ON A NAVIER–STOKES–CAHN–HILLIARD SYSTEM

with chemotaxis, active transport and nonlocal interaction

Hao Wu

School of Mathematical Sciences, Fudan University

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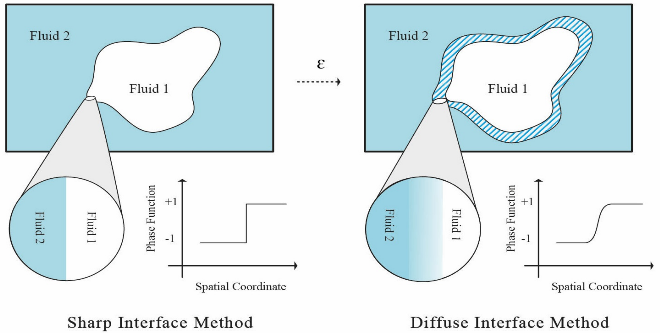
Introduction

- **Diffuse interface model** for binary mixtures:



⋆ The free interface between two components is replaced by a thin interfacial layer of finite width $\sim \varepsilon$

⋆ An efficient method to describe morphological evolution of interfaces: topological transitions can be easily handled

Introduction

- **Object**:

  A diffuse interface model for the motion of a binary mixture of incompressible viscous Newtonian fluids with a soluble chemical species

  ⋆ Navier–Stokes–Cahn–Hilliard dynamic for the fluid mixture

  ⋆ Advection–diffusion–reaction equation for the chemical species

- Related issues:

  modeling heterogeneous tumor growth with the presence of a nutrient

Basic Setting

- φ: the difference of volume fractions

$$\varphi \in [-1, 1]$$

- ρ: averaged density

$$\rho = \frac{\overline{\rho}_2 - \overline{\rho}_1}{2}\varphi + \frac{\overline{\rho}_2 + \overline{\rho}_1}{2}$$

- $\boldsymbol{v}$: (volume) averaged velocity

$$\boldsymbol{v} = \frac{1-\varphi}{2}\boldsymbol{v}_1 + \frac{1+\varphi}{2}\boldsymbol{v}_2$$

- σ: density of an unspecified chemical species

Basic Setting

- **Total energy** [1]

$$\mathcal{E} = \int_\Omega \underbrace{\frac{1}{2}\rho(\varphi)|\boldsymbol{v}|^2}_{\text{kinetic}} + \underbrace{\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\Psi(\varphi)}_{\text{mixing}} + \underbrace{N(\varphi,\sigma)}_{\text{chemical/interaction}} \, \mathrm{d}x$$

$$N(\varphi,\sigma) = \frac{\chi_\sigma}{2}\sigma^2 + \chi_\varphi\sigma(1-\varphi), \quad \text{for some } \chi_\sigma > 0,\ \chi_\varphi \geq 0.$$

$\longrightarrow$ **Chemotaxis & Active Transport**

- **Fluxes for generalized diffusion process**

$$\boldsymbol{q}_\varphi := -m(\varphi)\nabla\mu = -m(\varphi)\nabla\left(-\varepsilon\Delta\varphi + \varepsilon^{-1}\Psi'(\varphi)\right) + m(\varphi)\underbrace{\chi_\varphi\nabla\sigma}_{\text{chemotaxis effect}}$$

$$\boldsymbol{q}_\sigma := -n(\varphi)\nabla N_\sigma = -n(\varphi)\chi_\sigma\nabla\sigma + n(\varphi)\underbrace{\chi_\varphi\nabla\varphi}_{\text{active transport}}$$

