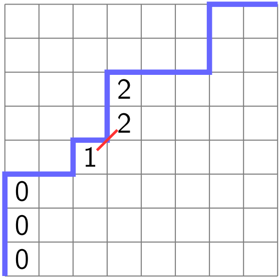

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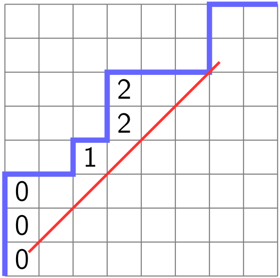

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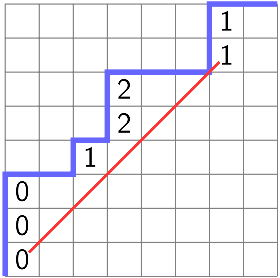

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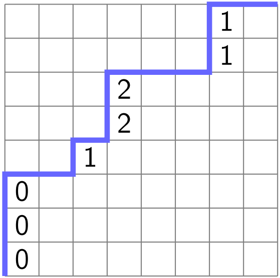

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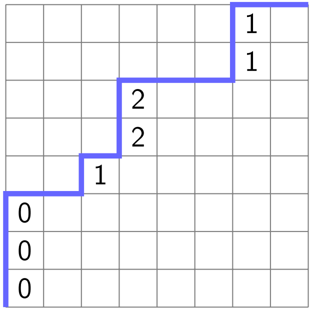

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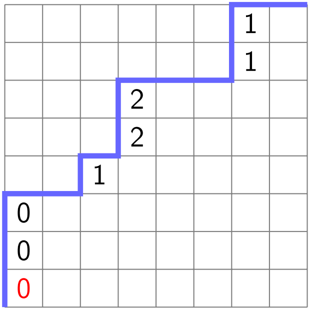

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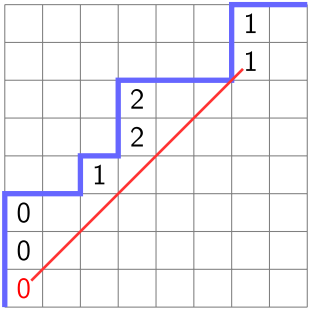

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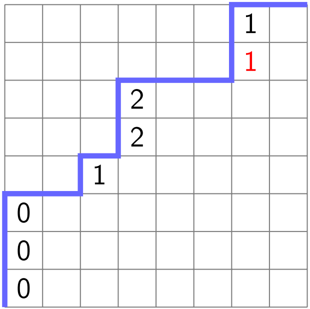

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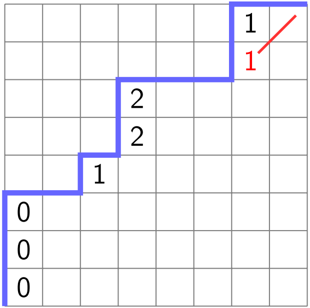

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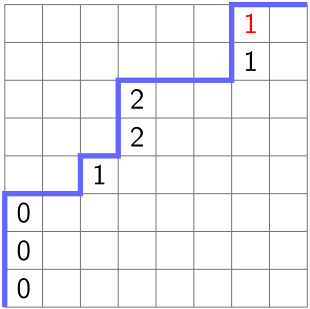

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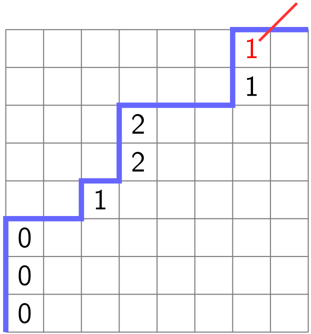

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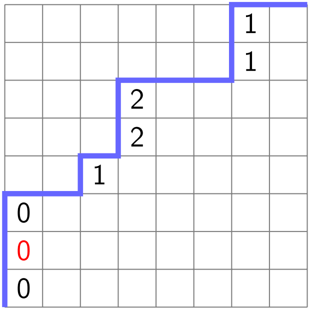

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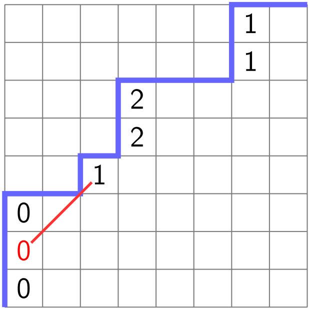

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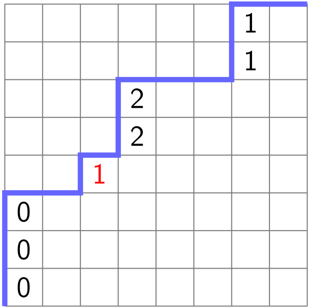

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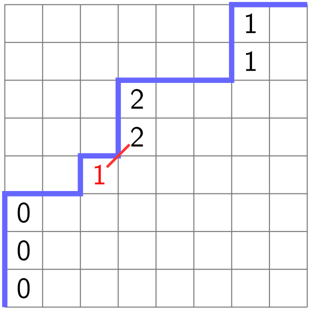

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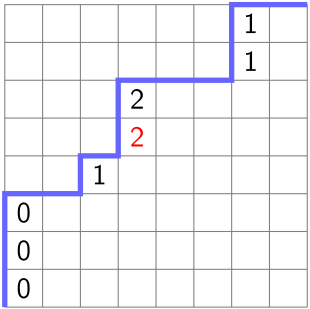

