Jumps in Besov spaces and fine properties of Besov and fractional Sobolev functions (Joint work with Paz Hashash)

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- For $u \in BV(\Omega)$, $\exists$ a "jump set", J_u of dimension $N-1$, with the normal ν_u s.t.:
  - (i) At $\mathcal{H}^{N-1}$-a.e. each $x \in \Omega \setminus J_u$, u is approximately continuous,
  - (ii) the approximate limits u^+, u^- exist on both sides of J_u.

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- The $W^{r,q}$ semi-norm: $[u]_{W^{r,q}}$.
- The $W^{r,q}$ norm: $\|u\|_{W^{r,q}} := [u]_{W^{r,q}} + \|u\|_{L^q}$.

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For any $x \in \Omega$ the vector z, uniquely determined by (1), is called the approximate limit of u at x and denoted by $\tilde{u}(x)$

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$$\lim_{\rho\to 0^+}\left(\inf_{c\in\mathbb{R}^d} \fint_{B_\rho(x)} |u(y) - c|dy\right) = 0 \quad \textit{if and only if}$$

$$\lim_{\rho\to 0^+} \fint_{B_\rho(x)} |u(y) - u_{B_\rho(x)}|dy = 0. \quad (3)$$

A classical result

Theorem (Adams-Hedberg, Ziemer, Evans-Gariepy)

Let $\Omega \subset \mathbb{R}^N$ *be an open set,* $1 \le p \le N$ *and* $u \in W^{1,p}(\Omega)$*.*

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Recall that for $1 \leq p < N$ there exists a constant C, such that $cap_p(E) \leq C\mathcal{H}^{N-p}(E)$ $\forall E \subset \mathbb{R}^N$.

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Remark

Recall that for $1 \le p < N$ there exists a constant C, such that $cap_p(E) \le C\mathcal{H}^{N-p}(E)$ $\forall E \subset \mathbb{R}^N$. However, if $1 < p < N$ then $\mathcal{H}^{N-p}(E) < \infty$ implies $cap_p(E) = 0$.

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Given $u \in L^1_{loc}(\Omega, \mathbb{R}^d)$, we say that $x \in \Omega$ is a generalized approximate limit-oscillation point of u if

$$\liminf_{\rho \to 0^+} \left(\inf_{c \in \mathbb{R}^d} \fint_{B_\rho(x)} |u(y) - c| dy \right) = 0. \tag{10}$$

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Obviously $\mathcal{S}''_u \subset \mathcal{S}'_u \subset \mathcal{S}_u$.

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Let $\Omega \subset \mathbb{R}^N$ be an open set, $u \in L^1_{loc}(\Omega, \mathbb{R}^d)$ and $x \in \Omega$. We say that x is an approximate jump point of u if $\exists a, b \in \mathbb{R}^d$ and $\exists \nu \in S^{N-1}$ such that $a \neq b$ and

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where

$$B^+_\rho(x,\nu) := \{y \in B_\rho(x) : (y - x) \cdot \nu > 0\}$$
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The triple (a, b, ν), uniquely determined, up to a permutation of (a, b) and the sign of ν, is denoted by $(u^+(x), u^-(x), \nu_u(x))$. The set of approximate jump points is denoted $\mathcal{J}_u$.

Remark

Obviously we have $\mathcal{J}_u \subset \mathcal{S}''_u \subset \mathcal{S}'_u \subset \mathcal{S}_u$.

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Theorem (Giacomo Del Nin)

Let $\Omega \subset \mathbb{R}^N$ *be an open set and let* $u \in L^1_{loc}(\Omega, \mathbb{R}^d)$. *Then the jump set* $\mathcal{J}_u$ *is countably* $(N-1)-$*rectifiable.*

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Theorem (Federer-Vol'pert)

Let $\Omega \subset \mathbb{R}^N$ *be a open set, and* $u \in BV_{loc}(\Omega, \mathbb{R}^d)$. *Then, the jump set* $\mathcal{J}_u$ *is countably* $(N-1)-$*rectifiable set, oriented with the jump vector* $\nu_u(x)$, *and moreover, we have* $\mathcal{H}^{N-1}(\mathcal{S}_u \setminus \mathcal{J}_u) = 0$.

Theorem (De Lellis-Otto, Ghiraldin-Lamy)

Let $\Omega \subset \mathbb{R}^2$ *be a open set, and* $u \in \left(B^{\frac{1}{3}}_{3,\infty}\right)_{loc}(\Omega, \mathbb{R}^2)$*, satisfying*

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Then, the jump set $\mathcal{J}_u$ *is countably* $1-$*rectifiable set, oriented with the jump vector* $\nu_u(x)$, *and moreover, we have* $\mathcal{H}^1\left(\mathcal{S}'_u \setminus \mathcal{J}_u\right) = 0$.

Our result for the Besov space $B^s_{q,\infty}$

Theorem (Hashash-P 2023)

Let $\Omega \subset \mathbb{R}^N$ be an open set, $s \in (0,1)$ and $q \geq 1$, such that $sq \leq N$ and let $u \in \left(B^s_{q,\infty}\right)_{loc}(\Omega)$.

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Example (Hashash-P 2023): $u_{B_\rho(x)}$ in (15) cannot be replaced by $\lim_{\varepsilon\to 0^+} u_{B_\varepsilon(x)}$ in the general case.

Thank you for your attention!