Aggregation-diffusion with (moderately) singular potentials: concentration and small-scale behaviour.

Alexandre Lanar (Boritchev), University of Lyon

Coauthors: Piotr Biler and Grzegorz Karch (Wrocław) and Philippe Laurençot (Chambéry).

Outline

What do we mean by small-scale behaviour? (1)

Basic idea: For example (case when the mass (L_1-norm) of $u \geq 0$ is conserved) most of the mass is concentrated on a small ball $B(\varepsilon)$ of radius ε.

More involved criterion: for $p \geq 1$ the L_p norms behave as $\varepsilon^{-c(p,N)}$.

Indeed, if $\int_{B(\varepsilon)} u \geq C$, then by Hölder's inequality,

$$\left(\int_{B(\varepsilon)} u^p \right)^{1/p} \geq C|B(\varepsilon)|^{-(p-1)/p} = C\varepsilon^{-N(1-1/p)}.$$

For a reverse inequality, we would need for example an upper estimate for $|u|_\infty$.
If we have weaker concentration in the limit $\varepsilon \to 0$ (on a surface of dimension k rather than a point), we obtain a different exponent for ε (equal to $-(N-k)(1-1/p)$).

What do we mean by small-scale behaviour? (2)

Oscillations beyond concentration: we study the small-scale behaviour of the Sobolev semi-norms

$$|u|_{m,p} := \left(\int_{\mathbb{R}^n} \left| \frac{\partial^m u}{\partial x^m} \right|^p \, dx \right)^{1/p}.$$

In the language of hydrodynamics/turbulence theory (Kolmogorov, Kraichnan, Frisch, Kuksin, Shirikyan...): typical small-scale quantities used to detect oscillations:

-$\hat{\mathbf{u}}(s)$ for large $|s|$.
-$\mathbf{u}(\mathbf{x}+\mathbf{r})-\mathbf{u}(\mathbf{x})$ for small $\mathbf{r}$.

Small-scale quantities are related to Sobolev norms:

-$H^m = W^{m,2}$ Sobolev norms defined through spectrum.

-Hölder, Sobolev-Slobodeckij... defined through increments; then Sobolev injections.

1D Periodic Generalised Burgers Equation

$$v_t + vv_x = \varepsilon v_{xx},\ t \geq 0,\ x \in S^1 = \mathbb{R}/\mathbb{Z}. \qquad \text{(1DB)}$$

"Pressureless turbulence" considered by many physicists, for instance Polyakov '95 (and Zeldovich in the multi-d case '89).

We assume that $\varepsilon > 0,\ \varepsilon \ll 1$. Again, only ε varies.

In [B. '14], sharp estimates for the (averaged) (1DB) solution.

$$\{|v|_{m,p}\} \overset{m,p}{\sim} \varepsilon^{-\gamma}, \quad \forall m \geq 1,\ 1 < p \leq \infty.$$

Here $\gamma(m,p) = m - 1/p$, and $\{\dots\}$ stands for averaging over a v_0-dependent time period $[T_1, T_2]$.

For more details (+randomness), see book [B.-Kuksin].

Aggregation-diffusion equation

Moderately singular aggregation-diffusion equation (pointy potential):

$$u_t - \varepsilon \Delta u + \nabla \cdot (u \nabla K * u) = 0, \ (ADE)$$

Radial kernel $K = k(|\cdot|)$ satisfying $k' \in L^\infty \cap C^0([0,\infty))$ (like 1D chemiotaxis).

Properties: Preservation of positivity; conservation of mass $M = \int u$; global well-posedness (in $L_1 \cap L_p, \ p < \infty, \ W^{m,1}$).

We assume that $k'(0) \neq 0$; therefore there is a mild singularity (pointy potential).

Typical examples $K(x) = -|x|; \ e^{-|x|}$.
From now on, we always assume $u \geq 0$.

Inviscid explosion

For $\varepsilon = 0$, i.e. for the aggregation equation

$$u_t + \nabla \cdot (u \nabla K * u) = 0,$$

short-time well-posedness and long-time explosion if the kernel is attractive.

Bertozzi, Laurent, Rosado; Carrillo, James, Lagoutière, Vauchelet; Carrillo, Di Francesco, Figalli, Laurent, Slepčev...

