Fractional Optimal Rearrangement Problems

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A classical rearrangement problem

Let us consider the stationary heat equation

$$\begin{cases} \underbrace{\partial_t u}_{=0} - \Delta u = f(x) & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

- force function modeled by $f(x)$
- $u(x)$ and $f(x)$ do not depend on time
- constant temperature on the boundary of Ω

Different force functions f result different heat distributions u_f.

Problem: Which f makes u be as equally distributed as possible, given that

$$\int_\Omega f(x)dx = B \text{ and } 0 \leq f(x) \leq M?$$

A classical rearrangement problem

Let u_f be the unique solution of the boundary value problem

$$\begin{cases} -\Delta u_f(x) = f(x) & \text{in } \Omega, \\ u_f = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider the minimization/maximization of the functional

$$\Phi(f) = \int_\Omega |\nabla u_f|^2 dx$$

over the class of admissible force functions

$$\mathcal{R}_\beta = \{f \ : \ f = 0 \text{ or } f = 1, \text{ and } \int_\Omega f dx = \beta\}.$$

The relaxed problem relates to the minimization/maximization over the weak-* closure of $\mathcal{R}_\beta$

$$\bar{\mathcal{R}}_\beta = \{f \ : \ 0 \leq f \leq 1, \text{ and } \int_\Omega f dx = \beta\}.$$

A classical rearrangement problem

Theorem

There exists a unique minimizer $\hat{f}$ of

$$\Phi(f) = \int_\Omega |\nabla u_f|^2 dx$$

over the set $f \in \bar{\mathcal{R}}$, and $\alpha > 0$ such that for the function $\hat{u} = u_{\hat{f}}$ the following is true

- $0 < \hat{u} \leq \alpha$ *in* Ω,
- $f = \chi_{\{\hat{u}<\alpha\}} \in \mathcal{R}$,
- $\hat{u} = \alpha$ *in* $\{\hat{f} = 0\}$.

Moreover, the function $U = \alpha - \hat{u}$ is the minimizer of the functional

$$J(w) = \int_\Omega |\nabla w|^2 + 2\max(w, 0)dx,$$

among functions $w \in W^{1,2}(\Omega)$ with boundary values α on $\partial\Omega$, and solves the obstacle problem

$$\Delta U = \chi_{\{U>0\}}.$$

The obstacle problem

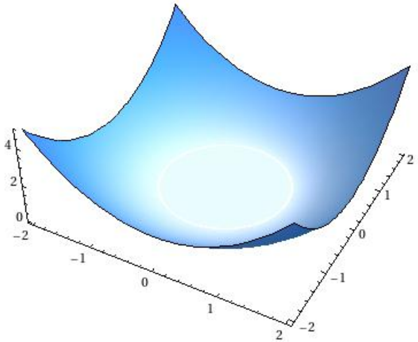
Consider the minimization of the functional

$$J(w) = \int_\Omega |\nabla w|^2 + 2\max(w,0)dx,$$

among functions $w \in W^{1,2}(\Omega)$ with boundary values $\alpha > 0$ on $\partial\Omega$.

There is a unique minimizer U to this problem, which solves the following equation

$$\Delta U = \chi_{\{U>0\}}.$$



A classical rearrangement problem

Theorem

There exists a maximizer $\hat{f}$ *of*

$$\Phi(f) = \int_\Omega |\nabla u_f|^2 dx$$

over the set $f \in \bar{\mathcal{R}}$, *and* $\alpha > 0$ *such that for the function* $\hat{u} = u_{\hat{f}}$ *the following is true*

$$f = \chi_{\{\hat{u} > \alpha\}} \in \mathcal{R}.$$

Moreover, the function $U = \alpha - \hat{u}$ *is the minimizer of the non-convex functional*

$$J(w) = \int_\Omega |\nabla w|^2 - 2\max(w, 0) dx,$$

among functions $w \in W^{1,2}(\Omega)$ *with boundary values* α *on* $\partial\Omega$, *and solves the unstable obstacle problem*

$$\Delta U = \chi_{\{U<0\}}.$$

Fractional setting

For a function u defined in $\mathbb{R}^n$ let us define the Gagliardo semi-norm

$$[u]_s^2 := \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dxdy,$$

where $0 < s < 1$.

