A priori estimates of solutions of local and nonlocal superlinear parabolic problems

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Abstract

We consider **a priori estimates** of possibly sign-changing solutions to superlinear parabolic problems and their applications (blow-up rates, energy blow-up, existence of nontrivial steady states etc.).

Method: **energy, interpolation** and **bootstrap** arguments.

We first discuss some known results on local problems and then consider problems with nonlocal nonlinearities and/or nonlocal differential operators.

$$u_t + \mathcal{A}u = \mathcal{F}(u(\cdot, t)) \quad \text{in} \quad \Omega \times (0, T),$$

$$u(\cdot, 0) = u_0, \qquad u = 0 \;\; \text{in } (\mathbb{R}^n \setminus \Omega) \times (0, T)$$

- $u : \mathbb{R}^n \to \mathbb{R}^N$, u_0 and $\Omega \subset \mathbb{R}^n$ are sufficiently smooth
- $\mathcal{A}$ is a differential operator (in x)
- $\mathcal{F}$ is a superlinear potential operator

} possibly nonlocal

Typical result:

$$\|u(\cdot, t)\| \leq C(\|u_0\|, \varepsilon), \quad t \in [0, T - \varepsilon) \tag{AE}$$

where

$$\varepsilon \begin{cases} = 0 & \text{if } T = \infty, \\ \geq 0 & \text{if } T < \infty, \end{cases}$$

and usually $\|\cdot\| = \|\cdot\|_\infty$

Model problem: (AE) and application to blow-up rate estimates

$$\left.\begin{aligned} u_t - \Delta u &= |u|^{p-1}u && \text{in } \Omega \times (0,T) \\ u &= 0 && \text{on } \partial\Omega \times (0,T) \\ u(\cdot,0) &= u_0 && \text{in } \Omega \end{aligned}\right\} \quad (1)$$

$N = 1$, $u_0 \in L^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ bounded, $T = \infty$, $p > 1$

[Cazenave, Lions 1984]: (AE) if $p < p_{CL} := \frac{3n+8}{(3n-4)_+}$ (**energy, interpolation**)

[Giga, Kohn 1987]: (AE) for a rescaled equation if $p < p_{CL}$
($\Rightarrow$ blow-up rate estimate for (1) if Ω convex, $T < \infty$)

[Q. 1999]: (AE) if $p < p_S := \frac{n+2}{(n-2)_+}$ (**bootstrap**)

(AE) for a rescaled equation if $p < p_S$ ($\Rightarrow$ blow-up rate estimates):

[Giga, Matsui, Sasayama 2004]: $\mathcal{F}(u) = |u|^{p-1}u$,

[V.T. Nguyen 2015]: $\mathcal{F}(u) = |u|^{p-1}u + h(u)$,

[Hamza, Zaag 2022]: $\mathcal{F}(u) = |u|^{p-1}u\log^a(2+u^2)$

[Zhanpeisov 2021-22]: $\mathcal{F}(u) = \nabla\left(\sum \beta_{ij}|u_i|^{\frac{p+1}{2}}|u_j|^{\frac{p+1}{2}}\right)$, $\beta_{ij} \geq 0$, $N > 1$

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More general or different problems

[Q. 2003]: (AE) for general second-order operators $\mathcal{A}$, $T \leq \infty$, $N = 1$,
Ω possibly unbounded, $\mathcal{F}(u) = f(\cdot, u)$,

$$c_1|u|^{p_1} - a_1(x) \leq f(x,u)\mathrm{sign}(u) \leq c_2|u|^{p_2} + a_2(x),$$

$p_1 < p_2 < p_S, \quad p_2 - p_1 < \kappa(p_2)$.

APPLICATIONS: **blow-up of energy**, **continuity of** $u_0 \mapsto T_{max}(u_0)$,
[Q. 2004]: existence of **nontrivial steady states** or **periodic solutions**.

[Q., Souplet 2003]: (AE) for global solutions of

$$u_t - \Delta u = -au \text{ in } \Omega \times (0,\infty), \quad \frac{\partial u}{\partial \nu} = |u|^{p-1}u \text{ on } \partial\Omega \times (0,\infty).$$

[Ackermann, Bartsch, Kaplický, Q. 2008] (AE) for
$u_t - \Delta u = a(x)|u|^{p-1}u + h(x,u)$, where a may change sign,
h has most linear growth in u, $p < \hat{p}$, $\hat{p} \in (p_{CL}, p_S)$, $n > 2$.

APPLICATIONS: **steady states and connecting orbits**.