- $(d_1(\pi), d_2(\pi), \ldots, d_n(\pi)) = (0, 1, 1, 0, 1, 2 \qquad)$

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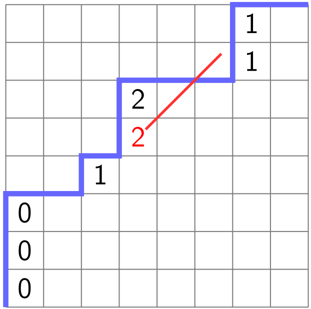

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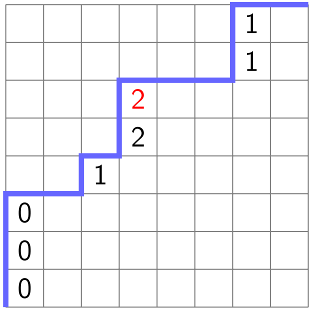

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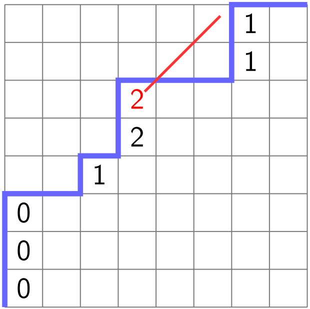

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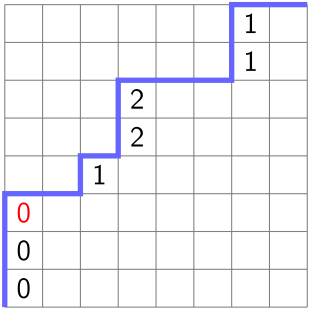

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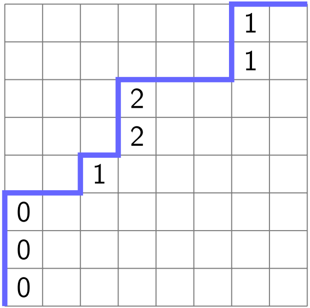

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- $\text{depth}(\pi) = 0 + 1 + 1 + 0 + 1 + 2 + 2 + 0 = 7$

New Statistics on Dyck Paths

Definition (P., Paul, S. 2021)

A depth diagonal inversion of π is a pair (i, j) such that

- $i < j$
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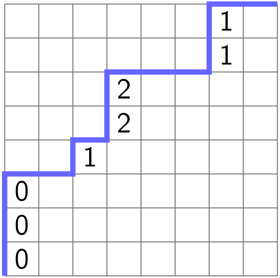

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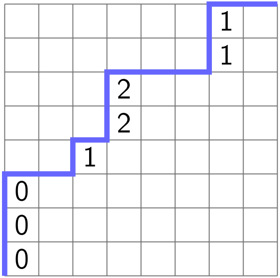

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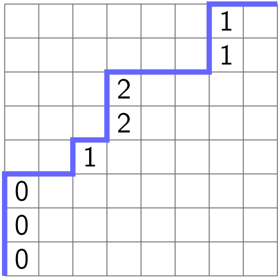

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Plane Trees

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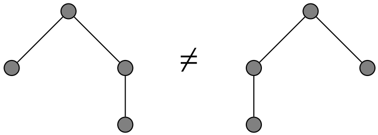

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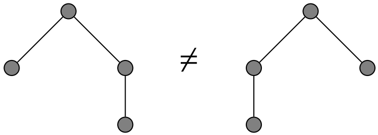

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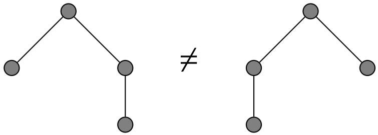

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Stanley Bijection

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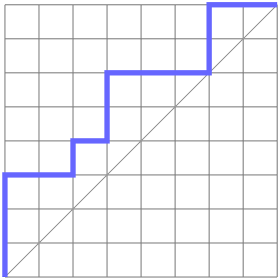

$\xrightarrow{\sigma}$

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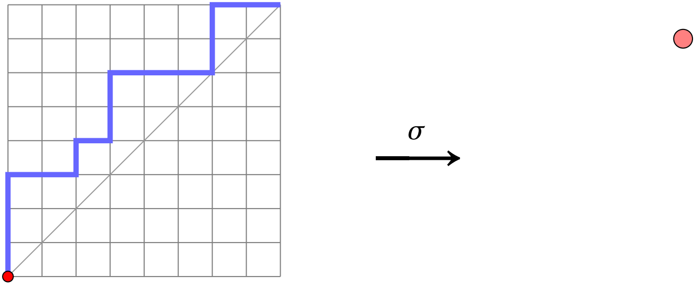

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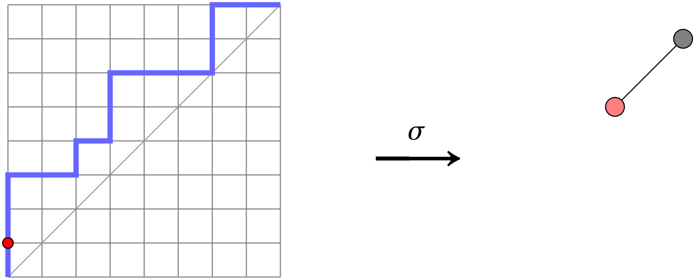

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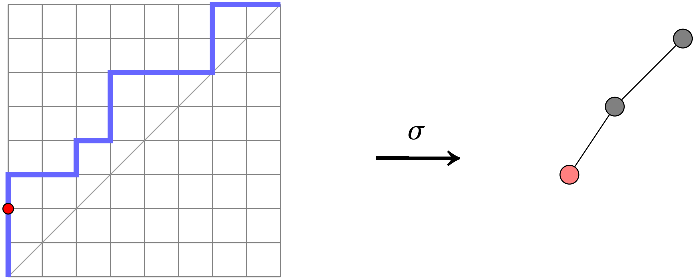

