Liquid crystal flows with free boundary

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Nematic liquid crystal flow in dimension two

We consider the following initial-boundary value problem of nematic liquid crystal flow in a bounded, smooth domain Ω in $\mathbb{R}^n$ $(n \geq 2)$, and $T > 0$

$$\text{(NLCF)} \quad \begin{cases} \partial_t v + v \cdot \nabla v + \nabla P = \Delta v - \varepsilon_0 \nabla \cdot \left(\nabla u \odot \nabla u - \frac{1}{2} |\nabla u|^2 \mathbb{I}_2 \right) \\ \nabla \cdot v = 0 \\ \partial_t u + v \cdot \nabla u = \Delta u + |\nabla u|^2 u, \end{cases}$$

$$(v, u)\big|_{t=0} = (v_0, u_0) \quad \text{in} \quad \Omega,$$

$$v = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \qquad u = u_0 \quad \text{on} \quad \partial\Omega \times (0, T),$$

- $v : \Omega \times (0, T) \to \mathbb{R}^n$ fluid velocity
- $P : \Omega \times (0, T) \to \mathbb{R}$ fluid pressure
- $u : \Omega \times (0, T) \to \mathbb{S}^2$ orientation field of nematic liquid crystal molecules
- $\varepsilon_0 > 0$: competition between kinetic energy and elastic energy

Nematic liquid crystal flow

(NLCF) is a simplified version of the Ericksen–Leslie model first proposed by Lin 1989. (NLCF) couples two important PDEs:

- Incompressible Navier–Stokes equation

$$\text{(iNS)} \quad \begin{cases} \partial_t v + v \cdot \nabla v + \nabla P = \Delta v & \text{in } \Omega \times (0,T) \\ \nabla \cdot v = 0 & \text{in } \Omega \times (0,T) \end{cases}$$

- Harmonic map heat flow

$$\text{(HMF)} \quad \begin{cases} \partial_t u = \Delta u + |\nabla u|^2 u & \text{in } \Omega \times (0,T) \\ u : \Omega \times (0,T) \to \mathbb{S}^2 \end{cases}$$

Coupling terms

- (HMF) provides forcing term to (iNS) $-\varepsilon_0 \nabla \cdot \left(\nabla u \odot \nabla u - \frac{1}{2} |\nabla u|^2 \mathbb{I}_2 \right)$
- (iNS) provides transport term to (HMF) $v \cdot \nabla u$

(iNS) and (HMF)

- (iNS)
  - Existence of suitable weak solutions: Leray 1934, Hopf 1951
    Leray–Hopf solution is regular in $\mathbb{R}^2$
  - Partial regularity results in $\mathbb{R}^3$: Caffarelli–Kohn–Nirenberg 1982, Lin 1998
- (HMF)
  - $n = 2$, Struwe (1985) established the existence of global weak solution, which has at most finitely many singular points
  - $n \geq 3$: existence of global weak solutions Chen–Struwe 1989, Chen–Lin 1993
  - $n \geq 3$: examples of finite time blow-up Coron–Ghidaglia 1989, Chen–Ding 1990
  - $n = 2$: critical dimension. Finite time blow-up
    Chang–Ding–Ye 1991
    van den Berg–Hulshof–King 2003 (formal analysis)
    Raphaël–Schweyer 2013
    Dávila–del Pino–Wei 2017 (blow-up at multiple points in general domains)

NLCF

- Existence of weak solutions and partial regularity results for $n = 2, 3$: Lin–Liu 1995, 1996
- $n = 2$: Lin-Lin-Wang (2010) proved the global existence of Leray–Hopf type weak solutions for (NLCF) that is smooth away from finitely many points.
- $n = 2$: Lin-Wang (2010) proved the uniqueness of Leray-Hopf weak solution to (NLCF)
- $n = 3$: Lin-Wang (2016) proved the global existence of weak solutions satisfying the global energy inequality under the assumption that the initial orientation field $d_0(\Omega) \subset \mathbb{S}^2_+$.
- Construction of blow-up solutions in two dimensions at a finite number of points by Lai-Lin-Wei-Wang-Zhou (2021)