Explosion in the radial attractive case: the quantity

$$D(u(t)) := u(0,t) \text{ if } \quad N = 1, \quad \int_{\mathbb{R}^n} \frac{u(x,t)}{|x|}\, dx \text{ if } \quad N \geq 2$$

explodes in finite time (Biler-Karch-Laurençot '09).

Small-scale behaviour

[Biler-B.-Karch-Laurençot 1] Assume that u_0 is radially symmetric, concentrated near 0 and K is attractive near 0. Then the solution u of (ADE) satisfies

$$\int_0^{T_*} \int_{B(\lambda_* \varepsilon)} u(x,t)\, dx\, dt \geq C_* \;\Rightarrow \text{(Hölder)}$$

$$\int_0^{T_*} \left(\int_{B(\lambda_* \varepsilon)} u(x,t)^p\, dx \right)^{1/p} \geq C(p)\varepsilon^{-N(1-\frac{1}{p})},\ 1 \leq p < \infty,$$

for all $\varepsilon \in (0, \varepsilon_*)$. The constants with the $*$ only depend on u_0, K through a finite number of parameters.

These L^p estimates are sharp; the corresponding upper estimates hold on the whole space $\mathbb{R}^n$ and without time averaging.

Proof of small-scale concentration

The upper estimates hold under very general conditions: radial symmetry is not needed. We use an energy method.

To prove lower estimates, we consider again the quantity:

$$D(u(t)) := u(0,t) \text{ if } \quad N = 1, \quad \int_{\mathbb{R}^n} \frac{u(x,t)}{|x|}\, dx \text{ if } \quad N \geq 2$$

If $\varepsilon > 0$, no explosion. However, integrating by parts and using a symmetrization trick we obtain that for some $T_* > 0$:

$$\int_0^{T_*} D(u(t)) \geq C\varepsilon^{-1}.$$

Combining this lower estimate with the upper ones in Lebesgue spaces (and in H^1 if $N = 1$) and using Hölder's inequality, we obtain the lower bounds for the mass over the small ball.

Sobolev norms

In [Biler-B.-Karch-Laurençot 2], we obtain ε-optimal Sobolev norms for u localised on a small ball as above.

Lower estimates: They follow from lower estimates for L_p norms and the GN (Gagliardo-Nirenberg) inequality.
For example, since (after averaging in time) we have (by conservation of the mass M):

$$C\varepsilon^{-N/2} \leq |u|_2 \leq CM^{2/(N+2)}|u|_{1,2}^{N/(N+2)},$$

we obtain that

$$|u|_{1,2} \geq CM^{-2/N}\varepsilon^{-(N/2+1)}.$$

Upper estimates: Energy method. Inequalities of Hölder, GN and HLS (Hardy-Littlewood-Sobolev), taking derivatives of K (convolution with $|x|^{-k}$ for $k < N$). As before, hold on the whole space and without time-averaging.

Analogy between Burgers and (ADE)

Formally, if v solves Burgers, $-v_x$ satisfies (ADE) with $K(x) = -|x|$.
See Bertozzi-Laurent; Bonaschi-Carillo-Di Francesco-Peletier.
However:

1. Periodic setting so $-v_x = u \geq 0$ is impossible.
2. This is a purely 1D analogy.

However, this suggests $|u|_p \sim \varepsilon^{-(1-1/p)}$.

And this is indeed true only in 1D. In general, for (ADE), $|u|_p \sim \varepsilon^{-N(1-1/p)}$ (no N for Burgers: cf. [B' 16]).

So dependence on N for (ADE). The explanation is that when $\varepsilon \to 0$, for Burgers (resp. (ADE)) we concentrate to a singularity of codimension 1 (resp. codimension N).

Concluding Remarks

Our results give precise and rigorously proved small-scale estimates for a broad class of (deterministic and random) models.

Until recently, such results were only available for Burgers-type equations, relying heavily on versions of Oleinik's estimate $v_x \leq t^{-1}$.

Many perspectives on aggregation-diffusion equations (which do not have inviscid upper estimates like Oleinik's, but have obvious inviscid lower ones since solutions are positive: remember formally $u = -v_x$!)

Bibliography

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