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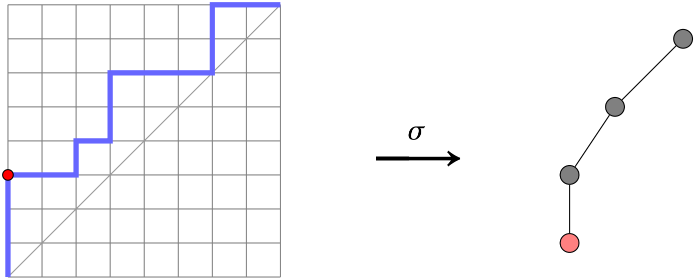

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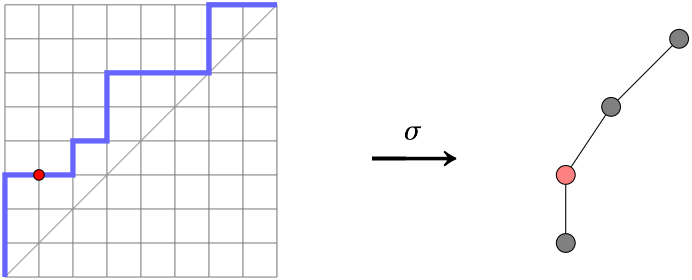

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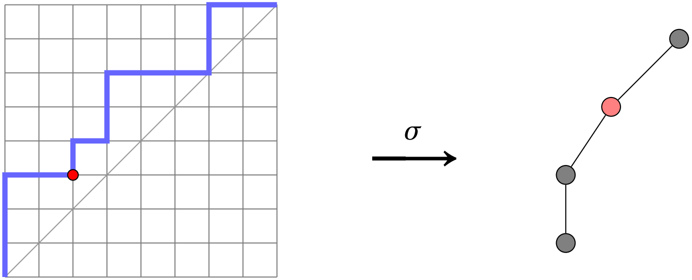

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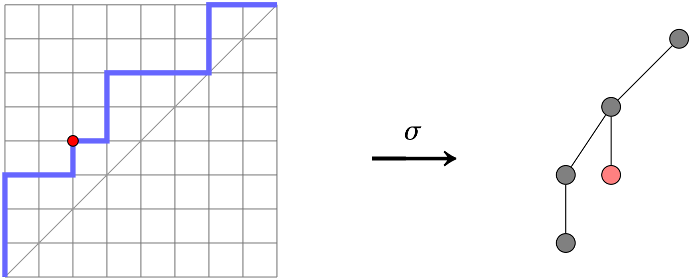

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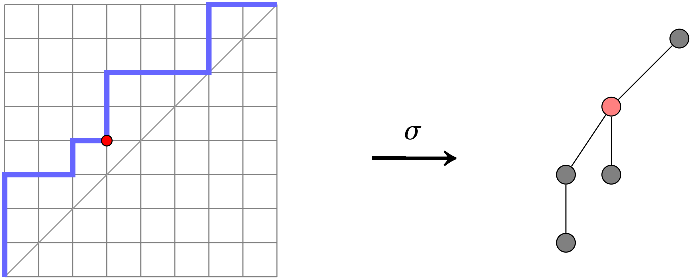

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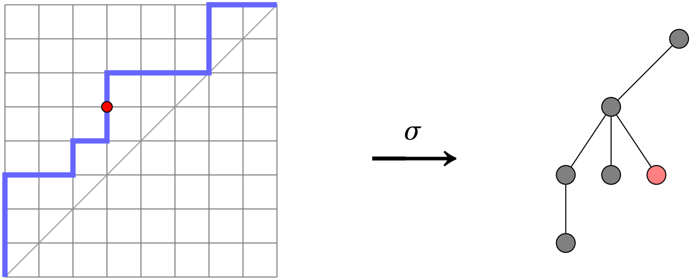

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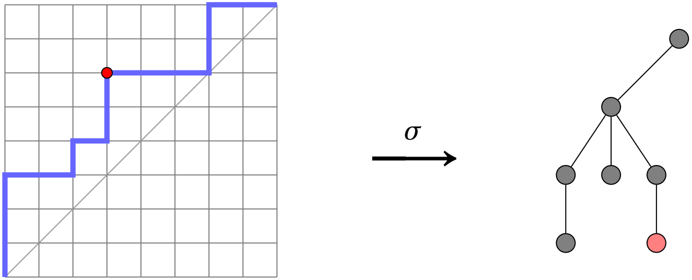

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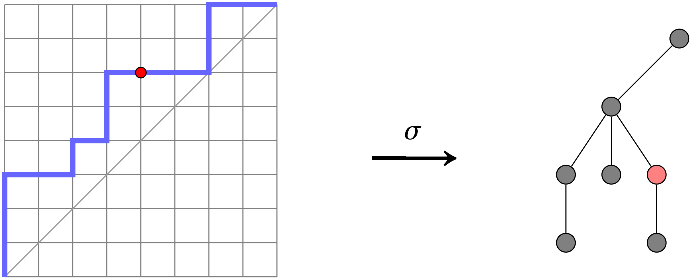

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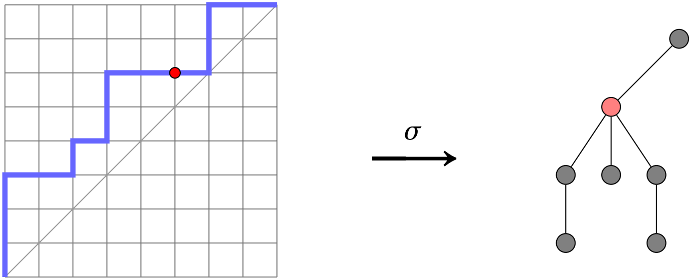