We also define fractional Sobolev spaces in $\mathbb{R}^n$

$$H^s(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) \colon [v]_s^2 < \infty\},$$

and in a bounded domain D

$$H_0^s(D) = \{v \in H^s(\mathbb{R}^n) \colon v = 0 \text{ a.e. in } D^c\},$$

as well as dual spaces

$$H^{-s}(\mathbb{R}^n) = (H^s(\mathbb{R}^n))', \quad H^{-s}(D) = (H_0^s(D))'.$$

Fractional setting

For a function $f \in H^{-s}(D)$ we say $u_f \in H_0^s(D)$ solves the fractional boundary value problem in D with homogeneous Dirichlet boundary condition

$$\begin{cases} (-\Delta)^s u_f = f & \text{in } D, \\ u_f = 0 & \text{in } D^c, \end{cases}$$

if the equation is satisfied in the sense of distributions

$$\frac{1}{2}\iint_{\mathbb{R}^{2n}} \frac{(u_f(x)-u_f(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy = \int_D fv\,dx$$

for any $v \in H_0^s(D)$.

Fractional Rearrangement Problem

Minimize/maximize

$$\Phi_s(f) = [u_f]_s^2$$

over $f \in \bar{\mathcal{R}}$.

Theorem (Bonder, Cheng, Mikayelyan, 2020)

There exists a unique minimizer $\hat{f} \in \bar{\mathcal{R}}_\beta \setminus \mathcal{R}_\beta$ *such that*

$$\Phi_s(\hat{f}) \leq \Phi_s(f)$$

for any $f \in \bar{\mathcal{R}}_\beta$*. Moreover, for some* $\alpha > 0$ *the function* $\hat{u} = u_{\hat{f}}$ *satisfies the following conditions*

$$0 \leq \hat{u} \leq \alpha \text{ in } D,$$

and

$$\hat{f} > 0, \quad \{\hat{f} < 1\} \subset \{\hat{u} = \alpha\}, \quad \{\hat{u} < \alpha\} \subset \{\hat{f} = 1\}.$$

The minimization problem

Moreover, the function $\hat{U} := \alpha - \hat{u}$ minimizes the functional

$$J(v) = [v]_s^2 + \int_D v^+ \, dx$$

over the set $H^s_\alpha = \{v \in H^s_{loc}(\mathbb{R}^n) \colon v - \alpha \in H^s_0(D)\}$, and the function $\hat{U}$ verifies the inequalities

$$\chi_{\{U>0\}} \leq -(-\Delta)^s U \leq \chi_{\{U\geq 0\}} \quad \text{in } D$$

in the sense of distributions.
Finally, the minimizer of J in H^s_α is unique and is the unique solution to the inequalities above.

Fractional normalized obstacle problem

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Theorem (Bonder, Cheng, Mikayelyan, 2020)

The function $U \in H_\alpha$ *satisfies*

$$\chi_{\{U>0\}} \leq -(-\Delta)^s U \leq \chi_{\{U\geq 0\}} \quad \textit{in } D$$

if and only if it satisfies the equation

$$\begin{cases} -(-\Delta)^s U - \chi_{\{U\leq 0\}} \min\{-(-\Delta)^s U^+; 1\} = \chi_{\{U>0\}}, & \textit{in } D, \\ U = \alpha & \textit{in } D^c, \end{cases}$$

among functions

$$H^s_{sub}(D) = \{u \in H^s_{loc}(\mathbb{R}^n) \colon (-\Delta)^s u \leq 0 \textit{ in } D\}.$$

Here

$$\min\{-(-\Delta)^s U^+; 1\} = 1 - (1 + (-\Delta)^s U)^+.$$

The maximization problem

Theorem (Bonder, Cheng, Mikayelyan, 2021)

