On two notions of distance between homotopy classes in $W^{1/p,p}(S^1, S^1)$

Itai Shafrir (Technion)

Based partially on joint works with Brezis and Mironescu

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- Rubinstein-Sh 07: $\text{dist}_{W^{1,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = \frac{2^{1+1/p}}{\pi^{1-1/p}} |d_2 - d_1|$.

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- Therefore, the degree is **uniform continuous** on both $W^{1,p}(S^1, S^1)$ and $C(S^1, S^1)$.

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Note: The degree is **continuous** on $W^{1/p,p}(S^1, S^1)$: if $\{u_n\} \in \mathcal{E}_d$ satisfy $u_n \to u$ then $u \in \mathcal{E}_d$.

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- The analogous problem in $W^{N/p,p}(S^N, S^N)$, $N \geq 2$ is still open (for $p > N+1$)!

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Then, $\lim_{n\to\infty}\inf_{v\in\mathcal{E}_{d_2}} |u_n-v|_{W^{1,p}} = \infty$.

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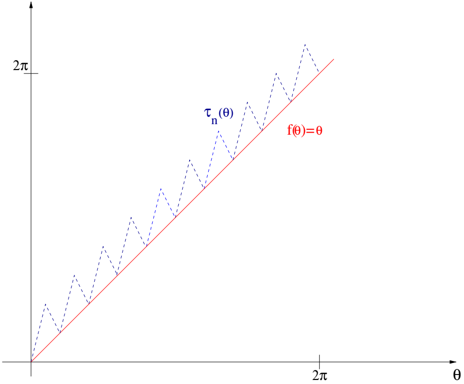
  (ii) τ_n' oscillates between n and $2 - n$ on intervals of length π/n^2.

  It satisfies:

$$\lim_{n\to\infty} \inf_{v\in\mathcal{E}_{d_2}} \int_{S^1} |\dot{u}_n - \dot{v}| = 2\pi |d_2 - d_1|$$


2π
τn(θ)
f(θ)=θ
2π
θ

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- $\mathrm{Dist}_{W^{1/p,p}(S^1,S^1)}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) = \sigma_{W^{1/p,p}}(d_2 - d_1)$

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Corollary

$\mathrm{Dist}_{W^{1/2,2}(S^1,S^1)}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = 2\pi|d_2 - d_1|^{1/2}.$

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  Lower bound is open ....

Thank you for your attention!