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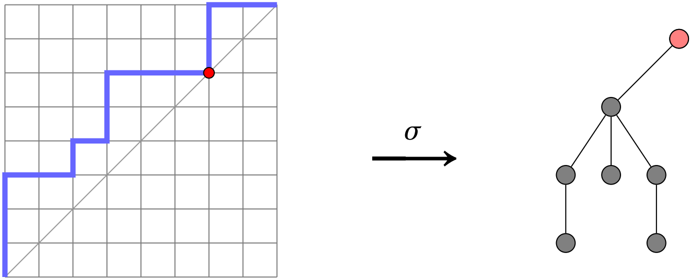

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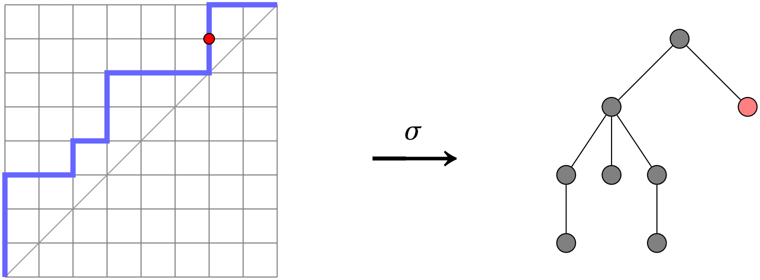

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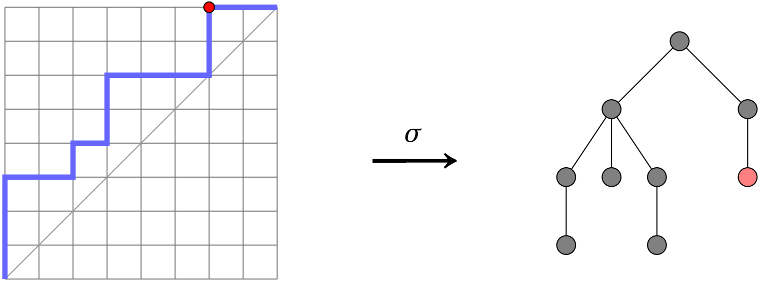

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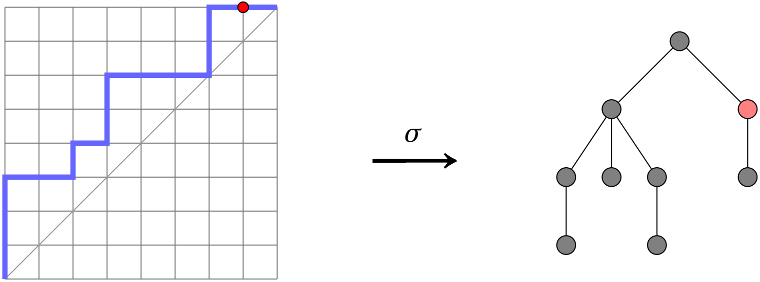

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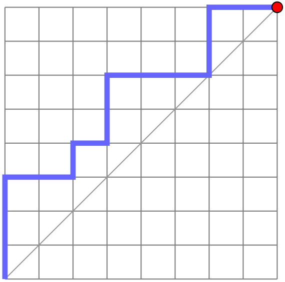
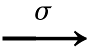
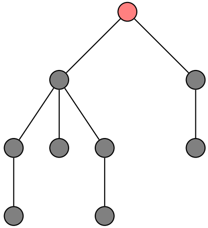
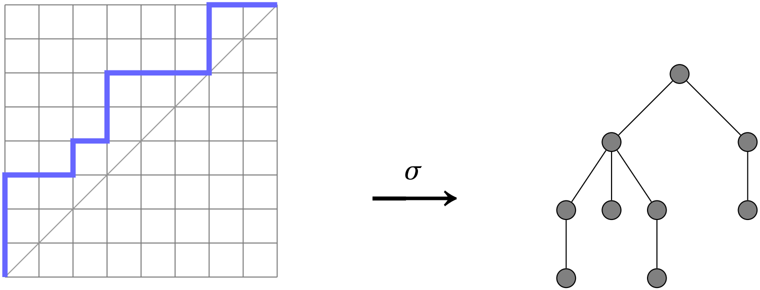






Area of Stanley Trees



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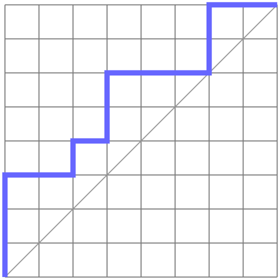

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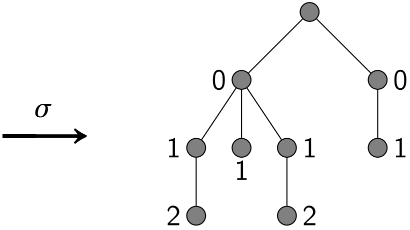
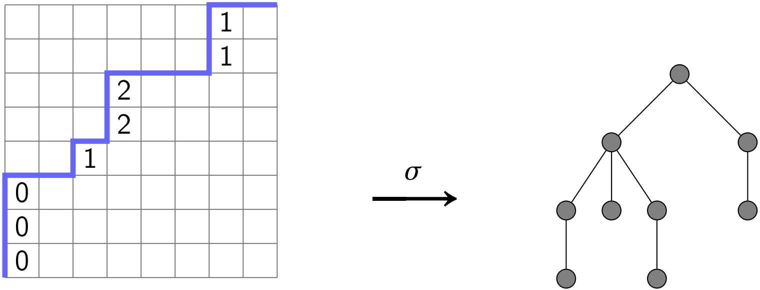


