Hardy type inequalities: critical cases

Nonlocal Problems in Mathematical Physics, Analysis and Geometry

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Boudary Hardy Inequality: Local case

[J.L.Lewis (1988)] Let $\Omega \subset \mathbb{R}^d$ Lipschitz bounded domain and $1 < p < \infty$. There exists a constant $C = C(d, p, \Omega) > 0$ such that

$$\int_\Omega \frac{|u(x)|^p}{\delta^p_{\partial\Omega}(x)} \leq C \int_\Omega |\nabla u(x)|^p, \quad \text{for all } u \in C_c^\infty(\Omega).$$

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To present the appropriate inequality for $p = 1$

Boundary Hardy Inequality: Non local case

B. Dyda [2004]. Let $s \in (0,1)$ and $sp > 1$. Let $\Omega \subset \mathbb{R}^d$ be Lipschitz bounded domain. There exists a constant $C = C(d,p,s,\Omega) > 0$ such that

$$\int_\Omega \frac{|u(x)|^p}{\delta_{\partial\Omega}^{sp}(x)} \leq C \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{sp+d}} dx, \quad \text{for all } u \in C_c^\infty(\Omega).$$

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Appropriate inequality for sp=1.

Some Other work on the Critical case :

Local Case: $\boxed{1 < p < d}$

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} \leq C \int_\Omega |\nabla u|^p, \quad \text{for all } u \in C_c^\infty(\Omega).$$

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critical case $p = d$

$$\int_\Omega \frac{|u(x)|^p}{\ln(x/R)^p |x|^p} \leq C \int_\Omega |\nabla u|^p, \quad \text{for all } u \in C_c^\infty(\Omega). \tag{1}$$

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Expected Inequality:

$$\int_\Omega \frac{1}{\psi(x)} \frac{|u(x)|^p}{\delta_{\partial\Omega}(x)} \leq C \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^2} + |u|_{p,\Omega}, \quad \text{for all } u \in C_c^\infty(\Omega)$$

$$\frac{1}{\psi(x)} \to 0, \; x \to \partial\Omega.$$

Our result: $d = 1$, Adimurthi, Jana and $d > 1$ Adimurthi, Sahu.)

Theorem

For $p > 1$, we have the optimal inequality

$$\int_\Omega \frac{|u(x)|^p}{\delta_{\partial\Omega}(x)} \frac{1}{\ln^p(\delta_{\partial\Omega}/R)} \leq C \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+1}} + C|u|_{p,\Omega}, \quad \forall\, u \in C_c^\infty(\Omega)$$

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Optimal in the sense that we cannot replace $\frac{1}{\ln^p(\delta_{\partial\Omega}(x)/R)}$ by any other function that goes faster to zero as $x \to 0$.

Idea of the proof of $\Omega = (0,2)$

- Diadic decomposition $A_k := \{x \mid 2^k < x \leq 2^{k+1}\}$, for $k = -1, -2, \cdots$.

$$\int_{A_k} \frac{|u|^p}{|x| \ln^p(2/x)}$$
$$\leq \frac{C}{(-k)^p |A_k|} \int_{A_k} (|u-(u)_{A_k}|^p + |(u_{A_k})|^p \leq \frac{C}{(-k)^p} \left\{ [u]_{s,p,A_k} + |(u_{A_k})|^p \right\}.$$

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C is independent of k. Sum $k = m \in \mathbb{Z}^-$ to $k = -1$,

$$\int_{2^m}^1 \frac{|u|^p}{|x| \ln^p(2/x)} \leq \sum_{k=m}^{-1} C[u]_{s,p,A_k} + \sum_{k=m}^{-1} C \frac{|(u_{A_k})|^p}{(-k)^p}$$

$$\leq C[u]_{s,p,(0,2)} + \sum_{k=m}^{-1} C \frac{|(u_{A_k})|^p}{(-k)^p}.$$

$$\left|(u)_{A_k}\right|^p \leq \left|(u)_{A_{k+1}} + C[u]_{s,p,A_k \cup A_{k+1}}\right|^p.$$