Results/arguments mentioned above used also in

[Busca, Jendoubi, Poláčik 2002]: Convergence to equilibria if $\Omega = \mathbb{R}^n$.

[Groisman, Rossi, Zaag 2003], [Chaves, Rossi 2004], ...: Properties of $u_0 \mapsto T_{\max}(u_0)$.

[Ackermann, Bartsch 2005], [Ackermann, Bartsch, Kaplický 2007]: Invariant sets.

[Gazzola, Weth 2005]: Solutions with high energy initial data.

[Lazzo, Schmidt 2005]: Saddle-point dynamics.

[Amann, Q. 2005]: Existence of optimal controls.

[Dancer 2007]: Multiplicity of equilibria (Conley index).

[Fiedler, Matano 2007]: Global dynamics of blow-up profiles ($n = 1$).

[Cazenave, Dickstein, Weissler 2009]: Structure of global solutions of (1).

[Chang, Jiang 2009]: Morse theory for indefinite elliptic problems.

$\cdots$ (etc) $\cdots$

[S.-Z. Du 2019]: Blow-up with bounded energy ($p \geq p_S$, (AE): $\|\cdot\|_{p+1-\varepsilon}$).

[H. Li, Z.-Q. Wang 2021]: $-\Delta u_i + \lambda u_i = \mu_i u_i^3 + \sum_{j\neq i} \beta_{ij} u_i u_j^2$ in $B \subset \mathbb{R}^n$ ($i = 1, 2, \ldots, N$, $n \leq 3$, $\lambda_i, \mu_i > 0$, $\sum_j \beta_{ij} \leq -\mu_i$): Radial solutions with prescribed number of sign changes for each component.

[Q. 2022]: (AE) for general power $p < p_S$ ($N = 2$, $\mu_i = 1$; Liouville).

Use of bootstrap arguments from [Q. 1999] in nonlocal problems

Assume $\mathcal{A} = -\Delta$, $N = 1$, Ω bounded:

[Q. 2003]: $\mathcal{F}_1(u) = \frac{|u|^{p-1}u}{(1+\int_\Omega |u|^{p+1})^\alpha}$, $0 < \alpha < \frac{p-1}{p+1}$, $p \in (1, p_S)$

$\mathcal{F}_2(u) = |u|^{p-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u$ $\left(\frac{\partial u}{\partial \nu}|_{\partial\Omega\times(0,T)} = 0\right)$, $p \in (1, p_S)$

more general problems if $n = 1$

[Rouchon 2006]: $\mathcal{F}_1$ with $\alpha < 0$

[Ianni 2012]: $\mathcal{F}_3(u)(x) = \left(\int_\Omega \frac{|u(y)|^p}{|x-y|^{n-2}}\,dy\right)|u(x)|^{p-2}u(x)$, $n = 3$, $p = 2$

$\mathcal{F}_4(u) = |u|^{q-1}u \pm \mathcal{F}_3(u)$, $n = 3$, $q \in (1, p_S)$ ($q > 3$ if "$-$")

REMARK: Consider $\mathcal{F}_3$. If $n = 3$, then $p = 2 < p_{CL} = \frac{17}{5} < p_S = 5$: bootstrap not needed. Problems with $\mathcal{F}_3$ (and general p, n) have been studied by many authors, but (AE) either have not been proved or only under very restrictive assumptions (e.g. [X. Li, B. Liu 2020]: $p < p_F := 1 + \frac{2}{n}$).

$$\left.\begin{array}{ll} u_t - \Delta u = \left(\int_\Omega \frac{|u(y)|^p}{|x-y|^{n-2}}\,dy\right)|u|^{p-2}u & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(\cdot,0) = u_0 & \text{in } \Omega \end{array}\right\} \qquad (2)$$

$$p^* := p_S - \frac{1}{n-2}\,\frac{8}{2(n-2)\sqrt{n(n+3)}+2n^2-n-4} \qquad p_{CL} = \frac{3n+8}{3n-4}$$