There exists a maximizer $\hat{f} \in \mathcal{R}_\beta$ *such that*

$$\Phi_s(\hat{f}) \geq \Phi_s(f)$$

for any $f \in \bar{\mathcal{R}}_\beta$*. Moreover, for some* $\alpha > 0$ *the function* $\hat{u} = u_{\hat{f}}$ *satisfies the following equation*

$$(-\Delta)^s \hat{u} = \chi_{\{\hat{u} > \alpha\}} = \hat{f}.$$

Open Problem: *(including the local case)*

D *convex* $\Rightarrow$ $\hat{f}$ *unique.*

Reinforced membrane problem

Hernot and Maillot have considered the following problem, where $f \geq 0$ is the external load and $\omega \subset D$ is the subset with increased stiffness:

For a fixed function $f \in L^2(D)$, let $\omega \subset D$ and $u_\omega \in W_0^{1,2}(D)$ be the unique solution of the following problem in a domain D

$$\begin{cases} -\Delta u_\omega(x) + \chi_\omega(x) u_\omega(x) = f(x) & \text{in} \;\; D, \\ u_\omega(x) = 0 & \text{on } \partial D. \end{cases}$$

Minimize the functional

$$F(\omega) = \int_D |\nabla u_\omega|^2 + \chi_\omega u_\omega^2 dx \;\; \left(= \int_D f u_\omega dx \right)$$

over all subsets with given volume $|\omega| = \beta$.

Q.: For which functions f does the optimal set ω exist?

Reinforced membrane problem

As in the previous examples, one has to consider the relaxed problem. Consider the weak-* closure of the set of characteristic functions χ_ω of given L^1-norm β:

$$\bar{\mathcal{R}}_\beta = \{l \ : \ 0 \le l \le 1, \text{ and } \int_D l dx = \beta\}.$$

Minimize the functional

$$F(l) = \int_D |\nabla u_l|^2 + l u_l^2 dx \ \left(= \int_D f u_l dx \right)$$

over functions $l \in \bar{\mathcal{R}}_\beta$, where function $f \in L^2(D)$, and $u_l \in W_0^{1,2}(D)$ be the unique solution of the BVP

$$\begin{cases} -\Delta u_l(x) + l(x) u_l(x) = f(x) & \text{in } \ D, \\ u_l(x) = 0 & \text{on } \partial D. \end{cases}$$

Reinforced membrane problem

Henrot and Maillot has shown the existence of the minimizer, as well as proven some properties.

Moreover, using the auxiliary function u_0

$$\begin{cases} -\Delta u_0(x) = f(x) & \text{in } \; D, \\ u_0(x) = 0 & \text{on } \partial D, \end{cases}$$

they have proven that the minimizer is a characteristic function, provided the function f satisfies one of the following conditions

(i) $u_0 \leq f$ in D,

(ii) $f \leq -\Delta f$ in D,

(iii) $|\{x \in D \,:\, u_0 > \gamma\}| < \beta$, where $\gamma = \inf\{f(x) \,:\, f(x) < u_0(x)\}$.

Furthermore, they prove that the minimizer is a characteristic function, in case of the ball and a non-increasing radial symmetric function f.

Fractional reinforced membrane

For a fixed function $f \in L^2(D)$, let $l \in \bar{\mathcal{R}}_\beta$ and $u_l \in H_0^s(D)$ be the unique solution of the following fractional analogue of the reinforced membrane problem in D

$$\begin{cases} (-\Delta)^s u_l(x) + l(x)u_l(x) = f(x) & \text{in } D, \\ u_l(x) = 0 & \text{in } \mathbb{R}^n \setminus D, \end{cases}$$

where the equation is satisfied in the sense of distributions

$$\iint_{\mathbb{R}^{2n}} \frac{(u_f(x) - u_f(y))(v(x) - v(y))}{|x-y|^{n+2s}}\, dxdy + \int_D l u_l v\, dx = \int_D f v\, dx$$

for any $v \in H_0^s(D)$.

Fractional reinforced membrane

Consider the minimization of the *design function*

$$F_s(l) := [u_l]_s^2 + \int_D l u_l^2 dx \quad \left(= \int_D f u_l dx \right)$$

over the set $l \in \bar{\mathcal{R}}_\beta$.