Depth of Stanley Trees



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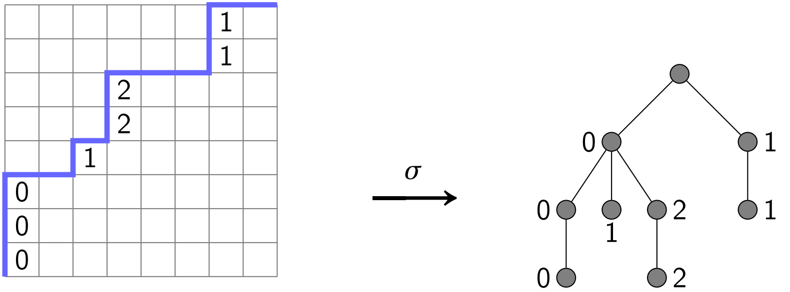

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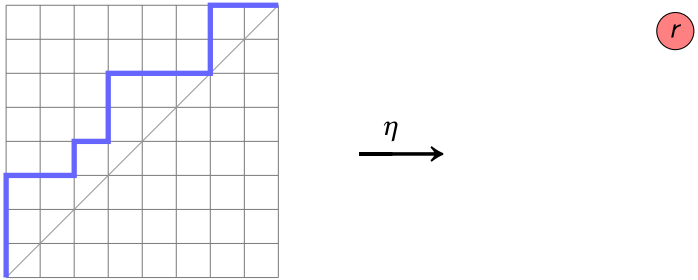
Haglund-Loehr Bijection



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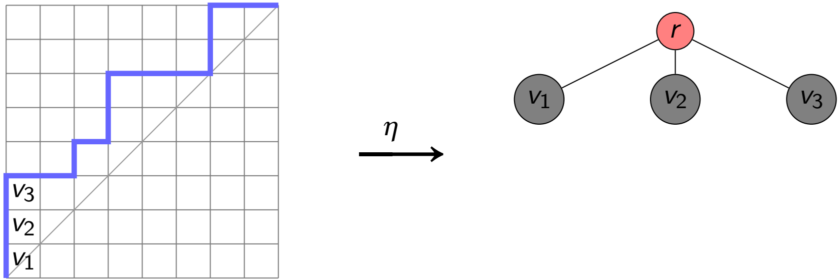

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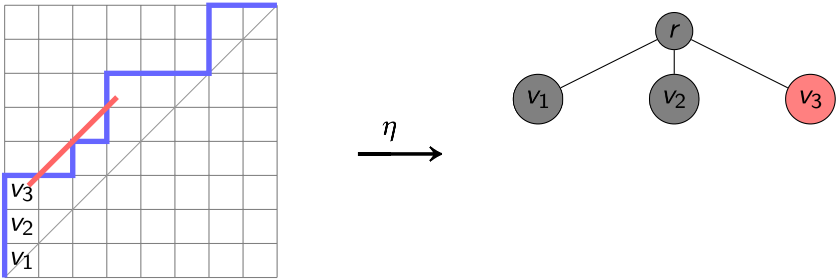

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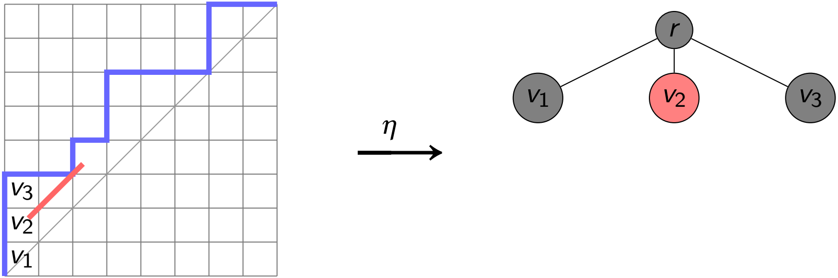

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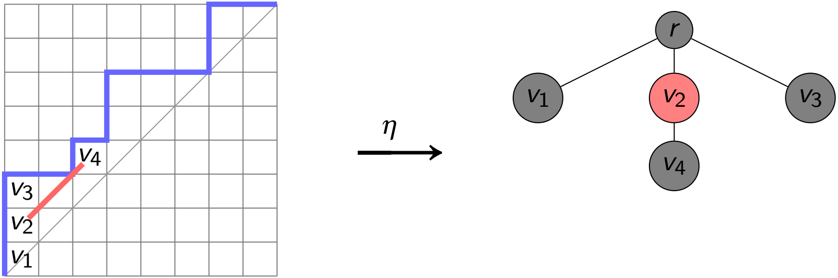

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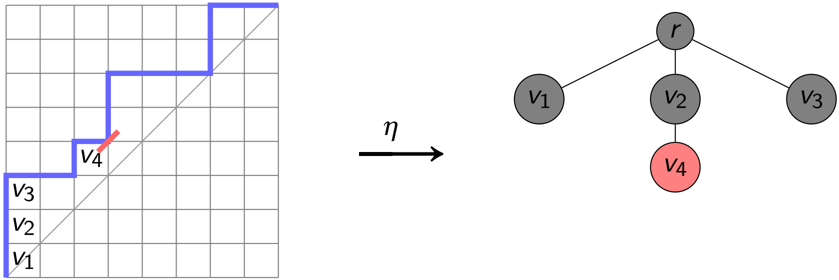