A new model with free boundary (F.H. Lin, Y. S., J. Wei, Y. Zhou)

$$\text{(LCF)} \quad \begin{cases} \partial_t v + v \cdot \nabla v + \nabla P = \Delta v - \varepsilon_0 \nabla \cdot \left(\nabla u \odot \nabla u - \frac{1}{2} |\nabla u|^2 \mathbb{I}_2 \right) \\ \nabla \cdot v = 0 \\ \partial_t u + v \cdot \nabla u = \Delta u + |\nabla u|^2 u, \end{cases}$$

$$\text{(FB)} \quad \begin{cases} v \cdot \nu = 0, & \text{on } \partial\Omega \times (0,T), \\ (Sv \cdot \nu)_\tau = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,t) \in \Sigma, & \text{on } \partial\Omega \times (0,T), \\ \frac{\partial u}{\partial \nu}(x,t) \perp T_{u(x,t)}\Sigma, & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where ν is the unit outer normal of $\partial\Omega$, S is the strain tensor (deformation tensor, shear stress)

$$Sv = \frac{1}{2}(\nabla v + (\nabla v)^T),$$

The blow-up result via parabolic gluing: informal statement

We construct both interior and boundary bubbling in the half-plane:

Theorem (F.H. Lin, Y. S., J. Wei, Y. Zhou (ARMA 2023))

For $T > 0$ sufficiently small and any given points in $\overline{\mathbb{R}^2_+}$, there exists initial data (u_0, v_0) such that the solution (u, v) to liquid crystal flow with free boundary conditions blows up at finite time T exactly at these given points. Moreover, u takes the form at leading order of the sharply scaled 1-corotational profile (equivariant harmonic map) with type II rate

$$\lambda(t) \sim \frac{T-t}{|\log(T-t)|^2}.$$

The blow-up result via parabolic gluing: formal statement

Theorem

For $T, \varepsilon_0 > 0$ sufficiently small and any given points $\{q_{\mathcal{B}}^{(j)}\}_{j=1}^{k_{\mathcal{B}}} \cup \{q_{\mathcal{I}}^{(j)}\}_{j=1}^{k_{\mathcal{I}}} \subset \overline{\mathbb{R}^2_+}$ with $q_{\mathcal{B}}^{(j)} \in \partial\mathbb{R}^2_+$ and $q_{\mathcal{I}}^{(j)} \in \mathring{\mathbb{R}}^2_+$, there exists initial data (u_0, v_0) such that the solution (u, v) blows up at finite time $t = T$ exactly at these given points, namely

$$u(x,t) - u_*(x) - \sum_{j=1}^{k_{\mathcal{B}}} \left[W_1\left(\frac{x - q_{\mathcal{B}}^{(j)}}{\lambda^{(j)}(t)} \right) - W_1(\infty) \right]$$

$$- \sum_{j=1}^{k_{\mathcal{I}}} Q_{\omega_{\mathcal{I}}^{(j)}} \left[W_2\left(\frac{x - q_{\mathcal{I}}^{(j)}}{\lambda^{(j)}(t)} \right) - W_2(\infty) \right] \to 0 \quad \text{as} \quad t \to T$$

in $H^1(\mathbb{R}^2_+) \cap L^\infty(\mathbb{R}^2_+)$, where $u_* \in H^1(\mathbb{R}^2_+) \cap C(\bar{\mathbb{R}}^2_+)$.

Continued

Theorem

The blow-up rates are

$$\lambda^{(j)}(t) \sim \kappa_j^* \frac{T-t}{|\log(T-t)|^2} \quad as \quad t \to T,$$

$$|\nabla u(\cdot,t)|^2\, dx \rightharpoonup |\nabla u_*|^2\, dx + 4\pi \sum_{j=1}^{k_{\mathcal{B}}} \delta_{q_{\mathcal{B}}^{(j)}} + 8\pi \sum_{j=1}^{k_{\mathcal{I}}} \delta_{q_{\mathcal{I}}^{(j)}} \quad as \quad t \to T$$

as convergence of Radon measures. Furthermore, the velocity field satisfies (for some $c > 0$ and $0 < \nu_j < 1, k = k_{\mathcal{B}} + k_{\mathcal{I}}$ and $\{q_j\}_{j=1}^k = \{q_{\mathcal{B}}^{(j)}\}_{j=1}^{k_{\mathcal{B}}} \cup \{q_{\mathcal{I}}^{(j)}\}_{j=1}^{k_{\mathcal{I}}}$)

$$|v(x,t)| \leq c \sum_{j=1}^{k} \frac{\lambda_j^{\nu_j - 1}(t)}{1 + \left|\frac{x-q_j}{\lambda_j(t)}\right|}, \quad 0 < t < T,$$