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $n \geq 2$, $u_0 \in L^\infty(\Omega)$, $p \in (1, p^*)$. If u is a solution of (2), then $\|u(t)\|_\infty \leq C(\|u_0\|_\infty)$ for any $t > 0$.*

n	p_S	p^*	p_{CL}	[Ianni 2012]	p_F [Li,Liu 2020]
2	∞	∞	7		
3	5	> 4.589	3.4	$p = 2$	$1.66\overline{6}$
4	3	> 2.911	2.5		1.5
5	$2.33\overline{3}$	> 2.299	$2.0\overline{909}$		1.4
$n \gg 1$	$\frac{n+2}{n-2}$	$\approx p_S - \frac{2}{n^3}$	$\approx p_S - \frac{8}{3n^2}$		$\approx p_S - \frac{2}{n}$

$$\left.\begin{array}{ll} u_t + (-\Delta)^\alpha u = |u|^{p-1} u & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{in } (\mathbb{R}^n \setminus \Omega) \times (0,\infty) \\ u(\cdot,0) = u_0 & \text{in } \Omega \end{array}\right\} \qquad (3)$$

$$\alpha \in (0,1), \qquad p_S(\alpha) := \frac{n+2\alpha}{n-2\alpha}$$

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $u_0 \in L^\infty(\Omega)$, $p \in (1, p_S(\alpha))$. If u is a solution of (3), then $\|u(t)\|_\infty \le C(\|u_0\|_\infty)$ for any $t > 0$.*

[Grubb 2015–2024]:

$(-\Delta)^\alpha_D : H_q^{\alpha,(2\alpha)}(\Omega) \to L^q(\Omega)$ isomorphism (α-transmission space)

L^p-maximal regularity $\Rightarrow \|u_t\|_{L^p(0,T;L^q(\Omega))} \le C \||u|^{p-1}u\|_{L^p(0,T;L^q(\Omega))}$

$H_q^{\alpha,(2\alpha)}(\Omega) \hookrightarrow L^r(\Omega)$ provided $2\alpha - \frac{n}{q} \ge -\frac{n}{r}$.

Remark. We could also consider $\mathcal{A}(u) = -a(\int_\Omega |\nabla u|^2)\Delta u$, for example, where $a(\cdot) \ge a_0 > 0$ is continuous, $\lim_{s\to\infty} a(s) = a_\infty$.

Remark. If u is only defined in $(0, T)$ with $T < \infty$, and $\varepsilon > 0$, then the same arguments as in the proofs of theorems above yield the estimate

$$\|u(t)\|_\infty \leq C(\|u_0\|_\infty, \varepsilon), \quad t \in (0, T - \varepsilon)$$

(we can take $\varepsilon = 0$ if the corresponding energy E stays bounded in $(0, T)$).

Corollary. *Consider problems (2), (3) in* $(0, T)$, $T \leq \infty$.

(i) If u *blows up at* $T < \infty$, *then* $E(u(t)) \to -\infty$ *as* $t \to T-$.

The maximal existence time $T_{max}(u_0)$ *depends continuously on* t.

(ii) Let $\mathcal{G} := \{u_0 : T_{max}(u_0) = \infty\}$. *Then* $\mathcal{G}$ *is closed.*

If $u_0 \in \partial\mathcal{G}$, *then* $\omega(u_0)$ *consists of nontrivial steady states.*

Idea of the proof for (1)–(3): Step 1 (energy and interpolation)

Assume: $\alpha \in (0,1]$, $1 < p < p_S(\alpha)$, $u_0 \in L^\infty(\Omega)$, Ω bounded, $T = \infty$,

$\mathcal{A} = (-\Delta)^\alpha$, $\quad \mathcal{F}(u) \in \Big\{ |u|^{p-1}u, \Big(\int_\Omega \frac{|u(y)|^p}{|x-y|^{n-2}}\,dy\Big)|u|^{p-2}u \text{ (if } \alpha{=}1) \Big\}$.

$$E(u) := \tfrac{1}{2}\|u\|_{H^\alpha}^2 - \Phi(u), \quad \Phi(u) := c_p \int_\Omega \mathcal{F}(u)u, \quad c_p \in \Big\{\tfrac{1}{p+1}, \tfrac{1}{2p}\Big\}$$

$$\tfrac{1}{2}\tfrac{d}{dt}\|u(\cdot,t)\|_2^2 = \int_\Omega u_t u = \int_\Omega (-\mathcal{A}(u) + \mathcal{F}(u))u$$

$$= -c_1 E(u) + c_2\|u\|_{H^\alpha}^2 + c_3\Phi(u) \geq -c_1 E(u) + c_4\|u\|_2^{p+1}$$

$$\Rightarrow E \geq 0,\ \|u\|_2 \leq C,\ \int_\delta^\infty \int_\Omega u_t^2 \leq E(u(\delta)) \leq C(\|u(\delta)\|_{H^\alpha}) \leq C(\|u_0\|_\infty)$$

$\delta \leq t_1 < t_2, \quad t_2 - t_1 \leq 2$:

$$\int_{t_1}^{t_2} \|u\|_{p+1}^{q(p+1)} \leq C \int_{t_1}^{t_2} (\|u\|_{H^\alpha}^2 + \Phi(u))^q \leq C \int_{t_1}^{t_2} \Big(1 + \Big|\int_\Omega u_t u\Big|^q\Big) \overset{q=2}{\leq} C$$