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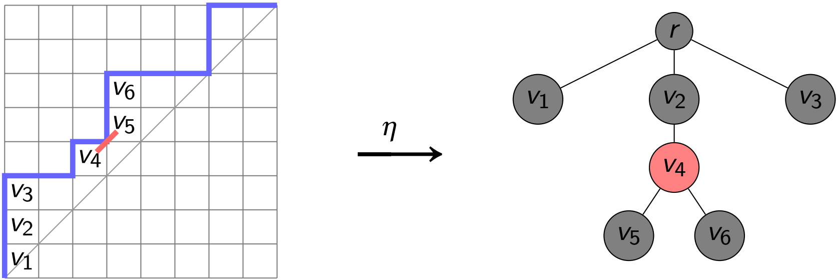

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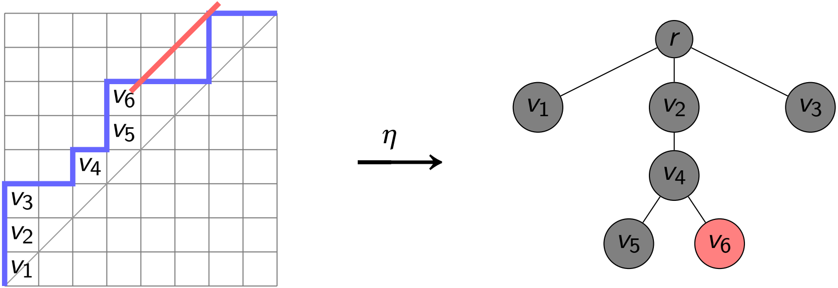

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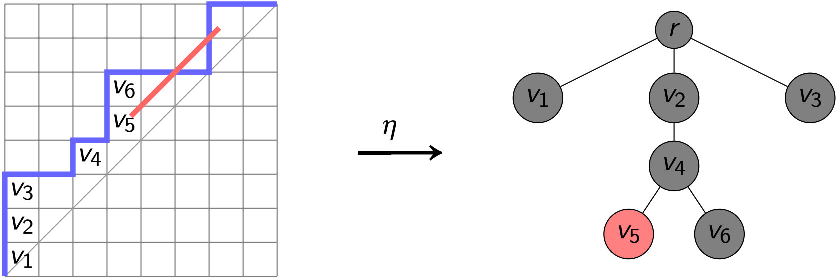

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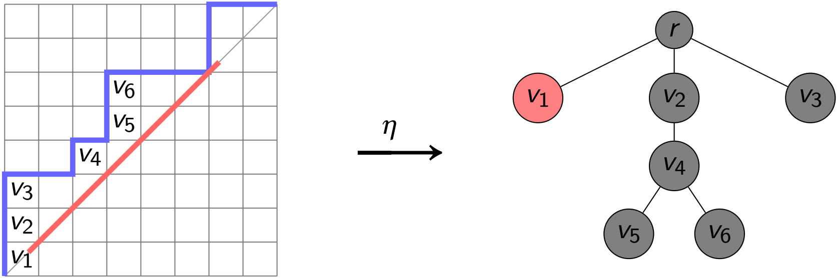

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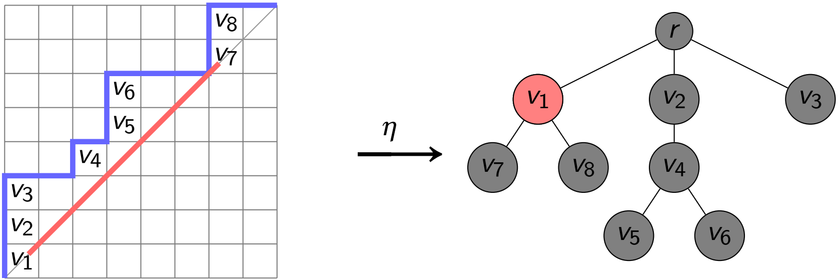

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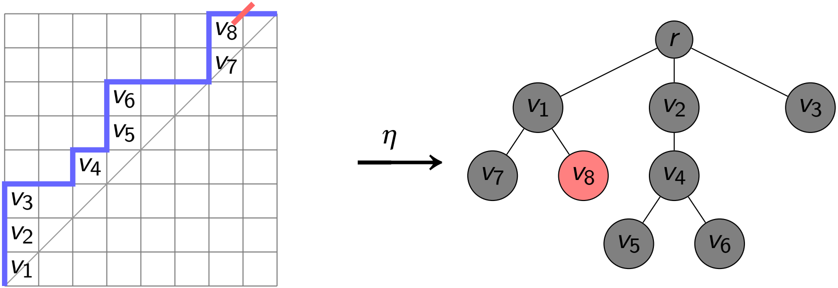

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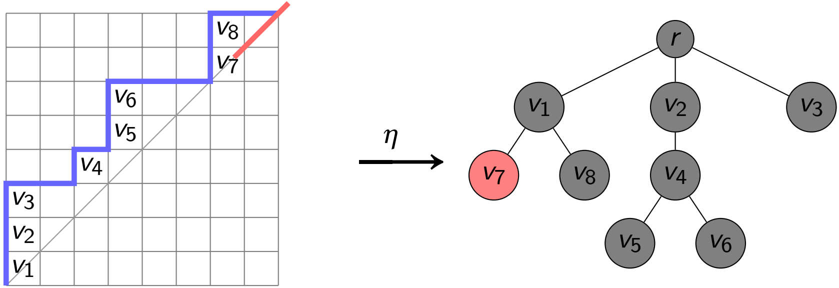