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$$\sum_{k=m}^{-1} \frac{\left|(u)_{A_k}\right|^p}{(-k)^{p-1}} \leq \sum_{k=m}^{-1} \frac{(u)_{A_{k+1}}^p}{(-k-(1/2))^{p-1}} + C\sum_{k=m}^{-1} [u]_{s,p,A_k\cup A_{k+1}}^p .$$

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$$\frac{|(u)_{A_m}|^p}{(-m)^{p-1}} + \sum_{k=m+1}^{-1}\left\{\frac{1}{(-k)^{p-1}} - \frac{1}{(-k+(1/2))^{p-1}} \sim \frac{1}{(-k)^p}\right\}|(u)_{A_k}|^p$$

$$\leq C|(u)_{A_0}|^p + C\sum_{k=m}^{-1}[u]^p_{s,p,A_k\cup A_{k+1}}.$$

$$\sum_{k=m}^{-1} \frac{|(u)_{A_k}|^p}{(-k)^{p-1}} \leq C|u|_{p,(0,2)} + C[u]_{s,p,(0,2)}.$$

Idea of the Proof of optimality

- $\Omega = (0, 2)$- $p = 2, sp = 1$

$$u_\epsilon(x) = \begin{cases} \frac{|\log \epsilon|}{|\log x|}, & x \in (0, \epsilon) \\ 1, & x \in (\epsilon, 1-\epsilon), \\ \frac{|\log(1-\epsilon)|}{|\log(1-x)|}, & x \in (1-\epsilon, 1). \end{cases}$$

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Then,

$$\int_0^1 \frac{(u_\epsilon - \bar{u_\epsilon})^2}{\delta_x \log^2 \delta_x} dx = \frac{1}{3|\log \epsilon|} + o\left(\frac{1}{|\log \epsilon|}\right)$$

and

$$\int_0^1 \int_0^1 \frac{(u_\epsilon(x) - u_\epsilon(y))^2}{|x-y|^2} dxdy \leq \frac{C}{|\log \epsilon|} + o\left(\frac{1}{|\log \epsilon|}\right).$$

For our notation $o(\delta)$ is a term such that $\frac{o(\delta)}{\delta} \to 0$.

Theorem

For $p > 1$, we have

$$\int_\Omega \frac{|u(x)|^p}{\delta_{\partial\Omega}(x)} \ln^{-p}(\delta_{\partial\Omega}/R) \leq C \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+1}} + |u|_{p,\Omega}, \quad \forall\, u \in C_c^\infty(\Omega)$$

- cannot put $p = 1$ above. what happens for the critical case $p = 1$??

Theorem

Let $\beta > 1$ $m \geq 2$

$$\int_\Omega \frac{|u(x)|}{\delta^s_{\partial\Omega}(x)} \mathcal{L}_1\left(\frac{\delta_{\partial\Omega}(x)}{R}\right) \cdots \mathcal{L}_{m-1}\left(\frac{\delta_{\Omega}(x)}{R}\right) \mathcal{L}^\beta_m\left(\frac{\delta_{\Omega}(x)}{R}\right)$$

$$\leq C2^m(1-s)\int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{d+s}} + |u|_{1\,\Omega}, \quad \forall\ u \in W^{s,1}(\Omega).$$

Local Boundary Hardy for $p = 1$

Letting $s \to 1-$ in the above inequality and using Brezis, Bourgain, Mironescu, Davilla,

$$\lim_{s \to 1-} (1-s) \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|}{|x-y|^{d+s}} dx = C[u]_{BV}(\Omega).$$

Theorem

$\beta > 1$

$$\int_{\Omega} \frac{|u(x)|}{\delta_{\partial\Omega}(x)} \mathcal{L}_1\left(\frac{\delta_{\partial\Omega}(x)}{R}\right) \ldots \mathcal{L}_{m-1}\left(\frac{\delta_{\Omega}(x)}{R}\right) \mathcal{L}_m^{\beta}\left(\frac{\delta_{\Omega}(x)}{R}\right) \leq C[u]_{BV(\Omega)}.$$

THANKS A LOT.