Principal eigenvalues of nonlocal diffusion operators with advection

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Main problem

Eigenvalue of nonlocal diffusion operators with advection

$$(a\varphi)'(x) + \tilde{r}(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)(\varphi(y) - \phi(x))dy = \lambda\varphi(x),$$

$\varphi \in L^1(\mathbb{S})$ ($\mathbb{S}$: Circle with length 1)

- $\int_{\mathbb{S}} K(x,y)(\varphi(y) - \varphi(x))dy$: Nonlocal diffusion
  e.g. $K(x,y) = k(x-y)$
- $(a\varphi)'(x)$: Advection (drift)
- $\tilde{r}(x)\varphi(x)$: Growth

Main problem

$$(a\varphi)'(x) + \tilde{r}(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)(\varphi(y) - \phi(x))dy = \lambda\varphi(x),$$

can be rewritten as

$$a(x)\varphi'(x) + r(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)\varphi(y)dy = \lambda\varphi(x),$$

with $r(x) = \tilde{r}(x) + a'(x) + \int_{\mathbb{S}} K(x,y)dy$ simply.

- $B\varphi := \int_{\mathbb{S}} K(x,y)\varphi(y)dy$
- $A\varphi := a(x)\varphi'(x) + r(x)\varphi(x)$

Main problem

$$(A+B)\varphi := a(x)\varphi'(x) + r(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)\varphi(y)dy$$

Some assumptions

- $K : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$ continuous and positive
- $a : \mathbb{S} \to \mathbb{R}$ and $r : \mathbb{S} \to \mathbb{R}$, of class C^1 and C^0 respectively.

Similar to random diffusion operator (Laplacian or general 2-order elliptic operator)

Principal eigenvalue probelm

Main problem

$$(A+B)\varphi := a(x)\varphi'(x) + r(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)\varphi(y)dy$$

Principal eigenpair (eigenvalue and eigenfunction)

- principal eigenvalue: "the largest eigenvalue"
- More precisely, consider a operator L
  - $\sigma(L)$: spectrum set
  - $\rho(L)$: resolvent set
  - the spectral bound of L:

$$s(L) = \sup\{\mathrm{Re}\,\lambda,\ \lambda \in \sigma(L)\}.$$

  - $s(L)$ is called the principal eigenvalue if $s(L)$ is an eigenvalue with algebraic multiplicity one

Rk: In many cases (elliptic operator, diffusion operator, nonnegative matrix...), because of the maximum principle, the respective eigenfunction of $s(L)$ is a positive one.

Main problem

$$(A+B)\varphi := a(x)\varphi'(x) + r(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)\varphi(y)dy$$

Main Question

- Is the spectral bound $s(A+B)$ the principal eigenvalue in $L^1(\mathbb{S})(C^0(\mathbb{S}))$ with a positive eigenfunction?

Main problem

$$(A+B)\varphi := a(x)\varphi'(x) + r(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)\varphi(y)dy$$

Rk. An equivalent operator in periodic media in $\mathbb{R}$:

$$L\phi := \bar{a}(x)\phi'(x) + \bar{r}(x)\phi(x) + \int_{\mathbb{R}} \bar{K}(x,y)\phi(y)dy$$

$\bar{a}, \bar{r}, \bar{K}(\cdot, y)$: 1-periodic in $\mathbb{R}$
$\phi \in L^1_{local}(\mathbb{R})$ and 1-periodic

Background

Heat equation, parabolic equation and diffusion in biology

$$\begin{cases} u_t = \nabla(A(x)\nabla u) + a(x)\cdot\nabla u + r(x)u, \ \ x \in \Omega \\ Homogeneous \ \ boundary \ \ condition \ \ on \ \partial\Omega \end{cases}$$

- $\boxed{\text{random diffusion}} + \boxed{\text{advection}} + \boxed{\text{local growth}}$
- Homogeneous boundary condition:
  Dirichlet, Neumann, Robin or periodic

Generator of the semigroup of above linear parabolic equation

$$\begin{cases} Lu := \nabla(A(x)\nabla u) + a(x)\cdot\nabla u + r(x)u, \ \ x \in \Omega \\ Homogeneous \ \ boundary \ \ condition \ \ on \ \partial\Omega \end{cases}$$

Background

$$\begin{cases} Lu := \nabla(A(x)\nabla u) + a(x)\cdot\nabla u + r(x)u, \ x \in \Omega \\ \textit{Homogeneous boundary condition on } \partial\Omega \end{cases}$$

With some assumptions on the smoothness and boundedness of A, a, r, Ω,
the existence of the principal eigenvalue λ of L is a standard result.