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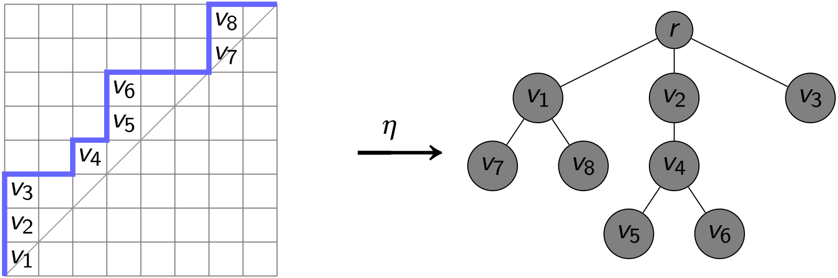
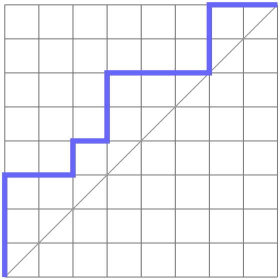


Area of Haglund-Loehr Trees



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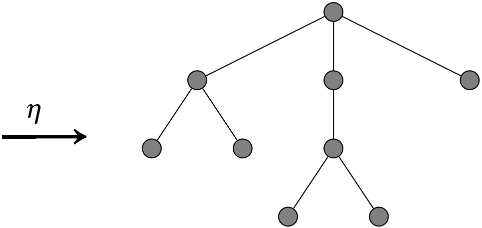

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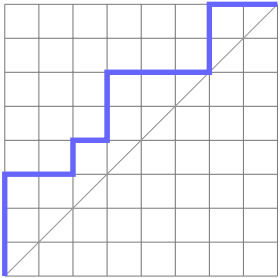

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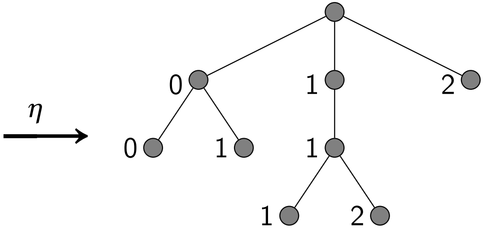
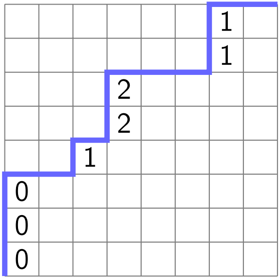


Depth of Haglund-Loehr Trees



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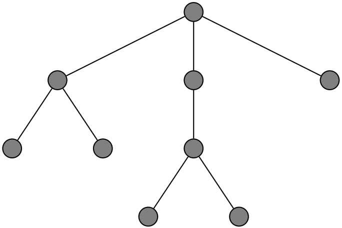

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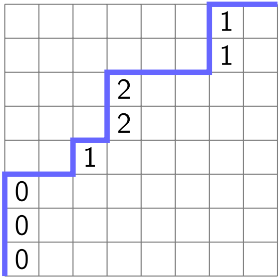

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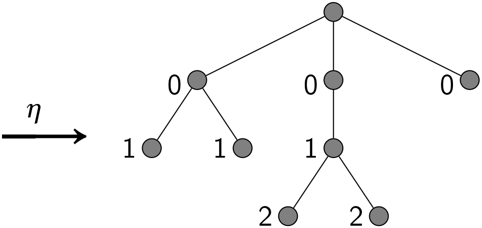

Dual Plane Trees

Definition (P., Paul, S. 2021)

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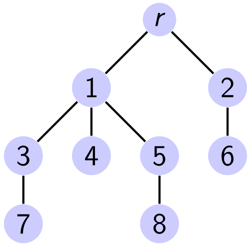

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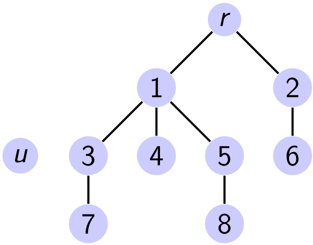

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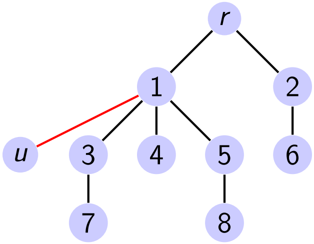

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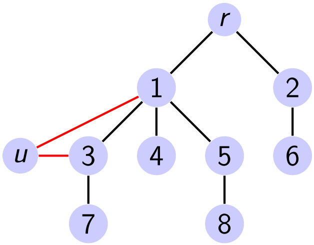
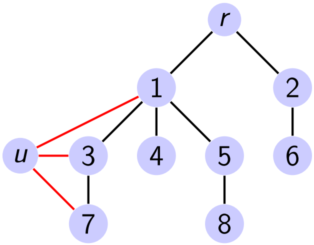


Dual Plane Trees

Definition (P., Paul, S. 2021)

The dual tree T^{dual} of a plane tree is given by the following algorithm:



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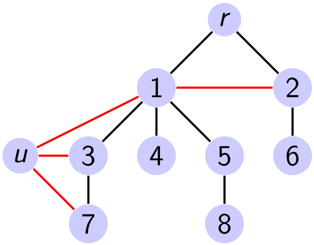

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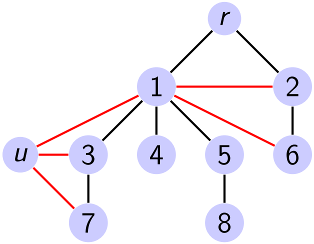

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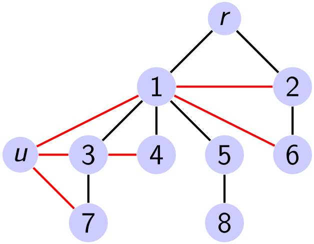

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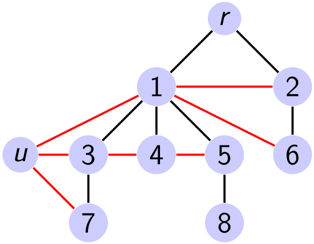