[1] Garcke et al 2016 M3AS

Full System

Lam & W. 2018 EJAM: via the mass balance law & the second law of thermodynamics ($\rightarrow$ generalization of the Abel–Garcke-Grün model)

$$
\begin{aligned}
\partial_t(\rho \boldsymbol{v}) + \operatorname{div}\left(\rho \boldsymbol{v} \otimes \boldsymbol{v} - \frac{\overline{\rho}_2 - \overline{\rho}_1}{2} m(\varphi) \boldsymbol{v} \otimes \nabla \mu\right) &= -\nabla p + \operatorname{div}\left(2\nu(\varphi) \mathrm{D} \boldsymbol{v}\right) \\
&\quad - \operatorname{div}\left(\varepsilon \nabla \varphi \otimes \nabla \varphi\right), \\
\operatorname{div} \boldsymbol{v} &= \frac{\Gamma_1}{\overline{\rho}_1} + \frac{\Gamma_2}{\overline{\rho}_2}, \\
\partial_t \varphi + \operatorname{div}(\varphi \boldsymbol{v} - m(\varphi) \nabla \mu) &= \frac{\Gamma_2}{\overline{\rho}_2} - \frac{\Gamma_1}{\overline{\rho}_1}, \\
\mu &= -\varepsilon \Delta \varphi + \frac{1}{\varepsilon} \Psi'(\varphi) + N_\varphi, \\
\partial_t \sigma + \operatorname{div}(\sigma \boldsymbol{v} - n(\varphi) \nabla N_\sigma) &= S(\varphi, \sigma),
\end{aligned}
$$

where

$$
N_\varphi = -\chi \sigma, \quad N_\sigma = \sigma + \chi(1 - \varphi), \quad \mathrm{D} \boldsymbol{v} = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T).
$$

Remark 1: Different Evolutions of $\boldsymbol{v}$

- Our model

  $\longrightarrow$ A full N-S description taking into account inertial effects.

- Darcy's equation (Garcke et al 2016 M3AS)

$$
\begin{aligned}
&\boldsymbol{v} = -K\left[\nabla p - (\mu + \chi\sigma)\nabla\varphi\right],\\
&\operatorname{div}\boldsymbol{v} = U_{\boldsymbol{v}}.
\end{aligned}
$$

- Brinkman's equation (Ebenbeck & Gacke 2019 JDE, SIMA)

$$
\begin{aligned}
&-\operatorname{div}\boldsymbol{T}(\boldsymbol{v},p) + \nu\boldsymbol{v} = (\mu + \chi\sigma)\nabla\varphi,\\
&\boldsymbol{T}(\boldsymbol{v},p) = 2\eta(\varphi)\mathrm{D}\boldsymbol{v} + \lambda(\varphi)\operatorname{div}\boldsymbol{v}\mathbb{I} - p\mathbb{I},\\
&\operatorname{div}\boldsymbol{v} = U_{\boldsymbol{v}}.
\end{aligned}
$$

Remark 2: Potential Functions for φ

- **Mixing energy** of Ginzburg–Landau type

$$E_{\mathrm{GL}}(\varphi) = \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\Psi(\varphi)\,\mathrm{d}x$$

- $\Psi \sim$ a **double-well** potential, e.g., the Flory–Huggins:

$$\Psi(s) = \frac{\Theta}{2}\left[(1+s)\ln(1+s) + (1-s)\ln(1-s)\right] - \frac{\Theta_0}{2}s^2, \quad s \in [-1,1],$$

where $0 < \Theta < \Theta_0$.

  - Regular potential

$$\Psi(s) = \frac{1}{4}(s^2-1)^2, \quad s \in \mathbb{R}.$$

  - Double obstacle potential

$$\Psi(s) = \begin{cases} \frac{\Theta_0}{2}(1-s^2), & \text{if } s \in [-1,1], \\ +\infty, & \text{else.} \end{cases}$$

Simplification

- **Assumptions**: matched densities & zero excess of total mass

$$\overline{\rho}_1 = \overline{\rho}_2 = 1, \quad \Gamma_2 = -\Gamma_1 = \Gamma, \quad m(\varphi) = n(\varphi) = 1.$$

$$\begin{cases} \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \operatorname{div}(2\nu(\varphi)\mathrm{D}\boldsymbol{v}) + \nabla p = (\mu + \chi\sigma)\nabla\varphi, \\ \operatorname{div} \boldsymbol{v} = 0, \\ \partial_t \varphi + \boldsymbol{v} \cdot \nabla \varphi = \Delta\mu - \alpha(\varphi - c_0), \\ \mu = -\varepsilon\Delta\varphi + \dfrac{1}{\varepsilon}\Psi'(\varphi) - \chi\sigma, \\ \partial_t \sigma + \boldsymbol{v} \cdot \nabla \sigma = \Delta(\sigma + \chi(1-\varphi)) - \mathcal{C}h(\varphi)\sigma + S, \end{cases} \quad \text{in } \Omega \times (0,T).$$

