# Propagation dynamics of the Fisher-KPP nonlocal diffusion equation with free boundary

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Nonlocal Problems in Mathematical Physics, Analysis and Geometry

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**Plan of the talk:**

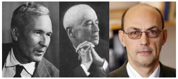
- Fisher-KPP with local diffusion
- Fisher-KPP with local diffusion and free boundary
- Fisher-KPP with nonlocal diffusion
- Fisher-KPP with nonlocal diffusion and free boundary in 1-D
- The nonlocal free boundary problem in N-D with radial symmetry
- Other extensions

**Survey paper** (open access):

Y. Du, Propagation and reaction diffusion models with free boundaries, Bull. Math. Sci., 12 (1) (2022), 2230001, pp 56.

## 1. Brief Review of Results with Local Diffusion

### (a) The classical Fisher-KPP equation

Starting from the pioneering works of **Fisher** (1937) and **KPP** (Kolmogorov-Petrovski-Piskunov, 1937), the Cauchy problem

$$
(1) \qquad \begin{cases} U_t = d\Delta U + f(U), & t > 0,\ x \in \mathbb{R}^N, \\ U(0,x) = U_0(x), & x \in \mathbb{R}^N, \end{cases}
$$

has been widely used to describe the spreading of a population with density $U(t,x)$ at time $t$ and space location $x$, where the initial population $U_0(x)$ is a nonnegative function with compact support, and the growth term $f(U)$ is usually a $C^1$ function satisfying $f(0) = 0$. The diffusion term

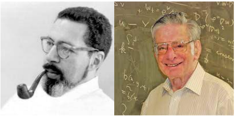
$$
d\Delta U
$$

is used to describe the dispersal of the population through random walk (following the rule of Brownian motion).

**Spreading Speed**. A striking feature of (1), with

$$f(U) = U(1-U) \text{ (a prototype Fisher-KPP nonlinearity)},$$

is the following result of Aronson and Weinberger [AM 1978]:



There exists a constant $c^* > 0$ such that

$$\begin{cases} \lim_{t\to\infty} U(t,x) = 1 & \text{uniformly in } |x| \le (c^* - \epsilon)t, \\ \lim_{t\to\infty} U(t,x) = 0 & \text{uniformly in } |x| \ge (c^* + \epsilon)t \end{cases}$$

for any small $\epsilon > 0$.

**Interpretation:** The population spreads into new territory with (asymptotic) speed $c^*$.

**Minimal speed of traveling waves** (Fisher and KPP 1937): For any $c \geq c^* := 2\sqrt{d}$, (1) has a traveling wave solution with velocity $c$: $U(t,x) := Q_c(ct - x)$, where $Q = Q_c$ satisfies the ODE

$$dQ'' - cQ' + f(Q) = 0, \; Q(-\infty) = 0, \; Q(+\infty) = 1.$$

There is no such solution for $c < c^*$.

**Q:** What is the spreading front determined by (1)?

Population range $\Omega(t) := \{x : U(t,x) > 0\} = \mathbb{R}^N$ for $t > 0$.

**Ramification:** Nominate a small $\delta > 0$ and regard the population range as

$$\Omega_\delta(t) := \{x : U(t,x) > \delta\},$$

which is a bounded set at any time $t > 0$, and so

$$\Gamma_\delta(t) := \partial\Omega_\delta = \{x : U(t,x) = \delta\}$$

is the spreading front. The Aronson-Weinberger result implies that for all large $t$ and arbitrarily small $\epsilon > 0$,

$$\Gamma_\delta(t) \subset A_\epsilon(t) := \{x \in \mathbb{R}^N : (c^* - \epsilon)t \leq |x| \leq (c^* + \epsilon)t\}.$$

## Logarithmic shifts – the radial case

*If the initial function in (1) is radially symmetric, then $U$ is radially symmetric in $x$ (i.e. $U = U(t, |x|)$) and*

$$(2) \qquad \lim_{t\to\infty} \left| U(t,|x|) - Q_{c^*}\left(c^* t - \frac{N+2}{c^*} d \log t + C - |x|\right)\right| = 0$$

*for some constant $C$, uniformly in $x \in \mathbb{R}^N$. Hence, for small $\delta > 0$,*

$$\Omega_\delta(t) := \left\{x : U(x,t) > \delta\right\} = \left\{x : |x| < R_\delta(t)\right\}$$

*with*

$$R_\delta(t) - \left(c^* t - \frac{N+2}{c^*} d \log t\right) \to C_\delta \in \mathbb{R}^1 \text{ as } t \to \infty.$$

## Remarks

- When $N = 1$, the logarithmic shift term in (2) has coefficient $\frac{3d}{c^*}$, which was first obtained in [**M. Bramson**, *Comm. Pure Appl. Math.* **1978**], by a probabilistic method for a problem concerning branching Brownian motion; it is now known as the Bramson correction term[1]
- For $N \geq 2$, (2) follows from [**J. Gartner**, *Math. Nachr.* **1982**] (probabilistic method).