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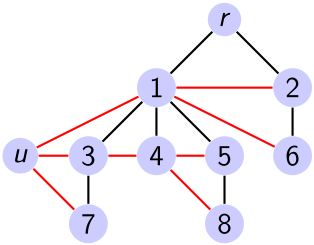

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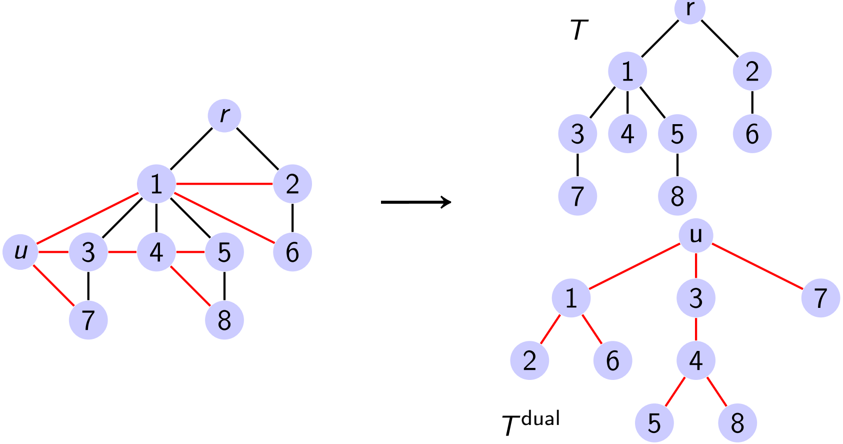

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Proposition (P., Paul, S. 2021)

The dual operator interchanges the "area" and "depth" sequences on plane trees.

Involution on Dyck Paths

Definition

Let $\omega = \sigma^{-1} \circ \eta \colon D_n \to D_n$. Equivalently $\omega = \sigma^{-1}(\sigma(\pi)^{\text{dual}})$ or $\eta^{-1}(\eta(\pi)^{\text{dual}})$.

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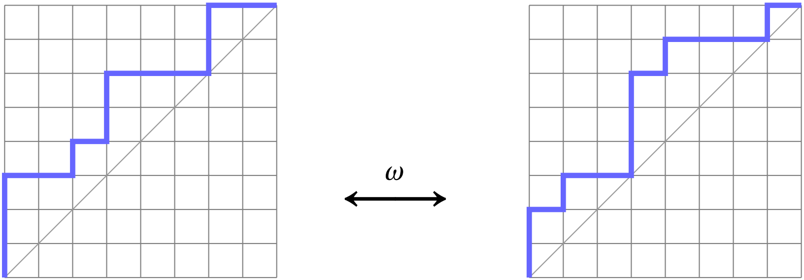

$\text{area} = (0,1,2,1,1,2,0,1)$
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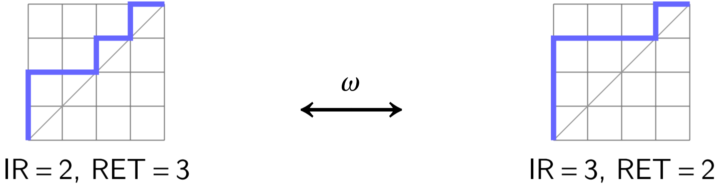

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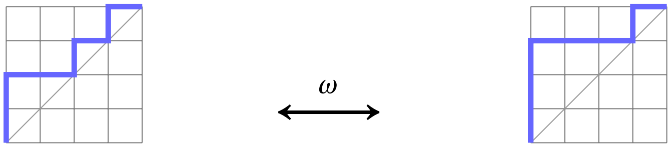

$\mathrm{IR} = 2,\ \mathrm{RET} = 3$ $\qquad$ $\mathrm{IR} = 3,\ \mathrm{RET} = 2$

Theorem (Ardila 2003)

The Tutte polynomial $T_{\mathrm{Cat}_n}(q,t) = \sum_{\pi \in D_n} q^{\mathrm{IR}(\pi)} t^{\mathrm{RET}(\pi)}$ of the Catalan Matroid is symmetric in q and t.

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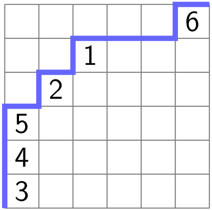

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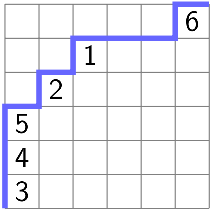

Parking Function p in P_6

- P_n - set of all parking functions on n cars
- $|P_n| = (n+1)^{(n-1)}$

Connection to Graphs

$\mathscr{C}_{n+1}$ - the set of all labelled connected graphs on vertices $\{0, 1, \ldots, n\}$

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Idea of proof:

- Look at parking functions as labelled trees under some bijection
- Associate edges to this tree based off the area statistic
- Show that all connected graphs can be obtained from this

Table of Contents

Open Problems

- Find a combinatorial proof that $C_n(q,t)$ is symmetric in q and t.
  - Can we find two maps from Dyck paths to plane trees such that their composition interchanges area and dinv?
- Is there a relation between $C_n(q,t)$ and $F_n(q,t)$?

Relation between C_n and F_n

As $F_n(q,1) = C_n(q,1)$, we have $F_n(q,t) - C_n(q,t) = (1-q)(1-t)M_n(q,t)$.

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$$M_n(1,1) = 4^{n-2} \sum_{j=0}^{4} (-1)^j \binom{4}{j} \binom{n + (j-1)/2}{n}$$

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- Is there a subspace of $\mathbb{C}[X_n, Y_n]$ such that $F_n(q,t)$ or $G_n(q,t)$ is its Hilbert series?

Thanks for listening!