- $0 < T \leq \infty$, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ bounded with smooth boundary $\partial\Omega$
- ν: fluid viscosity (e.g., the linear interpolation of ν_1 and ν_2)
- $2\Gamma = -\alpha(\varphi - c_0)$: interaction of Oono's type [2]

[2] the simplest in mathematical form; typical for block copolymers

Initial and Boundary Conditions

Initial conditions

$$\boldsymbol{v}|_{t=0} = \boldsymbol{v}_0(x), \quad \varphi|_{t=0} = \varphi_0(x), \quad \sigma|_{t=0} = \sigma_0(x), \qquad \text{in } \Omega.$$

Boundary conditions on $\partial\Omega$

- $\boldsymbol{v} = \mathbf{0}$: no-slip boundary condition
- $\partial_{\boldsymbol{n}}\mu = 0$: no-flux boundary condition
- $\partial_{\boldsymbol{n}}\varphi = 0$: free interface $\perp$ boundary
- $\partial_{\boldsymbol{n}}\sigma = 0$: no-flux boundary condition

$\boldsymbol{n}$: unit outer normal on $\partial\Omega$.

Other choices of B.C. are possible:

Periodic BC, Dirichlet BC, or

- $(2\eta(\varphi)\mathrm{D}\boldsymbol{v} - p\mathbf{I}) \cdot \boldsymbol{n} = \mathbf{0}, \quad \partial_{\boldsymbol{n}}\sigma = K(1-\sigma).$

Related Results

- **Navier–Stokes–Cahn–Hilliard system** $\sim$ **Model "H"**

  Huge literature

- **Navier–Stokes–Cahn–Hilliard–Oono system**

  Bosia, Grasselli & Miranville '12; Miranville & Temam '16 ...

  $\star$ Several open questions with singular potential !

- **NSCH system with chemotaxis and mass transport**:

  Lam & W. '18: with a general **regular potential** Ψ

> **Lack of maximum principle** for the 4th order Cahn–Hilliard eq.
>
> $\Downarrow$
>
> $\varphi \in [-1, 1]$ **cannot** be guaranteed with a regular potential Ψ.

Results under a Singular Potential

- Consider

$$\Psi(s) = \frac{\Theta}{2}\left[(1+s)\ln(1+s) + (1-s)\ln(1-s)\right] - \frac{\Theta_0}{2}s^2.$$

$\Longrightarrow$

$$\varphi \in [-1, 1].$$

(1) Existence of global weak solutions in 2D and 3D,

uniqueness of global weak solutions in 2D (He 2021 Nonlinearity)

(2) Existence of global strong solutions that are **strictly separated** from the pure states ± 1 over time in 2D (He & W. 2021 JDE)

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Question: how about **global weak/strong solutions in 3D** ?

$\longrightarrow$ require further understanding of the structure.

A Modified System

Consider

$$\begin{aligned}
&\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} - \mathrm{div}\big(2\nu(\varphi)D\boldsymbol{v}\big) + \nabla p = (\mu + \chi\sigma)\nabla\varphi,\\
&\mathrm{div}\ \boldsymbol{v} = 0,\\
&\partial_t\varphi + \boldsymbol{v}\cdot\nabla\varphi = \Delta\mu - \alpha(\overline{\varphi} - c_0),\\
&\mu = -\Delta\varphi + \Psi'(\varphi) - \chi\sigma + \beta\mathcal{N}(\varphi - \overline{\varphi}),\\
&\partial_t\sigma + \boldsymbol{v}\cdot\nabla\sigma = \Delta(\sigma - \chi\varphi),
\end{aligned}$$

subject to

$$\begin{aligned}
&\boldsymbol{v} = \mathbf{0},\ \partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\sigma = 0, && \text{on } \partial\Omega\times(0,T),\\
&\boldsymbol{v}|_{t=0} = \boldsymbol{v}_0,\ \varphi|_{t=0} = \varphi_0,\ \sigma|_{t=0} = \sigma_0, && \text{in } \Omega.
\end{aligned}$$