Continued

Theorem

The profiles are : let W_2 be the least energy (degree 1) harmonic map

$$W_2(x) = \begin{bmatrix} \frac{2x}{1+|x|^2} \\ \frac{|x|^2-1}{1+|x|^2} \end{bmatrix}, \ x \in \mathbb{R}^2, \tag{0.1}$$

namely, $\int_{\mathbb{R}^2} |\nabla W_2|^2 = 8\pi$. Our first approximation of the boundary bubble will be based on the degree 1 profile

$$W_1 := Q_* W_2, \tag{0.2}$$

with

$$Q_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{0.3}$$

Inner–outer gluing method for parabolic equations

- good approximation $\implies$ small error
- $u = \text{approximation} + \overbrace{\underbrace{\eta_R \phi(y,t)}_{\text{inner}} + \underbrace{\psi(x,t)}_{\text{outer}}}^{\text{perturbation}}, \quad y = \frac{x-\xi(t)}{\lambda(t)}$
- Inner problem: $\lambda^2 \phi_t = L[\phi] + \underbrace{\text{coupling}(\psi) + \text{error}}_{\mathcal{H}}$
- Outer problem (maximum principle):
  $\psi_t = \Delta_x \psi + \underbrace{(\phi \Delta \eta_R + 2\nabla \eta_R \cdot \nabla \phi)}_{\text{coupling}} + \text{nonlinear terms} + \text{error}$
- Orthogonality conditions ($\{Z_j\}$ span the kernel around a "bubble")
  - $\int \mathcal{H} Z_j dy = 0 \implies$ good inner solution
  - $\int \mathcal{H} Z_j dy = 0 \implies$ reduced equations for $\lambda,\ \xi$
- Fixed point argument: $\phi,\ \psi,\ \lambda,\ \xi$

A crucial tool for the study: estimates of Stokes operator with Navier B.C.

Consider the following Stokes system

$$\begin{cases} \partial_t v + \nabla P = \Delta v + F, & \text{in } \mathbb{R}^2_+ \times (0,\infty), \\ \nabla \cdot v = 0, & \text{in } \mathbb{R}^2_+ \times (0,\infty), \\ \partial_{x_2} v_1\Big|_{x_2=0} = 0, \quad v_2\Big|_{x_2=0} = 0, \\ v\Big|_{t=0} = 0, \end{cases} \tag{0.4}$$

F is solenoidal:

$$\nabla \cdot F = 0, \quad F_2\Big|_{x_2=0} = 0.$$

Theorem (F.H. Lin, Y. S., J. Wei, Y. Zhou)

The solution to (0.4) *with solenoidal forcing can be expressed in the form*

$$v(x,t) = \int_0^t \int_{\mathbb{R}^2_+} \mathcal{G}^0(x,y,t-\tau)F(y,\tau)dyd\tau + \int_0^t \int_{\mathbb{R}^2_+} \mathcal{G}^*(x,y,t-\tau) \int_0^\tau F(y,s)dsdyd\tau \tag{0.5}$$

$$P(x,t) = \int_0^t \int_{\mathbb{R}^2_+} \mathcal{P}(x,y,t-\tau) \cdot F(y,\tau)dyd\tau$$

$$|\partial_t^s D_x^k D_y^m P_j(x,y,t)| \lesssim t^{-1-s-\frac{m_2}{2}}(|x-y^*|^2+t)^{-\frac{1+|k|+|m'|}{2}} e^{-\frac{cy_2^2}{t}},$$

$$|\partial_t^s D_x^k D_y^m G_{ij}^*(x,y,t)| \lesssim t^{-1-s-\frac{m_2}{2}}(|x-y^*|^2+t)^{-\frac{2+|k|+|m'|}{2}} e^{-\frac{cy_2^2}{t}}. \tag{0.6}$$

See Solonnikov for others B.C.

