On improved Trudinger-Moser type inequalities involving the Leray potential

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Trudinger-Moser type inequality

The Trudinger-Moser inequality

$$\sup_{\int_\Omega |\nabla u|^2 dx \le 1,\ u \in C_c^\infty(\Omega)} \int_\Omega e^{4\pi u^2} dx < \infty, \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$, is an analogue of the following limiting Sobolev inequality in dimensions $N \ge 3$:

$$\sup_{\int_{\mathbb{R}^N} |\nabla u|^2 dx \le 1,\ u \in C_c^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u|^{2^*} dx < \infty,\ 2^* = \frac{2N}{N-2}.$$

Trudinger-Moser type inequality

Some extensions of (2.1), where $\int_\Omega |\nabla u|^2 dx \le 1$ is replaced by

$$\int_\Omega \left[|\nabla u|^2 - V(x)u^2\right] dx \le 1$$

with a suitable potential function $V(x)$.

- $V = V_1 := (1 - |x|^2)^{-2}$, Wang-Ye, (Adv. Math., 2012)
- $V = V_2 := \frac{V_{\mathrm{Leray}}(|x|)}{\max\{\sqrt{-\ln|x|},1\}}$, Tintarev, (JFA, 2014)
- Psaradakis-Spector, JFA, 2015.

where $V_{\mathrm{Leray}} := \frac{1}{4|x|^2(\ln\frac{1}{|x|})^2}$. Remark that

$$\lim_{r\to 1^-} V_1(r)/V_{\mathrm{Leray}}(r) = 1$$

$$\lim_{|x|\to 1^-} V_2(|x|)/V_{\mathrm{Leray}}(|x|) = 1, \quad \lim_{|x|\to 0^+} V_2(|x|)/V_{\mathrm{Leray}}(|x|) = 0.$$

Trudinger-Moser type inequality

In a different direction, Adimurthi and Druet (CPDE, 2004) proved the following result: For any bounded domain $\Omega \subset \mathbb{R}^2$,

$$\sup_{\int_\Omega |\nabla u|^2 dx \leq 1,\ u \in C_c^\infty(\Omega)} \int_\Omega e^{4\pi u^2 (1+\alpha \|u\|_2)} dx \begin{cases} < \infty \text{ if } \alpha \in [0, \lambda_1(\Omega)), \\ = \infty \text{ if } \alpha \geq \lambda_1(\Omega), \end{cases}$$

where $\lambda_1(\Omega)$ stands for the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and $\|u\|_2 = (\int_\Omega u^2 dx)^{1/2}$.

The Hardy-Leray operators

Assume $0 \in \Omega \subset \mathbb{R}^N$ $(N \geq 2)$, Ω is a bounded C^2-domain. The classical Hardy-Leray operators are defined by

$$\mathcal{L}_\mu := -\Delta + \frac{\mu}{|x|^2}, \quad \text{for} \quad \mu \geq \mu_0 := -\frac{(N-2)^2}{4}, \; N \geq 3$$

and

$$\mathcal{L}_\mu := -\Delta + \frac{\mu}{|x|^2(\ln|x|)^2}, \quad \text{for} \quad \mu \geq \mu_0 := -\frac{1}{4}, \; N = 2.$$

Hardy-Leray's inequality and its various improvements have been used in many contexts: The stability of solutions of elliptic and parabolic equations with singular potentials; The foundation of a large part harmonic analysis of singular integral operators such as the Hilbert transform or pseudo-differential operators.

[Ruzhansky-Suragan], Hardy Inequalities on homogeneous groups: 100 Years of Hardy Inequalities, Birkhäuser 2019.

Equations with the Hardy-Leray potentials

We study the nonhomogeneous linear problem

$$\begin{cases} \mathcal{L}_\mu u = f & \text{in} \quad \Omega \setminus \{0\}, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.2}$$

where $0 \in \Omega \subset \mathbb{R}^N$ $(N \geq 3)$, $\mu \geq \mu_0 := -\frac{(N-2)^2}{4}$ and $\mathcal{L}_\mu := -\Delta + \frac{\mu}{|x|^2}$. A complete picture of the existence and/or non-existence, classification of singularities.