- ⋆ $\overline{\varphi}$: spatial average of φ
- ⋆ $\mathcal{N} = (-\Delta)^{-1}$: inverse of the minus Neumann Laplacian
- ⋆ $\alpha = \beta$: recover Oono's interaction

Assumptions

(H1) Viscosity function: $\nu \in C^2(\mathbb{R})$ and

$$\nu_* \le \nu(s) \le \nu^*, \quad |\nu'(s)| \le \nu_0, \quad |\nu''(s)| \le \nu_1, \quad \forall\, s \in \mathbb{R}.$$

(H2) Singular potential: $\Psi \in C([-1,1])$ and **real analytic** in $(-1,1)$,

$$\Psi(s) = \Psi_0(s) + \frac{\theta_c}{2}(1-s^2),$$

such that

$$\lim_{s\to\pm 1} \Psi_0'(s) = \pm\infty, \quad \Psi_0''(s) \ge \theta > 0,$$

with $\theta_c - \theta > 0$. There exists $\epsilon_0 \in (0,1)$ such that Ψ_0'' is nondecreasing in $[1-\epsilon_0, 1)$ and nonincreasing in $(-1, -1+\epsilon_0]$.

(H3) Coefficients:

$$\chi \in \mathbb{R}, \quad c_0 \in (-1,1), \quad \alpha \ge 0, \quad \beta \in \mathbb{R}.$$

Structure I

Mass dynamics:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overline{\varphi} - c_0) + \alpha(\overline{\varphi} - c_0) = 0,$$

$\Longrightarrow$

$$\overline{\varphi}(t) - c_0 = (\overline{\varphi_0} - c_0)e^{-\alpha t}, \quad \forall\, t \geq 0.$$

- If $\alpha = 0$, or $\overline{\varphi_0} = c_0$ for $\alpha > 0$: $\overline{\varphi}(t)$ is **conserved** in time.
- Otherwise, $\overline{\varphi}(t)$ converges **exponentially fast** to c_0 provided that $\alpha > 0$, the so-called off-critical case.

Structure II

Modified free energy:

$$\mathcal{F}(\varphi,\sigma) = \int_\Omega \Big(\frac{1}{2}|\nabla\varphi|^2 + \Psi(\varphi) + \frac{1}{2}|\sigma|^2 - \chi\sigma\varphi \underbrace{+ \frac{\beta}{2}|\nabla\mathcal{N}(\overline{\varphi}-\varphi)|^2}_{\text{nonlocal interaction}}\Big)\mathrm{d}x.$$

Energy balance:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{1}{2}|\boldsymbol{v}|^2\,\mathrm{d}x + \mathcal{F}(\varphi,\sigma)\Big) \\
&\quad + \int_\Omega \big(2\nu(\varphi)|D\boldsymbol{v}|^2 + |\nabla\mu|^2 + |\nabla(\sigma-\chi\varphi)|^2\big)\,\mathrm{d}x \\
&= -\alpha\int_\Omega(\overline{\varphi}-c_0)\mu\,\mathrm{d}x = \underbrace{-\alpha(\overline{\varphi}-c_0)\int_\Omega \mu\,\mathrm{d}x}_{\text{"better" energy production rate}}
\end{aligned}$$

Global Strong Solutions in 3D

Theorem (He & W. 2024 Math. Ann.)