Heat flow of harmonic maps with free boundary

Let (M, g) be an m-dimensional smooth Riemannian manifold with boundary ∂M and N be another smooth compact Riemannian manifold without boundary. Suppose Σ is a k-dimensional submanifold of N without boundary. Any continuous map $u_0 : M \to N$ satisfying $u_0(\partial M) \subset \Sigma$ defines a relative homotopy class in maps from $(M, \partial M)$ to (N, Σ). A map $u : M \to N$ with $u(\partial M) \subset \Sigma$ is called homotopic to u_0 if there exists a continuous homotopy $h : [0,1] \times M \to N$ satisfying $h([0,1] \times \partial M) \subset \Sigma$, $h(0) = u_0$ and $h(1) = u$. An interesting problem is that whether or not each relative homotopy class of maps has a representation by harmonic maps, which is equivalent to the following problem:

$$\begin{cases} -\Delta u = \Gamma(u)(\nabla u, \nabla u), \\ u(\partial M) \subset \Sigma, \\ \frac{\partial u}{\partial \nu} \perp T_u\Sigma. \end{cases} \tag{0.7}$$

Here ν is the unit normal vector of M along the boundary ∂M, $\Delta \equiv \Delta_M$ is the Laplace-Beltrami operator of (M, g), Γ is the second fundamental form of N (viewed as a submanifold in $\mathbb{R}^\ell$ via Nash's isometric embedding), T_pN is the tangent space in $\mathbb{R}^\ell$ of N at p and $\perp$ means orthogonal in $\mathbb{R}^\ell$. (0.7) is the Euler-Lagrange equation for critical points of the Dirichlet energy functional

$$E(u) = \int_M |\nabla u|^2 \, dv_g$$

defined over the space of maps

$$H^1_\Sigma(M, N) = \{u \in H^1(M, N) : u(x) \subset \Sigma \text{ a.e. } x \in \partial M\}.$$

Existence by flow (see Eells-Sampson for standard harmonic maps)

$$\begin{cases} \partial_t u - \Delta u = \Gamma(u)(\nabla u, \nabla u) & \text{on } M \times [0, \infty), \\ u(x,t) \in \Sigma & \text{on } \partial M \times [0, \infty), \\ \frac{\partial u}{\partial \nu}(x,t) \perp T_{u(x,t)}\Sigma & \text{on } \partial M \times [0, \infty) \\ u(\cdot, 0) = u_0 & \text{on } M. \end{cases} \tag{0.8}$$

Weak solutions of the harmonic map heat flow with FB

Take $M = \mathbb{R}^{n+1}_+$ and $N = \mathbb{R}^\ell$. We will try to solve the following regularized version of the heat flow (extrinsic version):

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \ \mathbb{R}^{n+1}_+ \times \mathbb{R}_+, \\ u(x,0,t) \in \Sigma & x \in \mathbb{R}^n, t > 0, \\ -\lim\limits_{y \to 0^+} \dfrac{\partial u}{\partial y}(x,y,t) \perp T_{u(x,0,t)}\Sigma & x \in \mathbb{R}^n, t > 0, \\ u(x,y,0) = u_0(x,y) & (x,y) \in \mathbb{R}^{n+1}_+. \end{cases} \tag{0.9}$$

We focus on the study of (0.9) for

$$\boxed{\Sigma = \mathbb{S}^{\ell-1}}$$

Intrinsic version: Hamilton, Struwe, Chen-Lin

Harmonic maps with free boundary and their geometric interest

$$\begin{cases} -\Delta u = \Gamma(u)(\nabla u, \nabla u), \\ u(\partial M) \subset \Sigma, \\ \frac{\partial u}{\partial \nu} \perp T_u\Sigma. \end{cases}$$

- Existence and regularity: Nitsche, Hildebrandt, Jost, Duzaar-Steffen, Hardt-Lin, etc...
- New point of view via half-harmonic maps: Da Lio-Rivière, Millot-S., Da Lio-Rivière-Laurain
- Branched minimal immersions with free boundary and spectral geometry of extremal Steklov eigenvalues: Fraser-Schoen, Karpukhin-Stern, Laurain-Petrides, etc..