- $u(x,t)$: density of some species,
- the principal eigenvalue λ
  - averaged growth rate
    (it may be larger than $\bar{r}$, the averaged value of r)
  - to describe the basic reproduction number R_0
  - to describe the propagation speed of one invaded species

One obsevion

Let $T(t)$ be the solution semigroup of

$$\begin{cases} u_t = \nabla(A(x)\nabla u) + a(x)\cdot\nabla u + r(x)u, \ x\in\Omega \\ Homogeneous\ boundary\ condition\ on\ \partial\Omega \end{cases}$$

- $T(t)$: compact (smoothness of the solution), strongly positive (maximum principle, order-preserving) for fixed $t > 0$
- The spectral radius of $T(t)$, $\rho(T(t))$ is the principal eigenvalue of $T(t)$: Krein-Rutmann theorem.
- The spectral bound $s(L)$ of L is the principal eigenvalue of

$$\begin{cases} Lu := \nabla(A(x)\nabla u) + a(x)\cdot\nabla u + r(x)u, \ x\in\Omega \\ Homogeneous\ boundary\ condition\ on\ \partial\Omega \end{cases}$$

$s(L) = \frac{\ln\rho(T(t)))}{t}, t > 0$

One obsevion

Let $T(t)$ be the solution semigroup of

$$\begin{cases} u_t = \nabla(A(x)\nabla u) + a(x)\cdot\nabla u + r(x)u, \ x\in\Omega \\ Homogeneous\ boundary\ condition\ on\ \partial\Omega \end{cases}$$

- The advection $a(x)\cdot\nabla u$ and growth $r(x)u$ may yield some singularity of $T(t)$
- The random diffusion $\nabla(A(x)\nabla u)$ can eliminate such singularity, and yield smoothness and compactness...

Known result about Nonlocal diffusion

To consider the principal eigenvalues of

$$(A+B)\varphi := a(x)\varphi'(x) + r(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)\varphi(y)dy$$

we transfer it to the principal eigenvalues of the solution semigroup $T_{A+B}(t)$ of

$$u_t = a(x)u_x + r(x)u + \int_{\mathbb{S}} K(x,y)u(t,y)dy$$

Rk. Comparing with $A+B$, $T_{A+B}(t)$ has better properties. Moreover, we can explain the concepts, results and methods clearer in natural language.

Known result about Nonlocal diffusion

$$(A+B)\varphi := a(x)\varphi'(x) + r(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)\varphi(y)dy$$

- $a \equiv 0$, non-advection; $x \in \mathbb{S}$
  W. Shen and A. Zhang; J.Coville...

Theorem (See W.Shen and A.Zhang)

Suppose that r is $\boxed{\textit{local Lip-continuous}}$ *near its maximum points. Then the spectral bound of $r(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)\varphi(y)dy$ is an eigenvalue with algebraic multiplicity one. Moreover, the respective eigenfunction is positive.*

- $a > 0$ but $x \in [0,1]$ not $\mathbb{S}$,
  J.Coville, F.Li and X.Wang

Nonlocal diffusion with positive advection

$$(A + B)\varphi := a(x)\varphi'(x) + r(x)\varphi(x) + \int_{\mathbb{S}} K(x, y)\varphi(y)dy$$

- $a > 0$; $x \in \mathbb{S}$

Theorem (A.Ducrot, G. Griette, X.L.)

Suppose that $a > 0$. Then the spectral bound of $A + B$ is an eigenvalue with algebraic multiplicity one. Moreover, the respective eigenfunction is positive.

Rk. Comparing with the result in the case without advection (W.Shen and A.Zhang), here the effect of advection yields we do not need thorough assumption on r.

Nonlocal diffusion with positive advection

Before prove this theorem, we explain how the growth part $\boxed{r(x)u}$ generates singularity and how the advection can help to eliminate such singularity.

- $L_1 u := r(x)u$. $\sigma(L_1) = \{r(x_0), x_0 \in \mathbb{R}\}$. $s(L_1) = \max_{x \in \mathbb{S}} r(x)$ is not an isolate spectral points
- Consider the evolution equation $u_t = r(x)u$. For example, let $r(x_0) = 0$ and $r(x) < 0$ for $x \neq x_0$. $u(+\infty, x_0) = u(0, x_0)$ and $u(+\infty, x) = 0, x \neq x_0$. The spatial heterogeneity of the growth generates singularity finally.