Let (φ_*, σ_*) be a ***local minimizer*** *of $\mathcal{F}$. For any $\epsilon > 0$, there exist η_1, η_2, $\eta_3 \in (0,1)$ such that for any regular initial data $(\boldsymbol{v}_0, \varphi_0, \sigma_0)$ satisfying*

$$\boldsymbol{v}_0 \in \boldsymbol{H}^1_{0,\mathrm{div}}(\Omega), \quad \varphi_0 \in H^3(\Omega) \cap H^2_N(\Omega), \quad \sigma_0 \in H^1(\Omega),$$
$$\|\boldsymbol{v}_0\|_{\boldsymbol{L}^2} \le \eta_1, \quad \|\varphi_0 - \varphi_*\|_{H^2} \le \eta_2, \quad \|\sigma_0 - \sigma_*\|_{L^2} \le \eta_3,$$

the IBVP admits ***a unique global strong solution*** *$(\boldsymbol{v}, \varphi, \mu, \sigma, p)$ on $[0, +\infty)$. Moreover,*

$$\|\varphi(t) - \varphi_*\|_{H^2} \le \epsilon, \quad \|\sigma(t) - \sigma_*\|_{L^2} \le \epsilon, \quad \forall\, t \ge 0,$$

and there exists $\delta > 0$ such that

$$\|\varphi(t)\|_{C(\overline{\Omega})} \le 1 - \delta, \quad \forall\, t \ge 0,$$

Long-time Behavior in 3D

Theorem (He & W. 2024 Math. Ann.)

There exists $\kappa \in (0, 1/2)$ *such that*

$$\|\boldsymbol{v}(t)\|_{\boldsymbol{L}^2} + \|\varphi(t) - \varphi_\infty\|_{H^1} + \|\sigma(t) - \sigma_\infty\|_{L^2} \le C(1+t)^{-\frac{\kappa}{1-2\kappa}}, \quad \forall\, t \ge 0,$$

where $(\varphi_\infty, \sigma_\infty) \in \big(H^3(\Omega) \cap H^2_N(\Omega)\big) \times H^2_N(\Omega)$ *is a steady state:*

$$\begin{aligned} &-\Delta\varphi_\infty + \Psi'(\varphi_\infty) - \chi\sigma_\infty + \beta\mathcal{N}(\varphi_\infty - c_0) = \overline{\Psi'(\varphi_\infty)} - \chi\overline{\sigma_\infty}, && \text{in } \Omega,\\ &\Delta(\sigma_\infty - \chi\varphi_\infty) = 0, && \text{in } \Omega,\\ &\partial_{\boldsymbol{n}}\varphi_\infty = \partial_{\boldsymbol{n}}\sigma_\infty = 0, && \text{on } \partial\Omega, \end{aligned}$$

subject to the constraints $\overline{\varphi_\infty} = c_0, \quad \overline{\sigma_\infty} = \overline{\sigma_0}.$

Proof: Energy dissipation structure + **Łojasiewicz–Simon approach**

Global Weak Solutions in 3D – Existence

Theorem (He & W. arXiv:2408.09514)

*For any initial data (∼ **finite initial energy**)*

$$\boldsymbol{v}_0 \in \boldsymbol{L}^2_{0,\mathrm{div}}(\Omega),\ \varphi_0 \in H^1(\Omega),\ \sigma_0 \in L^2(\Omega),\ \|\varphi_0\|_{L^\infty} \le 1,\ |\overline{\varphi}_0| < 1,$$

*the IBVP admits **at least one global weak solution** $(\boldsymbol{v}, \varphi, \mu, \sigma)$ on $[0, +\infty)$:*

$$\begin{aligned}
&\boldsymbol{v} \in L^\infty(0,+\infty; \boldsymbol{L}^2_{0,\mathrm{div}}(\Omega)) \cap L^2(0,+\infty; \boldsymbol{H}^1_{0,\mathrm{div}}(\Omega)),\\
&\varphi \in L^\infty(0,+\infty; H^1(\Omega)) \cap L^4_{\mathrm{uloc}}(0,+\infty; H^2_N(\Omega)) \cap L^2_{\mathrm{uloc}}(0,+\infty; W^{2,6}(\Omega)),\\
&\mu \in L^2_{\mathrm{uloc}}(0,+\infty; H^1(\Omega)), \quad \nabla\mu \in L^2(0,+\infty; \boldsymbol{L}^2(\Omega)),\\
&\varphi \in L^\infty(\Omega \times (0,+\infty)) \ \textit{with}\ |\varphi(x,t)| < 1 \ \ \textit{a.e. in}\ \Omega \times (0,+\infty),\\
&\sigma \in L^\infty(0,+\infty; L^2(\Omega)) \cap L^2_{\mathrm{uloc}}(0,+\infty; H^1(\Omega)),
\end{aligned}$$