Ginzburg-Landau approximation

Given $U_0 \in \dot{H}^1(\mathbb{R}^{n+1}_+, \Sigma)$ and $\varepsilon > 0$, consider

$$\begin{cases} (\partial_t - \Delta) U_\varepsilon(x,y,t) = 0 & \text{in } \mathbb{R}^{n+1}_+ \times (0,\infty), \\ U_\varepsilon(x,y,0) = U_0(x,y) & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial U_\varepsilon}{\partial y} = -\dfrac{1}{\varepsilon^2}(1 - |U_\varepsilon|^2) U_\varepsilon & \text{on } \partial\mathbb{R}^{n+1}_+ \times (0,\infty). \end{cases} \tag{0.10}$$

For fixed $\varepsilon > 0$, (0.10) is the gradient flow of

$$E_\varepsilon(U) = \int_{\mathbb{R}^{n+1}_+} \frac{1}{2} |\nabla U|^2 \, dxdy + \int_{\partial\mathbb{R}^{n+1}_+} \frac{(1-|U|^2)^2}{4\varepsilon^2} \, dx.$$

There exist smooth solutions $U_\varepsilon : \mathbb{R}^{n+1}_+ \times (0,\infty) \to \mathbb{R}^\ell$ of (0.10):

$$\begin{aligned} & E_\varepsilon(U_\varepsilon)(t) + \int_0^t \int_{\mathbb{R}^{n+1}_+} |\partial_t U_\varepsilon|^2 \, dxdydt \\ & \leq E_\varepsilon(U_0) = \int_{\mathbb{R}^{n+1}_+} \frac{1}{2} |\nabla U_0|^2 \, dxdy. \end{aligned} \tag{0.11}$$

For $U_0 \in H^1(\mathbb{R}^{n+1}_+, \Sigma)$, let $u_0 = U_0|_{\partial\mathbb{R}^{n+1}_+}$. Let $\mathcal{P}^k$ denote the k-dimensional Hausdorff measure on $\mathbb{R}^{n+1} \times \mathbb{R}$ with respect to

$$\delta((X,t),(Y,s)) = \max\left\{|X-Y|, \sqrt{|t-s|}\right\}.$$

Theorem (Hyder, Segatti, Y. S., Wang, (CPDE 2022))

1) $\exists\, U_* \in L^\infty(\mathbb{R}_+, H^1(\mathbb{R}^{n+1}_+, \mathbb{S}^{\ell-1}))$ *with* $\partial_t U_* \in L^2(\mathbb{R}^{n+1}_+ \times \mathbb{R}_+)$ *solving*

$$\begin{cases} (\partial_t - \Delta)U_* = 0 & \text{in } \mathbb{R}^{n+1}_+ \times (0,\infty), \\ U_*\big|_{t=0} = U_0 & \text{on } \mathbb{R}^{n+1}_+, \\ U_*(x,0,t) \in \Sigma; \quad \frac{\partial U_*}{\partial y}(x,0,t) \perp T_{U_*(x,0,t)}\Sigma & \text{on } \mathbb{R}^n \times (0,\infty). \end{cases}$$

such that $U_\varepsilon \rightharpoonup U_*$ *in* $H^1(\mathbb{R}^{n+1}_+ \times \mathbb{R}_+)$.
2) $\exists\, \mathcal{S} \subset \partial\mathbb{R}^{n+1}_+ \times (0,\infty)$, *with* $\mathcal{P}^n(\mathcal{S}) < \infty$, *such that*

$$U_\varepsilon \to U_* \in C^2_{loc}(\overline{\mathbb{R}^{n+1}_+} \times (0,\infty) \setminus \Sigma).$$

Theorem (Continued)

3) Set $u_* = U_*\big|_{\partial\mathbb{R}^{n+1}_+\times[0,\infty)}$. *Then* $u_* \in C^\infty(\mathbb{R}^n \times (0,\infty)\setminus \mathcal{S})$ *solves the* $\frac{1}{2}$*-harmonic map heat flow:*

$$\begin{cases}(\partial_t - \Delta)^{\frac{1}{2}} u_* \perp T_{u_*}\Sigma & \text{in} \;\; \mathbb{R}^n \times (0,\infty), \\ u_*(x,0) = u_0(x) & \text{in} \;\; \mathbb{R}^n. \end{cases} \tag{0.12}$$

4) For any $C_0 > 0$, $\exists \epsilon_0 > 0$ *such that if*

$$\|\nabla U_0\|_{L^\infty(\mathbb{R}^{n+1}_+)} \le C_0, \quad E(U_0) \le \epsilon_0,$$

$U_* \in C^\infty\big(\overline{\mathbb{R}^{n+1}_+} \times (0,\infty)\big)$ $(\Rightarrow u_* = U_*\big|_{\partial\mathbb{R}^{n+1}_+\times[0,\infty)} \in C^\infty)$.