Proof

- $\partial_t \pi(t,s;x) = -a(\pi(t,s;x)), \;\; \pi(s,s;x) = x, x \in \mathbb{S}$;
  Specially, let $t_0 = \int_{\mathbb{S}} \frac{1}{a(x)} dx > 0$.
  Then $\pi(t+t_0, t; x) = x, x \in \mathbb{S}$
- $\int_0^{t_0} r(\pi(s,t;y))ds = \int_{\mathbb{S}} r(\pi(s(x),t;y)(-1/a(x))dx =: \bar{r}$
  $\bar{r}$: independent y
- Solve

$$\begin{cases} \partial_t u(t,x) = Au := a(x)\partial_x u(t,x) + r(x)u(t,x), x \in \mathbb{S} \\ u(0,.) = \varphi. \end{cases}$$

$$T_A(t)\varphi(x) := u(t,x) = \varphi(\pi(0,t;x)) \exp\left[\int_0^t r(\pi(s,t;x)ds\right]$$

- Fix $t = t_0$, $T_A(t_0)\varphi(x) = \varphi(x)e^{\bar{r}}$, that is, $T_A(t_0) = e^{\bar{r}} I$

Proof: continued

- $$\begin{cases} \partial_t u(t,x) = (A+B)u, x \in \mathbb{S} \\ Au = a(x)\partial_x u(t,x) + r(x)u(t,x) \\ Bu = \int_{\mathbb{S}} K(x,y)u(t,y)dy \\ u(0,.) = \varphi. \end{cases}$$
- The solution semigroup $T_{A+B}(t)\varphi(x) = u(t,x)$
- Use variation of constant. At $t = t_0$ $[T_{A+B}(t_0)\varphi](x) = [T_A(t_0)\varphi](x) + \int_0^{t_0} T_A(s)(B(u(t_0 - s, \cdot)))ds$ $=: e^{\bar{r}}\phi(x) + [L_2\phi](x)$

Proof: continued

- For any $t \in \mathbb{R}$, $T_A(t)$:
  strictly positive bounded linear operator on $C(\mathbb{S})$
- The integral operator B:
  strongly positive compact linear operator on $C(\mathbb{S})$
  (Ascoli-Arzel theorem)
- $L_2\varphi := \int_0^{t_0} T_A(s)(B(u(t_0 - s, \cdot)))ds$: strongly positive compact
- Krein-Rutmann theorem can be applied for L_2,
  the spectral radius λ of L_2 is its principal eigenvalue with a respective positive eigenfunction φ_0
- $e^{\bar{r}} + \lambda$, the spectral radius of $T_{A+B}(t_0) = e^{\bar{r}} I + L_2$ is its principal eigenvalue with the respective positive eigenfunction φ_0.
- $\mu := \frac{\ln(e^{\bar{r}} + \lambda)}{t_0}$, the spectral bound of $A + B$ is its principal eigenvalue with the respective positive eigenfunction φ_0.

Remark

Rk. 1. For mass conservation, a evolution equation on advection should be

$$u_t = (a(x)u)_x, \ x \in \mathbb{S}$$

2. Our theorem may be proved by the "touch lemma" developed in the work of J.Coville, F.Li and X.Wang

Nonlocal diffusion with sign-change advection

A more interesting but difficult case is $\boxed{a \text{ change sign}}$.
For some species with density $u(t, x)$, $u_t = a(x)u_x$.

- Let $a(x_0) = 0$, $a(x) < 0$ for $x < x_0$ and $a(x) > 0$ for $x > x_0$.
- the species gather to the sink x_0 in long term. Dirac's δ function may appear at x_0

On the other hand,

- Let $a(x_1) = 0$, $a(x) > 0$ for $x < x_1$ and $a(x) < 0$ for $x > x_1$.
- the species leave from the source x_1. To support such motion in long term, the density of the species at x_1 should be large enough,Dirac's δ function may appear at x_1

Nonlocal diffusion with sign-change advection

$$(A+B)\varphi := a(x)\varphi'(x) + r(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)\varphi(y)dy$$

or rewrite A as

$$A\varphi = (a\varphi)' + \lambda(x)\varphi$$

with $\lambda = r - a'$

- Assumption1. There exists $x_0 < x_1$ such that

  $$a(x_0) = a(x_1) = 0, a(x) \neq 0, x \neq x_0, x_1$$

  and

  $$a'(x_0) > 0, \quad a'(x_1) < 0.$$

- Assumption2.

  $$\lambda(x_0) < \lambda(x_1)$$

Nonlocal diffusion with sign-change advection

We first give a rough description of our results.