Theorem (Cheng, Mikayelyan, 2024)

F_s is convex and is weak$^*-$continuous in $\{f \in L^\infty(\Omega) \, : \, f \geq 0$ a.e.$\}$. In particular, there exists $\hat{l}$ in $\bar{\mathcal{R}}_\beta$ such that

$$\inf_{\omega \in \mathcal{R}_\beta} F_s(\omega) = \min_{l \in \bar{\mathcal{R}}_\beta} F_s(l) = F_s(\hat{l}).$$

Fractional reinforced membrane

Theorem (Cheng, Mikayelyan, 2024)

Let $\hat{u}$ solve (**) with a design function $\hat{l} \in \bar{\mathcal{R}}_\beta$, and

$$\Omega_0 = \left\{ x \in \Omega \ : \ \hat{l}(x) = 0 \right\},$$

$$\Omega_1 = \left\{ x \in \Omega \ : \ \hat{l}(x) = 1 \right\},$$

$$\Omega_* = \left\{ x \in \Omega \ : \ 0 < \hat{l}(x) < 1 \right\}.$$

Then $\hat{l}$ minimizes F_s if and only if the following two conditions hold
$\gamma_{\hat{l}} = \sup_{x \in \Omega_0} \hat{u}(x) = \inf_{x \in \Omega_1} \hat{u}(x)$.
If $|\Omega_*| > 0$, then $\hat{u}(x) = \gamma_{\hat{l}}$ a.e. in Ω_*.

Theorem (Cheng, Mikayelyan, 2024)

Let $\Omega = B_1$. Assume that $f = f(r)$ is non-negative, radially symmetric and decreasing in $r = |x|$. Then, for every $R \in [0, 1]$, the characteristic function $\hat{l} = \chi_{B_R}$ is a minimizer of F_s over $\bar{\mathcal{R}}_\beta$ with $\beta = |B_R|$.

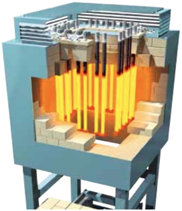
Cylindrical rearrangement problem

Let us consider the cylindrical domain $\Omega = D_{x'} \times (0,1)_{x_n}$ and the subclass of force functions which are independent of x_n

$$\bar{\mathcal{R}}_\beta^D = \{f \in \bar{\mathcal{R}}_\beta \ : \ f(x) = f(x')\}.$$

Let u_f be the unique solution of the boundary value problem

$$\begin{cases} -\Delta u_f(x) = f(x') & \text{in } \Omega, \\ u_f = 0 & \text{on } \partial\Omega. \end{cases}$$



Consider the minimization of the functional

$$\Phi(f) = \int_\Omega |\nabla u_f|^2 dx$$

over the class of admissible force functions $\bar{\mathcal{R}}_\beta^D$.

Cylindrical rearrangement problem

Theorem (Mikayelyan, 2018)

The minimization problem

$$\min_{f\in\bar{\mathcal{R}}_D} \Phi(f)$$

has a unique solution $\hat{f}\in\bar{\mathcal{R}}_D\setminus\mathcal{R}_D$, $\hat{f}>0$ *in* D, *and there exists a constant* $\alpha>0$ *such that for the function*

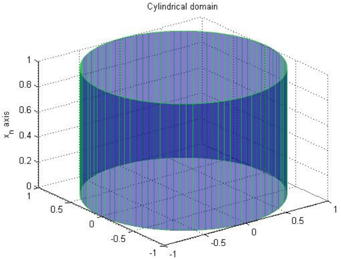
$$\hat{v}(x')=v_{\hat{f}}(x')=\int_0^1 u_{\hat{f}}(x',t)dt$$

the following is true

$\hat{v}=v_{\hat{f}}\le\alpha,$

$\{\hat{f}<1\}\subset\{\hat{v}=\alpha\},$

$\{\hat{v}<\alpha\}\subset\{\hat{f}=1\}.$



The non-local obstacle

Theorem (Mikayelyan, 2018)