Proof: Energy dissipation structure + semi-Galerkin scheme

Global Weak Solutions in 3D – Regularity

Question: Regularity propagation of global weak solutions ?

- **Expectation:** similar behavior like that for the 3D Navier–Stokes system: eventual regularity for large time.
- **Difficulty:** A weak-strong type uniqueness is only available **if φ is strictly separated from** ± 1 (not valid for Cahn–Hilliard eq. in 3D)
- **Idea:** Abels, Garcke & Giorgini for the NSCH system (2024 Math. Ann.)

Regularity Propagation

Theorem (He & W. arXiv:2408.09514)

Let $(\boldsymbol{v}, \varphi, \mu, \sigma)$ be a global weak solution.

(1) ***Instantaneous regularity of*** (φ, μ, σ)*:*

for any $\tau \in (0,1)$, it holds

$$
\begin{aligned}
&\varphi \in L^\infty(\tau,+\infty;W^{2,6}(\Omega)), \quad \partial_t\varphi \in L^2(\tau,+\infty;H^1(\Omega)),\\
&\mu \in L^\infty(\tau,+\infty;H^1(\Omega)) \cap L^2_{\mathrm{uloc}}(\tau,+\infty;H^3(\Omega)),\\
&\Psi'(\varphi) \in L^\infty(\tau,+\infty;L^6(\Omega)),\\
&\sigma \in L^\infty(\tau,+\infty;L^6(\Omega)), \quad \partial_t\sigma \in L^2(\tau,+\infty;(H^1(\Omega))'),
\end{aligned}
$$

- We do not have $\sigma \in L^\infty(\tau,+\infty;H^1(\Omega))$ due to low regularity of $\boldsymbol{v}$.

Regularity Propagation

Theorem (He & W. arXiv:2408.09514)

(2) ***Eventual strict separation property of*** φ*:*

there exist $T_{\mathrm{SP}} \geq 1$ *and* $\delta \in (0,1)$ *such that*

$$|\varphi(x,t)| \leq 1-\delta, \quad \forall\,(x,t) \in \overline{\Omega} \times [T_{\mathrm{SP}}, +\infty).$$

(3) ***Eventual regularity of*** $(\boldsymbol{v}, \sigma)$*:*

there exist $T_{\mathrm{R}} \geq T_{\mathrm{SP}}$ *such that*

$$\boldsymbol{v} \in L^\infty(T_{\mathrm{R}}, +\infty; \boldsymbol{H}^1_{0,\mathrm{div}}(\Omega)) \cap L^2(T_{\mathrm{R}}, +\infty; \boldsymbol{H}^2(\Omega)),$$
$$\sigma \in L^\infty(T_{\mathrm{R}}, +\infty; H^1(\Omega)) \cap L^2_{\mathrm{uloc}}(T_{\mathrm{R}}, +\infty; H^2_N(\Omega)).$$

Moreover, $\varphi \in L^\infty(T_{\mathrm{R}}, +\infty; H^3(\Omega))$*.*

Strategy of Proof

Step 1. Given a divergence-free velocity

$$\boldsymbol{v} \in L^2(0,+\infty; \boldsymbol{H}^1_{0,\mathrm{div}}(\Omega)),$$

study a **convective Cahn–Hilliard–diffusion system for** (φ, σ):

$$\begin{aligned}
&\partial_t \varphi + \boldsymbol{v} \cdot \nabla \varphi = \Delta \mu - \alpha(\overline{\varphi} - c_0), \\
&\mu = -\Delta \varphi + \Psi'(\varphi) - \chi \sigma + \beta \mathcal{N}(\varphi - \overline{\varphi}), \\
&\partial_t \sigma + \boldsymbol{v} \cdot \nabla \sigma = \Delta(\sigma - \chi \varphi),
\end{aligned}$$