Theorem (A.Ducrot, G. Griette, X.L.)

$$(A + sB)\varphi := (a\varphi)' + \lambda(x)\varphi + \int_{\mathbb{S}} sK(x, y)\varphi(y)dy$$

Under Assumptions 1,2, we have the following conclusion:
There is some parameter $s_0 > 0$, *a function* $\rho(s)$ *and some* $\varphi_s \in L^1(\mathbb{S})$ *positive almost everywhere for any* $s > 0$, *such that*

- $\rho(s) \equiv \lambda(x_1)$ *for* $s \leq s_0$, $\rho(s) > \lambda(x_1)$ *for* $s \leq s_0$
- $(A + sB)\varphi = \rho(s)\varphi$ *for* $s \geq s_0$
- $(A + sB)(\varphi + \delta_{x_1}(x)) = \rho(s)(\varphi + \delta_{x_1}(x)) = \lambda(x_1)(\varphi + \delta_{x_1}(x))$ *in the weak sense for* $s < s_0$.

Rk. If the diffusion is strong enough, we have a real principal eigenvalue, otherwise, we only have a weak one.

Some useful conclusion and techniques

- Theorem (Krein-Rutmann theorem for noncompact positive operator and semigroup)

  Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{T_L(t)\}_{t \geq 0}$ on the Banach lattice X such that T_L is positive and irreducible. Assume furthermore that $\rho_{ess}(T_L(t)) < \rho(T_L(t))$ for $t > 0$ then there exists $y \in X^+$, strictly positive such that

  $$Ly = s(L)y.$$

  $\rho_{ess}(T_L(t))$: essential spectral radius,
  $\rho(T_L(t))$: spectral radius,
  $s(L)$: spectral bound, $s(L) = \frac{\ln \rho(T_L(t))}{t}$

- Theorem

  Let$O = O_1 + O_2$,where O_1, O_2 are linear bounded operators on some Banach space. Moreover, suppose that O_2 is compact, then $\rho_{ess}(O) = \rho_{ess}(O_1)$

  ,

Some useful conclusion and techniques

Techniques

$$(A+B)\varphi := a(x)\varphi'(x) + r(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)\varphi(y)dy$$

- Use variation of constant.
  $T_{A+B}(t)\varphi(x) = \boxed{T_A(t)\varphi(x)} + \boxed{\int_0^t T_A(s)(B(u(t-s,\cdot)))ds}$
- Since the second part $\int_0^t T_A(s)(B(u(t-s,\cdot)))ds$ is compact, $\rho_{ess}(T_{A+B}(t)) = \rho_{ess}(T_A(t)) \leq \rho(T_A(t))$
- Then after estimating $\rho(T_A(t))$ and $\rho(T_{A+B}(t))$, we can find some conditions on $\rho_{ess}(T_{A+B}(t)) < \rho(T_{A+B}(t))$
- $\boxed{\text{the abstract theorem}}$ +
  $\boxed{\text{the condition on} \rho_{ess}(T_{A+B}(t)) < \rho(T_{A+B}(t))}$
  $\Rightarrow s(A+B)$ is the principal eigenvalue.

Some useful conclusion and techniques

Techniques

$$(A+B)\varphi := a(x)\varphi'(x) + r(x)\varphi(x) + \int_{\mathbb{S}} K(x,y)\varphi(y)dy$$

- Define $(A-\lambda(x_1))^{-1}$. In other words, solve

$$(a\varphi)' + (\lambda(x) - \lambda(x_1))\varphi(x) = f$$

- Difficulty: $a(x_0), a(x_1) = 0$, and the solution are not unique. We need to choose a suitable one.

Some useful conclusion and techniques

Lemma (Special strongly ergodic theorem)

For all continuous function Φ on $\mathbb{S}$, one has

$$\limsup_{t\to\infty}\sup_{x\in\mathbb{S}}\frac{1}{t}\int_0^t \Phi(\pi(\ell,0;x))d\ell = \max\{\Phi(x_0),\Phi(x_1)\}.$$

where $\pi(\ell,0;x)$ is the solution of $dx/dt = -a(x)$ with the initial value x.

Rk. This useful lemma can be considered as one on the ergodic property of the solution of $dx/dt = a(x)$.

Lemma

$\rho(T_A(t)) = e^{\lambda(x_1)t}$.
Then
$\rho_{ess}(T_{A+B}(t)) = \rho_{ess}(T_A(t)) \le \rho(T_A(t)) = e^{\lambda(x_1)t}$

Some new problem

- finitely many zero-points of $a(x)$, all zero-points are non-degenerate (trivial extension)
- degenerate zero-points of $a(x)$ (on-going)
- high-dimensional problem
- fractional diffusion ($(-\Delta)^s$)

Thank you for your attention!