Moreover, the function $\hat{U}(x) = \alpha - \hat{u}(x)$ *is the minimizer of the convex functional*

$$J(U) = \int_\Omega |\nabla U|^2 dx + 2 \int_D \max(V, 0) dx',$$

among functions in $U \in W^{1,2}(\Omega)$ *such that* $U = \alpha$ *on* $\partial\Omega$*, where*

$$V(x') = \int_0^1 U(x', t) dt.$$

Theorem (Mikayelyan, 2018)

$$\hat{u} = u_{\hat{f}} \in W^{2,2}(D' \times (0,1))$$

for any $D' \Subset D$*.*

The non-local obstacle

Theorem (Mikayelyan, 2018)

Consider the minimization of the convex functional in the domain $\Omega = D \times (0,1)$

$$J(u) = \int_\Omega |\nabla u|^2 dx + 2\int_D v^+ dx'$$

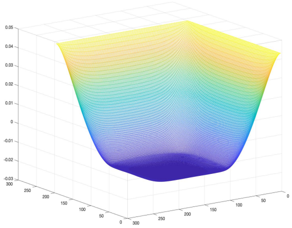
among functions with prescribed boundary values $u = g = const$ *on* $\partial\Omega$, *where* $v(x') = \int_0^1 u(x', x_n) dx_n$.

Then the functional J *has a unique minimizer* u, *and*

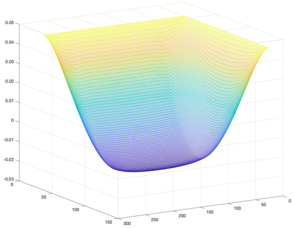
$$\Delta u(x) = \chi_{\{v>0\}} + 2\partial_\nu u(x', 0)\chi_{\{v=0\}} \text{ in } \Omega.$$

Numerical simulations

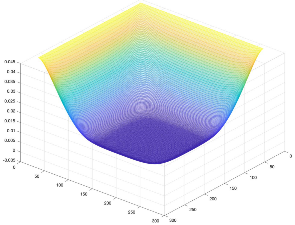
joint work with Zhilin Li (North Carolina State University)



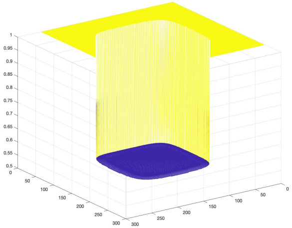
$u(x, y, 0)$



$u(x, 0, z)$



$v(x, y)$



$f(x, y)$

Comparison principle

Lemma

Let u_1 and u_2 minimize

$$J(u) = \int_\Omega |\nabla u|^2 dx + 2 \int_D v^+ dx'$$

among functions with constant boundary data α_1 and α_2 respectively, and $0 < \alpha_1 < \alpha_2$. Then the comparison principle does **not** *hold for the functions u_1 and u_2.*

$$u_1(x) \le u_2(x) \text{ is not true for all } x \in \Omega.$$

Conjecture

For $v_j(x') = \int_0^1 u_j(x', t)dt$, $j = 1, 2$,

$$v_1(x') \le v_2(x') \text{ in } D.$$

Comparison principle

Theorem (Chipot, Mikayelyan, 2022)

For $0 < \alpha_1 < \alpha_2$ *and* $v_j(x') = \int_0^1 u_j(x', t)dt$, $j = 1, 2$,

$$v_1(x') \leq v_2(x') \ \textit{in} \ D.$$

$$\Updownarrow$$

$$u_2 - \alpha_2 \leq u_1 - \alpha_1$$

$$\Updownarrow$$

$$\partial^2_{x_n x_n}(u_2 - u_1) \geq 0$$

Remarks on free boundary regularity

Theorem

Consider the minimizer u *of the convex functional*

$$J(u) = \int_\Omega |\nabla u|^2 dx + 2\int_D v^+ dx'.$$

in the domain $\Omega = D \times (0,1)$*, where* $v(x') = \int_0^1 u(x', x_n)dx_n$*.*

Then

$$\Delta v = h(x')\chi_{\{v>0\}},$$

where

$$h(x') = 1 - 2\partial_\nu u(x', 0) \in C^\alpha(D),$$

and

$$h \geq 0 \ \ \textit{in} \ \ \{v \geq 0\}.$$

Remarks on free boundary regularity

If $x' \in \partial\{v > 0\}$ and $h(x') > 0$, then we have same regularity as for the classical obstacle problem.