- Instantaneous regularization of weak solution (φ, σ) for $t > 0$
- Due to the second order nature of σ-equation & low regularity of $\boldsymbol{v}$:

⋆ σ only **partially regularizes**

⋆ No uniqueness! But a weak-strong uniqueness under

$$\sigma \in L^8(0,T; L^4(\Omega)).$$

Strategy of Proof

Step 2. Eventual strict separation of φ: **a dynamic approach via ω-limit set**

$$\lim_{t\to+\infty} \mathrm{dist}_{W^{1,6}}\big(\varphi(t),\omega(\varphi)\big) = 0.$$

For every small $\delta > 0$, there is $T_{\mathrm{SP}} \gg 1$ such that

$$\|\varphi(t)\|_{C(\overline{\Omega})} \leq 1-\delta, \quad \forall\, t \geq T_{\mathrm{SP}}.$$

Step 3. Weak-strong uniqueness for $(\boldsymbol{v},\varphi,\sigma)$: **a relative energy method**

$(\boldsymbol{v}_1,\varphi_1,\mu_1,\sigma_1)$ is a weak solution, $(\boldsymbol{v}_2,\varphi_2,\mu_2,\sigma_2)$ is a strong solution, they satisfy the same initial data and both φ_1, φ_2 are strictly separated from ± 1, then

$$(\boldsymbol{v}_1,\varphi_1,\mu_1,\sigma_1) = (\boldsymbol{v}_2,\varphi_2,\mu_2,\sigma_2), \quad \text{in } [0,T].$$

Strategy of Proof

Step 4. Eventual regularity of $\boldsymbol{v}$:

Lemma (Abels 2009, ARMA)

$$
\begin{aligned}
&\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \operatorname{div}\big(2\nu(c)D\boldsymbol{u}\big) + \nabla p = \boldsymbol{f}, && \text{in } \Omega\times(0,T),\\
&\operatorname{div}\boldsymbol{u} = 0, && \text{in } \Omega\times(0,T),\\
&\boldsymbol{u} = \boldsymbol{0}, && \text{on } \partial\Omega\times(0,T),\\
&\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, && \text{in } \Omega,
\end{aligned}
$$

for given data c, $\boldsymbol{u}_0$, $\boldsymbol{f}$. Suppose $c\in BUC([0,+\infty);W^{1,6}(\Omega))$, $\boldsymbol{u}_0\in \boldsymbol{H}^1_{0,\mathrm{div}}(\Omega)$ and $\boldsymbol{f}\in L^2(0,+\infty;\boldsymbol{L}^2_{0,\mathrm{div}}(\Omega))$. There is $\varepsilon_0>0$ such that, if

$$
\|\boldsymbol{u}_0\|_{\boldsymbol{H}^1_{0,\mathrm{div}}(\Omega)} + \|\boldsymbol{f}\|_{L^2(0,+\infty;\boldsymbol{L}^2_{0,\mathrm{div}}(\Omega))} \le \varepsilon_0,
$$

then there is a unique global strong solution $\boldsymbol{u}$ on $[0,+\infty)$.

Set $c=\varphi$, $\boldsymbol{u}=\boldsymbol{v}$, $\boldsymbol{f}=-\varphi\nabla\mu-\chi\varphi\nabla(\sigma-\chi\varphi)$ and conclude on $[T_{\mathrm{R}},+\infty)$

Summary and Outlook

For a 3D NSCH system with chemotaxis, mass transfer and Oono's interaction, we have shown

(1) Existence of global weak solutions,

(2) Existence and uniqueness of a global solution (near energy minimizers)

(3) Regularity propagation of global weak solutions

(4) Long-time behavior: convergence to a single equilibrium

Future work:

- Unmatched densities
- Nonlocal CH dynamics
- General mass scourses

End

Thank You !