$\Rightarrow u$ bounded in $H^1((t_1,t_2), L^2(\Omega)) \cap L^{q(p+1)}((t_1,t_2), L^{p+1}(\Omega))$

Interpolation: $\quad \|u(\cdot,t)\|_s \leq C$ if $s < s_q := p + 1 - \frac{p-1}{q+1}$

- $s_q > \frac{n}{2\alpha}(p-1)$ if $[q = 2,\ \alpha = 1,\ p < p_{CL}]$ or $[q \gg 1,\ p < p_S(\alpha)]$

Idea of the proof for (1)–(3): Step 2 (bootstrap)

Recall: $\int_{t_1}^{t_2} \Phi(u)^q \le C \int_{t_1}^{t_2}(1+|\int_\Omega u_t u|^q) \overset{q=2}{\le} C$, $\quad \|u(\cdot,t)\|_s \le C$, $s < s_q$

$$
\begin{aligned}
\tilde q > q: &\qquad \int_{t_1}^{t_2} \Phi(u)^{\tilde q} \le C\Big(1+\int_{t_1}^{t_2} \big|\int_\Omega u_t u\big|^{\tilde q}\Big)\\
\|u\|_s \le C: &\qquad \le C\Big(1+\int_{t_1}^{t_2} \|u_t\|_{s'}^{\tilde q}\Big)\\
r < s',\ \frac{\theta}{r}+\frac{1-\theta}{2}=\frac{1}{s'}: &\qquad \le C\Big(1+\int_{t_1}^{t_2} \|u_t\|_r^{\theta\tilde q}\|u_t\|_2^{(1-\theta)\tilde q}\Big)\\
\beta(1-\theta)\tilde q = 2,\ \int \|u_t\|_2^2 \le C: &\qquad \le C\Big(1+\big(\int_{t_1}^{t_2} \|u_t\|_r^{\beta'\theta\tilde q}\big)^{1/\beta'}\Big)\\
&\qquad \le C\Big(1+\big(\int_{t_1}^{t_2} \|\mathcal{F}(u)\|_r^{\beta'\theta\tilde q}\big)^{1/\beta'}\Big)
\end{aligned}
$$

Bootstrap condition: $\quad \|\mathcal{F}(u)\|_r^{\beta'\theta\tilde q} \le C\Phi(u)^q$

$\mathcal{F}(u) = |u|^{p-1}u$: $\quad r := \frac{p+1}{p}, \quad \|\mathcal{F}(u)\|_r = C\Phi(u)^{1-\frac{1}{p+1}}$

$\mathcal{F}(u) = \Big(\int_\Omega \frac{|u(y)|^p}{|x-y|^{n-2}}\,dy\Big)|u|^{p-2}u$: $\quad \frac{1}{r} := 1-\frac{n+2}{2n}\frac{1}{p}, \quad \|\mathcal{F}(u)\|_r \le \Phi(u)^{1-\frac{1}{2p}}$

A priori estimates based on Liouville theorems

$\mathcal{A} = (-\Delta)^\alpha$, $u \geq 0$:

$f(u) = u^p$, $1 < p < p_S$:

- elliptic Liouville theorem (and related a priori estimates) known: $\alpha = 1$: [Gidas, Spruck 1981], [W. Chen, C. Li 1991]; $\alpha < 1$: [W. Chen, C. Li 2016], ...
- parabolic Liouville thm known if $\alpha = 1$ ([Q. 2021]), but not if $\alpha < 1$ (only Fujita-type results for $p \leq \frac{n+2\alpha}{n}$). Missing: Monotonicity formula for rescaled equation (cf. [Deng, Sire, Wei, Wu 2021], $\alpha = 1/2$)

$f(u) = h(x_1)u^p$, $p > 1$, h satisfies some monotonicity assumptions:

- $\alpha = 1$: elliptic & parabolic Liouville known [Y. Du, S. Li 2005], [Poláčik, Q. 2005] (combined with Liouville for $f(u) = u^p$ implies a priori estimates for $f(u) = a(x)u^p + \lambda u$, where a changes sign),
- $\alpha < 1$: elliptic & parabolic Liouville known (moving planes). Parabolic results 2021–2024: W. Chen, L. Wu, P. Wang, L. Luo, Z. Zhang, G. Wang, Y. Liu, J.J. Nieto, L. Zhang, W. Dai, Y. Guo, H. Huang, Y. Zhong, ...