- Connection with the weak solution of

$$\begin{cases} \mathcal{L}_\mu u = f + c_\mu k \delta_0 & \text{in} \quad \Omega \setminus \{0\}, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.3}$$

where $k \in \mathbb{R}$, in the $d\mu$-distribution sense, that is $u \in L^1(\Omega, d\mu)$, $d\mu(x) := \Gamma_\mu(x)dx$, $\mathcal{L}_\mu^* := -\Delta - \frac{2\tau_+(\mu)}{|x|^2} x \cdot \nabla$.

$$\int_\Omega u_k \mathcal{L}_\mu^*(\xi)\, d\mu = \int_\Omega f\xi \, d\mu + c_\mu k \xi(0), \quad \forall \xi \in C_0^{1.1}(\Omega). \tag{3.4}$$

Functions space and some embedding results

- We denote by $\mathcal{H}^1_{\mu,0}(B_1)$, for $\mu \geq -\frac{1}{4}$, the completion of $C_c^\infty(B_1)$ under the norm

$$\|u\|_\mu = \sqrt{\int_{B_1} \left(|\nabla u|^2 dx + \mu \frac{u^2}{|x|^2(-\ln|x|)^2}\right) dx},$$

and so $\mathcal{H}^1_{\mu,0}(B_1)$ is a Hilbert space with inner product

$$\langle u, v\rangle_\mu = \int_{B_1} \left(\nabla u \cdot \nabla v\, dx + \mu \frac{uv}{|x|^2(-\ln|x|)^2}\right) dx.$$

Set

$$\mathcal{H}^1_0(B_1) = \mathcal{H}^1_{0,0}(B_1) \quad \text{and} \quad \hat{\mathcal{H}}^1_0(B_1) = \mathcal{H}^1_{-\frac{1}{4},0}(B_1).$$

- Denote $\mathcal{H}^1_0(B_1) = W^{1,2}_0(B_1)$. Then

$$\mathcal{H}^1_{\mu,0}(B_1) = \mathcal{H}^1_0(B_1) \text{ for } \mu > -\frac{1}{4}, \text{ but } \mathcal{H}^1_0(B_1) \subsetneqq \hat{\mathcal{H}}^1_0(B_1). \tag{3.5}$$

Some functions spaces

Let $\Omega \subset B_1$ be a bounded domain containing the origin, $\mu \geq -\frac{1}{4}$, and $V : (0,1) \to [0,\infty)$ a continuous function such that

$$\mu V(r) \geq -V_{\mathrm{Leray}}(r) = -\frac{1}{4r^2(\ln \frac{1}{r})^2} \text{ for } r \in (0,1).$$

We denote by $\mathcal{H}^1_{V,\mu,0}(\Omega)$ the completion of $C_c^\infty(\Omega \setminus \{0\})$ under the norm

$$\|u\|_{V,\mu} = \sqrt{\int_\Omega \left(|\nabla u|^2 dx + \mu V u^2\right) dx},$$

which is a Hilbert space with inner product

$$\langle u, v\rangle_{V,\mu} = \int_\Omega \left(\nabla u \cdot \nabla v \, dx + \mu V u v\right) dx.$$

For $\mu \geq -\frac{1}{4}$, we denote

$$m_\mu := 4\pi\sqrt{1+4\mu}.$$

Theorem 1.6. *Assume that $\mu > 0$ and $V : (0,1) \to [0,+\infty)$ is a continuous function satisfying*

$$V(r) \geq \frac{1}{r^2(-\ln r)^2} \quad \text{in } (0,1). \tag{1.13}$$

Then the following conclusions hold:

(i) *For* **radially symmetric** *functions in $\mathcal{H}^1_{V,\mu,0}(B_1)$ we have*

$$\sup_{u \,\text{is radial}, \|u\|_{V,\mu} \leq 1} \int_{B_1} e^{m_\mu |u|^2} dx < \infty,$$

and this result is optimal: If $\alpha > m_\mu$ and

$$\lim_{r \to 0^+} V(r) r^2 (-\ln r)^2 = 1, \tag{1.14}$$

then there exists a sequence of radially symmetric functions which concentrate at the origin such that $\|u_n\|_{V,\mu} \leq 1$ and

$$\int_{B_1} e^{\alpha |u_n|^2} dx \to \infty \quad \text{as } n \to +\infty.$$

(ii) *For general functions in $\mathcal{H}^1_{V,\mu,0}(B_1)$ we have*

$$\sup_{\|u\|_{V,\mu} \leq 1} \int_{B_1} e^{4\pi |u|^2} dx < \infty,$$

Main results

and this result is optimal: If $\alpha > 4\pi$ *and* (1.14) *holds, then there exists a sequence of functions concentrating at some point away from the origin, such that* $\|u_n\|_{V,\mu} \le 1$ *and*

$$\int_{B_1} e^{\alpha|u_n|^2} dx \to +\infty \quad \text{as } n \to +\infty.$$

Here a sequence $\{u_n\}$ is said to be concentrating at some point x_0, if for any $r \in (0,1)$ and any $\epsilon > 0$ there exists $n_0 > 0$ such that

$$\int_{B_1 \setminus B_r(x_0)} \left(|\nabla u_n|^2 dx + \mu V u_n^2\right) dx < \epsilon.$$

Next we consider the case $\mu \in (-\frac{1}{4}, 0)$.

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Theorem 1.7. *Suppose* $\mu \in (-\frac{1}{4}, 0)$ *and* $V : (0,1) \to [0, +\infty)$ *is continuous and verifies*

$$V(r) \leq \frac{1}{r^2(-\ln r)^2}. \tag{1.15}$$

Then the following conclusions hold:

(i) *For* **radially symmetric** *functions,*

$$\sup_{u \text{ is radial}, \|u\|_{V,\mu} \leq 1} \int_{B_1} e^{m_\mu |u|^2} dx < \infty,$$

and this result is optimal: If $\alpha > m_\mu$ *and* (1.14) *holds, then there exists a sequence of radially symmetric functions which concentrate at the origin such that* $\|u_n\|_{V,\mu} \leq 1$ *and*

$$\int_{B_1} e^{\alpha |u_n|^2} dx \to \infty \quad \text{as } n \to +\infty.$$

(ii) *For general functions, if* V *is* **decreasing** *in* $(0, 1)$ *and verifies* (1.15), *then*

$$\sup_{\|u\|_{V,\mu} \leq 1} \int_{B_1} e^{m_\mu |u|^2} dx < \infty.$$

(iii) *The result in* (ii) *is optimal: If* (1.14) *holds, then for any* $\alpha > m_\mu$, *there exists a sequence* $\{u_n\}_n$ *concentrating at the origin such that* $\|u_n\|_{V,\mu} \leq 1$ *and*

$$\int_{B_1} e^{\alpha |u_n|^2} dx \to \infty \quad \text{as } n \to \infty.$$

Main results

Let us note that $V(r) := \frac{1}{r^2(1-\ln r)^2}$ is decreasing in $(0,1]$ and satisfies both (1.14) and (1.15).

Corollary 1

Let $\mu \in (-\frac{1}{4}, 0)$, $r_0 = 1/e$ and $V(r) = \frac{1}{r^2(\ln \frac{1}{r})^2}$. Then

$$\sup_{\|u\|_{\mathcal{H}^1_{V,\mu,0}(B_{r_0})} \le 1} \int_{B_{r_0}} e^{m_\mu u^2} dx < \infty,$$

and the exponent m_μ is optimal.

Finally we consider the critical case $\mu = -\frac{1}{4}$.