Open questions:

1. Is it possible to have $h(x') = 0$ on $\partial\{v > 0\}$?

2. What happens if $h(x') = 0$?

Technique (unconstrained case)

Lemma

$$\Phi(f) = \int_\Omega |\nabla u_f|^2 dx = \int_\Omega f u_f dx = \sup_{u \in W_0^{1,2}(\Omega)} \int_\Omega 2fu - |\nabla u|^2 dx.$$

Lemma

The functional Φ is

(i) weakly sequentially continuous in L^2,

(ii) strictly convex,

(iii) Gâteaux differentiable, and $\Phi'(f)$ can be identified with $2u_f$.

Technique (unconstrained case)

Step 1:

The minimizer $\hat{f}$ of Φ over $\bar{\mathcal{R}}$ exists and is unique.

Step 2:

The minimality condition is

$$0 \in \partial\Phi(\hat{f}) + \partial\xi_{\bar{\mathcal{R}}}(\hat{f}),$$

where $\partial\Phi$ is the sub-differential of Φ and

$$\xi_{\bar{\mathcal{R}}}(g) = \begin{cases} 0 & \text{if } g \in \bar{\mathcal{R}} \\ \infty & \text{if } g \notin \bar{\mathcal{R}} \end{cases}.$$

Thus

$$-2\hat{u} \in \partial\xi_{\bar{\mathcal{R}}}(\hat{f}) = \left\{ w \in L^2(\Omega) \ : \ \xi_{\bar{\mathcal{R}}}(f) - \xi_{\bar{\mathcal{R}}}(\hat{f}) \geq \int_\Omega (f - \hat{f}) w dx' \right\}$$

and for any $f \in \bar{\mathcal{R}}$

$$\int_\Omega \hat{f}\hat{u} dx \leq \int_\Omega f\hat{u} dx.$$

Technique (unconstrained case)

Lemma

For $f, g \in L^2_+(D)$ there exists $\widetilde{f} \in ext(\bar{\mathcal{R}}(f))$ such that functional

$$\int_D \widetilde{f} g dx \leq \int_D h g dx,$$

for all $h \in \bar{\mathcal{R}}(f)$.

Step 3:
There exists $\tilde{f} \in \mathcal{R}$ such that for any $f \in \bar{\mathcal{R}}$

$$\int_\Omega \hat{f} \hat{u} dx = \int_\Omega \widetilde{f} \hat{u} dx \leq \int_\Omega f \hat{u} dx.$$

Step 4:
Prove that

$$\widetilde{f} = \hat{f}.$$

Technique (constrained case)

For $f \in \bar{\mathcal{R}}_D$ we have $f(x) = f(x')$ and thus

$$\Phi(f) = \int_\Omega |\nabla u_f|^2 dx = \int_\Omega f u_f dx = \int_D f(x') v_f(x') dx',$$

where

$$v_f(x') = \int_0^1 u_f(x', t) dt.$$

We can consider Φ in $L^2_D(\Omega)$ or in $L^2(D)$.

Lemma

The functional Φ is

(i) weakly sequentially continuous in $L^2_D(\Omega)$ and in $L^2(D)$,

(ii) strictly convex,

(iii) Gâteaux differentiable. Moreover, $\Phi'(f)$ can be identified with $2u_f$ if we consider Φ in $L^2(\Omega)$ or $2v_f$ if we consider Φ in $L^2(D)$.

Technique (constrained case)

Lemma

Let $\Omega = D_{x'} \times (0,1)_{x_n}$ *and*

$$\begin{cases} -\Delta u(x) = f(x') & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Then

$$u(x', x_n) = u(x', 1 - x_n), \tag{2}$$

and the function $v(x') = \int_0^1 u(x', x_n) dx_n$ *satisfies the following equation*

$$\begin{cases} -\Delta_{x'} v = f(x') + 2\partial_\nu u(x', 0) & \text{in } D, \\ v = 0 & \text{on } \partial D. \end{cases} \tag{3}$$

Technique (constrained case)

How does the proof work?