A consequence of the bubbling construction

Y. Chen and F.-H. Lin (JGEA) raised the following question: When M is a smooth domain in $\mathbb{R}^2$, $N = \mathbb{R}^n$ and Σ a smooth compact submanifold of $\mathbb{R}^n$, is there a smooth initial datum u_0 such that the harmonic map heat flow has no global smooth solutions ?

Theorem (S.-Wei-Zheng, AJM 2021)

Given points $q = (q_1, \cdots, q_k) \in \left(\partial\mathbb{R}^2_+\right)^k := (\mathbb{R} \times \{0\})^k$ and any sufficiently small $T > 0$, there exists u_0 such that the solution $u_q(x,t)$ blows-up at exactly those k points as $t \nearrow T$. More precisely, there exist numbers $k_i^* > 0$ and a function $u_* \in H^1(\mathbb{R}^2_+) \cap C(\mathbb{R}^2_+)$, such that in the H^1 sense

$$u_q(x,y,t) - u_*(x,y) - \sum_{j=1}^{k} \left[\omega\left(\frac{x-q_i}{\lambda_i}, \frac{y}{\lambda_i}\right) - \omega(\infty)\right] \to 0 \text{ as } t \nearrow T,$$

with $\lambda_i(t) = k_i^* \frac{T-t}{|\log(T-t)|^2}(1+o(1))$ as $t \nearrow T$.

Another solution for Chen-Lin question

Theorem (Lin, S., Wei, Zhou)

Assume $M = \mathbb{R}^2_+$, $N = \mathbb{S}^2$, *and* $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{S}^2 : x_3 = 0\}$ *in* (0.8). *Given any finitely many distinct points* q_k *in* $\mathbb{R}^2_+$ *or on* $\partial\mathbb{R}^2_+$, *for* $T > 0$ *sufficiently small, there exists initial data* u_0 *such that the solution to* (0.8) *blows up exactly at these prescribed points at time* $t = T$. *Moreover, the blow-up profile takes the form of sharply scaled 1-corotational profile around each point* q_k *with type II blow-up rate*

$$\lambda_k(t) \sim \frac{T-t}{|\log(T-t)|^2} \quad as \quad t \to T.$$

Why is the LCF with FB model physical ?

We first derive the energy law. Multiply by v and integrate over Ω:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2+\int_\Omega (v\cdot\nabla v)\cdot v+\int_\Omega \nabla P\cdot v = -\int_\Omega |\nabla v|^2-\int_\Omega (\Delta u\cdot\nabla u)\cdot v,$$

where we have used

$$\nabla\cdot\left(\nabla u\odot\nabla u-\frac{1}{2}|\nabla u|^2\mathbb{I}_2\right)=\Delta u\cdot\nabla u.$$

And

$$\int_\Omega (v\cdot\nabla v)\cdot v=\int_\Omega \nabla P\cdot v=0.$$

So we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2=-\int_\Omega |\nabla v|^2-\int_\Omega (\Delta u\cdot\nabla u)\cdot v. \tag{0.13}$$

Multiply with $\Delta u + |\nabla u|^2 u$ and integrate over Ω

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 + \int_\Omega (v\cdot\nabla u)\cdot(\Delta u + |\nabla u|^2 u) = \int_\Omega \left|\Delta u + |\nabla u|^2 u\right|^2.$$

Since

$$\int_\Omega (v\cdot\nabla u)\cdot(|\nabla u|^2 u) = \int_\Omega |\nabla u|^2 v\cdot\frac{\nabla(|u|^2)}{2} = 0,$$

we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 + \int_\Omega (\Delta u\cdot\nabla u)\cdot v = \int_\Omega \left|\Delta u + |\nabla u|^2 u\right|^2. \quad (0.14)$$

Combining (0.13) and (0.14), we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |v|^2 + |\nabla u|^2\right) = -\int_\Omega |\nabla v|^2 - \int_\Omega \left|\Delta u + |\nabla u|^2 u\right|^2 \quad (0.15)$$

which is called *the basic energy law* (energy dissipation).