Theorem 1.9. *Suppose that* $\mu = -\frac{1}{4}$ *and* $V \in C((0,1))$ *is nonnegative and verifies* (1.15). *Then the following conclusions hold:*

(i) *For* **radially symmetric** *functions and* $p \in (0,1)$, $\alpha > 0$,

$$\sup_{u \text{ is radial}, \ \|u\|_{V,-1/4} \le 1} \int_{B_1} e^{\alpha |u|^p} dx < \infty.$$

(ii) *For general functions, if* V *is* **decreasing** *in* $(0,1)$, *then for* $p \in (0,1)$ *and* $\alpha > 0$,

$$\sup_{\|u\|_{V,-1/4} \le 1} \int_{B_1} e^{\alpha |u|^p} dx < \infty.$$

(iii) *If there exist* $\theta > 0$ *and* $C > 0$ *such that*

$$|V(r) r^2 (-\ln r)^2 - 1| \le C(-\ln r)^{-\theta} \quad \text{for } r \in (0, \frac{1}{4}), \tag{1.16}$$

then there exists a sequence $\{u_n\} \subset \mathcal{H}^1_{V,-1/4,0}(B_1)$ *such that* $\|u_n\|_{\mathcal{H}^1_{V,-1/4}(B_1)} = 1$ *and for any* $p \ge 1$ *and any* $\alpha > 0$,

$$\int_{B_1} e^{\alpha |u_n|^p} dx \to \infty \quad \text{as } n \to \infty.$$

Trudinger-Moser type inequalities for radial functions

Define

$$\mathcal{H}^1_{\mathrm{rad},\mu,0}(B_1) := \left\{w \in \mathcal{H}^1_{\mu,0}(B_1) : \ w \text{ is radially symmetric}\right\}.$$

We will prove the following two theorems in this section.

Theorem 2

Let $\mu > -\frac{1}{4}$. Then

$$\sup_{u\in\mathcal{H}^1_{\mathrm{rad},\mu,0}(B_1),\ \|u\|_\mu\le 1} \int_{B_1} e^{4\pi\sqrt{1+4\mu}\ u^2} dx < \infty,$$

and the result fails when $4\pi\sqrt{1+4\mu}$ is replaced by any $\alpha > 4\pi\sqrt{1+4\mu}$.

Trudinger-Moser type inequalities for radial functions

Note that $4\pi\sqrt{1+4\mu} = m_\mu \to 0$ as $\mu \to -\frac{1}{4}$, which suggests that the inequality should be different for $\mu = -\frac{1}{4}$.

Theorem 3

(i) For any $p \in (0,1)$ and any $\alpha > 0$, there exists $c = c_{p,\alpha}$ depending on p and α such that for every $u \in \mathcal{H}^1_{\mathrm{rad},-1/4,0}(B_1)$ with $\|u\|_{-1/4} \le 1$, there holds

$$\int_{B_1} e^{\alpha |u|^p} dx \le c_{p,\alpha}.$$

(ii) For any $p \ge 1$ and any $\alpha > 0$, there exists a sequence $\{u_n\}$ such that $\|u_n\|_{-1/4} \le 1$,

$$\int_{B_1} e^{\alpha |u_n|^p} dx \to +\infty \quad \text{as} \quad n \to +\infty.$$

Leray inequality

The well-known Leray inequality states:

$$\int_{B_1} |\nabla w|^2 dx - \frac{1}{4}\int_{B_1} \frac{w^2}{|x|^2(\ln \frac{1}{|x|})^2} dx > 0, \quad \forall\, w \in C_c^\infty(B_1),\ w \not\equiv 0, \tag{5.6}$$

where B_1 is the unit ball in $\mathbb{R}^2$

- $\frac{1}{|x|^2(-\ln|x|)^2}$ has a weaker singularity at 0 than $1/|x|^2$, has a singularity of order $(1-|x|)^{-2}$ at the boundary ∂B_1.