Steps 1-3 are similar to unconstrained case

There exists a unique minimizer $\hat{f} \in \bar{\mathcal{R}}_D$ of Φ.

There exists $\widetilde{f} = \chi_{D_0}(x') \in ext(\bar{\mathcal{R}}_D) = \mathcal{R}_D$ such that

$$\int_D \hat{f}\hat{v}dx' = \int_\Omega \widetilde{f}(x')\hat{u}(x)dx \leq \int_\Omega f(x')\hat{u}(x)dx$$

for any function $f \in \bar{\mathcal{R}}_D$.

Main challenge: $\widetilde{f} \neq \hat{f}$.

Technique (constrained case)

Step 4:

Claim 1:

$$\alpha = \sup_{D_0} \hat{v} \leq \inf_{D \setminus D_0} \hat{v}.$$

Claim 2:

$$\hat{f} = \widetilde{f} = 1, \text{ in } \{\hat{v} < \alpha\}.$$

Claim 3:

$$\{\hat{v} > \alpha\} \subset D^{\#} := \{\hat{f} = 0\}.$$

Technique (constrained case)

Claim 4:

$$D^{\#} \text{ has no interior. Thus } \hat{v} \leq \alpha.$$

From (3) and the Hopf's lemma it follows that

$$\Delta_{x'} \hat{v}(x') = -2\partial_\nu u(x', 0) > 0 \text{ in } \text{int}(D^{\#})$$

and $\hat{v} \geq \alpha$ in int$(D^{\#})$. This means that there exists $y \in \partial(\text{int}(D^{\#}))$ such that $\hat{v}(y) = \beta > \alpha$, which contradicts Claim 3 and continuity of $\hat{v}$.

Claim 5:

$$\hat{f} > 0.$$

We need to verify this only in int$(\{\hat{v} = \alpha\})$ where

$$0 = \Delta_{x'} \hat{v} = -\hat{f}(x') - 2\partial_\nu \hat{u}(x', 0).$$

and the outer normal derivative of $\hat{u}$ is not vanishing in D by Hopf lemma.

References

Julián F. Bonder, Zhiwei Cheng and Hayk Mikayelyan
Optimal rearrangement problem and normalized obstacle problem in the fractional setting.
Advances in Nonlinear Analysis. 9 (2020), 1, p. 1592-1606 15 p.

Julián F. Bonder, Zhiwei Cheng and Hayk Mikayelyan
Fractional optimal maximization problem and the unstable fractional obstacle problem.
Journal of Mathematical Analysis and Applications. 495 (2021), 1, 124686.

Zhiwei Cheng and Hayk Mikayelyan
Optimization of the shape for a non-local control problem.
Fract Calc Appl Anal (2024).
https://doi.org/10.1007/s13540-024-00318-9

References

Hayk Mikayelyan
Cylindrical optimal rearrangement problem leading to a new type obstacle problem.
ESAIM: Control, Optimisation and Calculus of Variations. 24 (2018) 859-872

Michel Chipot and Hayk Mikayelyan
On some nonlocal problems in the calculus of variations
Nonlinear Analysis, Theory, Methods and Applications. 217 (2022), 112754.

Zhilin Li and Hayk Mikayelyan
Numerical analysis of a free boundary problem with non-local obstacles
Applied Mathematics Letters. 135 (2023), 108414.

References

G. R. Burton
Rearrangements of functions, maximization of convex functionals, and vortex rings.
Math. Ann., 276(2):225–253, 1987.

G. R. Burton and E. P. Ryan
On reachable sets and extremal rearrangements of control functions.
SIAM J. Control Optim., 26(6):1481–1489, 1988.

Henrot, A., Maillot, H.
Optimization of the shape and the location of the actuators in an internal control problem.
Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. 4(3), 737–757 (2001)

Ivan Blank.
Sharp results for the regularity and stability of the free boundary in the obstacle problem.
Indiana Univ. Math. J., 50(3):1077–1112, 2001.