On the other hand, the physical compatibility condition should be satisfied

$$\left\langle \left(\frac{\nabla v + (\nabla v)^T}{2} - P\mathbb{I}_2 - \nabla u \odot \nabla u \right) \nu, \tau \right\rangle = 0, \qquad \text{on} \;\; \partial\Omega, \tag{0.16}$$

where

$$\nabla \cdot \left(\frac{\nabla v + (\nabla v)^T}{2} - P\mathbb{I}_2 - \nabla u \odot \nabla u \right)$$

is called *stress tensor*. It is easy to see that $< P\mathbb{I}_2 \nu, \tau >= 0$ as $< \nu, \tau >= 0$. Also,

$$\left\langle \frac{\nabla v + (\nabla v)^T}{2} \nu, \tau \right\rangle = 0$$

is the *Navier boundary condition* and

$$0 = \langle (\nabla u \odot \nabla u)\nu, \tau \rangle = \langle \nabla_\nu u, \nabla_\tau u \rangle$$

implies the free boundary condition

$$\frac{\partial u}{\partial \nu} \perp T_u \Sigma \;\; \text{on} \;\; \partial\Omega \times (0, T).$$

Symmetry encoded in the free boundary condition

Since on $\partial\mathbb{R}^2_+$ one has

$$\begin{cases} u(x,t) \in \Sigma, \\ \frac{\partial u}{\partial \nu}(x,t) \perp T_{u(x,t)}\Sigma, \end{cases} \implies \begin{cases} \partial_{x_2} u_1 = 0, \\ \partial_{x_2} u_2 = 0, \\ u_3 = 0, \end{cases} \tag{0.17}$$

and

$$\begin{cases} v \cdot \nu = 0, \\ (Sv \cdot \nu)_\tau = 0, \end{cases} \implies \begin{cases} \partial_{x_2} v_1 = 0, \\ v_2 = 0, \end{cases} \tag{0.18}$$

then even reflection for u_1, u_2, v_1 and odd reflection for u_3, v_2:

$$\tilde{u}(x_1, x_2, t) = \begin{pmatrix} u_1(x_1, -x_2, t) \\ u_2(x_1, -x_2, t) \\ -u_3(x_1, -x_2, t) \end{pmatrix}, \quad \tilde{v}(x_1, x_2, t) = \begin{pmatrix} v_1(x_1, -x_2, t) \\ -v_2(x_1, -x_2, t) \end{pmatrix}, \tag{0.19}$$

is such that the free boundary conditions are satisfied.

With the previous reflections and

$$\tilde{P}(x_1, x_2, t) = P(x_1, -x_2, t), \tag{0.20}$$

the structure of the equation is preserved, i.e.,

$$\begin{cases} \partial_t \tilde{u} + \tilde{v} \cdot \nabla \tilde{u} = \Delta \tilde{u} + |\nabla \tilde{u}|^2 \tilde{u}, \\ \partial_t \tilde{v} + \tilde{v} \cdot \nabla \tilde{v} + \nabla \tilde{P} = \Delta \tilde{v} - \nabla \cdot \left(\nabla \tilde{u} \odot \nabla \tilde{u} - \frac{1}{2} |\nabla \tilde{u}|^2 \mathbb{I}_2 \right), \\ \nabla \cdot \tilde{v} = 0. \end{cases} \tag{0.21}$$

Open problems

- Caffarelli-Kohn-Nirenberg partial regularity of suitable solutions in two dimensions (on-going with Yantao Wu)
- Global Weak solutions in three dimensions
- Coupling surface diffusion with heat flows of harmonic maps (Vorticity formulation with compensated-compactness phenomena with the Hopf differential)
- (Heat flow of) Harmonic maps with free boundary: Rigidity *à la Siu-Sampson* for manifolds with boundary, singular domains/targets, Teichmuller flow on moduli space of hyperbolic metrics on surfaces with boundary

THANK YOU