Leray inequality with a remainder term

- Barbatis, Filippas and Tertikas proved an improved version with a remainder term:

$$\int_{B_1} |\nabla w|^2 dx - \frac{1}{4}\int_{B_1} \frac{w^2}{|x|^2(\ln\frac{e}{|x|})^2} dx \geq \frac{1}{4}\sum_{i=2}^{\infty}\int_{B_1}\frac{|w|^2}{|x|^2}\prod_{j=1}^{i} X_j^2(|x|)dx, \quad (5.7)$$

$\forall\, w \in C_c^\infty(B_1)$, where

$$X_1(r) = (\ln\tfrac{e}{r})^{-1}, \; X_k(r) = X_1(X_{k-1}(r)) \text{ for } k = 2, \ldots. \quad (5.8)$$

Theorem 1.1. *The following inequalities hold:*

(i) (Leray's inequality with a remainder term) *There exists* $\mu_2 > 0$ *such that for any* $u \in C_c^1(B_1)$

$$\int_{B_1} \Big(|\nabla u|^2 - \frac{1}{4} \frac{|u|^2}{|x|^2 (\ln \frac{1}{|x|})^2} \Big) dx \geq \mu_2 \int_{B_1} \frac{|u|^2}{|x|^2 (\ln \frac{1}{|x|})^2 \big(1 + |\ln \ln \frac{1}{|x|}|\big)^2} dx. \tag{1.6}$$

(ii) (Leray's inequality with a remainder term for radial functions) *For any* $q > 2$*, there exists* $\mu_q > 0$ *such that for every* $u \in C^1_{\mathrm{rad},c}(B_1)$,

$$\int_{B_1} \Big(|\nabla u|^2 - \frac{1}{4} \frac{|u|^2}{|x|^2 (\ln \frac{1}{|x|})^2} \Big) dx \geq \mu_q \Big(\int_{B_1} \frac{|u|^q}{|x|^2 \Big[(\ln \frac{1}{|x|}) \big(1 + |\ln \ln \frac{1}{|x|}|\big) \Big]^{1+\frac{q}{2}}} dx \Big)^{\frac{2}{q}}. \tag{1.7}$$

(iii) (Leray's inequality with a remainder term and singularity at 0 only) *For any* $q > 2$ *and* $r_0 = e^{-1}$*, there exists* $\mu_q > 0$ *such that for every* $u \in C_c^1(B_{r_0})$,

$$\int_{B_{r_0}} \Big(|\nabla u|^2 - \frac{1}{4} \frac{|u|^2}{|x|^2 (\ln \frac{1}{|x|})^2} \Big) dx \geq \mu_q \Big(\int_{B_{r_0}} \frac{|u|^q}{|x|^2 \Big[\big(\ln \frac{1}{|x|} \big) \big(1 + |\ln \ln \frac{1}{|x|}|\big) \Big]^{1+\frac{q}{2}}} dx \Big)^{\frac{2}{q}}. \tag{1.8}$$

Some embedding results

It follows in particular that

(i) for any $\beta > 1$, the following embedding is compact:

$$\hat{\mathcal{H}}_0^1(B_1) \hookrightarrow L^2\Big(B_1, |x|^{-2}(-\ln|x|)^{-2}\Big(1 + \big|\ln\ln\frac{1}{|x|}\big|\Big)^{-2\beta}dx\Big).$$

Moreover, the embedding inequality (1.6) holds for $u \in \hat{\mathcal{H}}_0^1(B_1)$.

(ii) For any $q > 2$, $r \in (0, 1)$ and $\beta > 1$, the following embedding is compact:

$$\hat{\mathcal{H}}_0^1(B_r) \hookrightarrow L^q\Big(B_r, |x|^{-2}(-\ln|x|)^{1+\frac{q}{2}}\Big(1+\big|\ln\ln\frac{1}{|x|}\big|\Big)^{-\beta(1+\frac{q}{2})}dx\Big).$$

Moreover, the embedding inequality (1.8) holds for $u \in \hat{\mathcal{H}}_0^1(B_1)$.

Thanks for your attention!