## Logarithmic shifts – general case

*If $U$ is not radially symmetric, then it follows from the radial result that, for any $\delta \in (0,1)$, there exist $C_1^\delta \leq C_2^\delta$, such that for all large $t$, the boundary of $\Omega_\delta(t) := \left\{x : U(x,t) > \delta\right\}$ is a smooth closed hypersurface contained in the spherical shell*

$$\left\{x \in \mathbb{R}^N : C_1^\delta \leq |x| - \left[c^* t - \frac{N+2}{c^*} d \log t\right] \leq C_2^\delta\right\}.$$

[1] A recent result [J. Nolen, J.-M. Roquejoffre, L. Ryzhik, CCM 2019] shows that the correction term in 1-d can be further sharpened to $-\frac{3d}{2}\log t + C_{u_0} - \frac{3\sqrt{d\pi}}{a} t^{-1/2} + O(t^{-1+\epsilon})$.

## (b) The Fisher-KPP equation with free boundary

$$(3) \qquad \begin{cases} u_t - d\Delta u = f(u) & \text{for} \quad x \in \Omega(t), t > 0, \\ u = 0 \text{ and } u_t = \mu |\nabla_x u|^2 & \text{for} \quad x \in \partial\Omega(t), t > 0, \\ u(0, x) = u_0(x) & \text{for} \quad x \in \Omega_0. \end{cases}$$

- Population range: $\Omega(t) \subset \mathbb{R}^N$, $\Omega(0) = \Omega_0$,
- Range boundary (free boundary): $\Gamma(t) := \partial\Omega(t)$,
- $f(u) = u(1-u)$ (for simplicity),
- $\Omega_0$ bounded domain with smooth boundary,
- $u_0 \in C^1(\overline{\Omega}_0)$, positive in $\Omega_0$, $u_0|_{\partial\Omega_0} = 0$.

**Remark:** If $f(u) \equiv 0$, then (3) reduces to the well-known one-phase Stefan problem for ice melting, which has been studied extensively since the 1960s by many people including A. Friedman, D. Kinderlehrer, L. Nirenberger, L. Caffaralli, ...

- **Physical meaning of the free boundary condition:** Each point $x \in \Gamma(t)$ moves in the direction of the outer normal to $\Gamma(t)$ at $x$, with velocity $\mu|\nabla_x u(t,x)|$.
- In the spherically symmetric setting, where

$$\Gamma(t) = \{x : |x| = h(t)\} \text{ and } u = u(t,r),\ r = |x|,$$

this can be simplified to $h'(t) = -\mu u_r(t, h(t))$.
- The free boundary condition can be deduced from the assumption that $k$ units of the species is lost per unit volume at the front [**G. Bunting, Y. Du and K. Krakowski**, *Netw. Heterogeneous Media* **2012**]. ($\mu = d/k$.)
- It was shown in [**Y. Du and Z.M. Guo**, J. Diff. Eqns. **2012**] that problem (3) has a unique weak solution defined for all $t > 0$. The free boundary is understood as

$$\Gamma(t) = \partial\Omega(t),\ \Omega(t) := \{x : u(t,x) > 0\}.$$

## Basic results

**Theorem 1.** ([Y. Du, H. Matano and K. Wang, ARMA **2014**])

1. *$\Omega(t)$ is expanding: $\overline{\Omega}_0 \subset \Omega(t) \subset \Omega(s)$ if $0 < t < s$.*
2. *$\Gamma(t) \setminus$ (convex hull of $\overline{\Omega}_0$) is smooth.*
3. *Spreading-vanishing dichotomy:*
   *Let $\Omega_\infty := \cup_{t>0}\Omega(t)$. Then either*

   $$\textbf{(a)}\ \underline{\Omega_\infty \text{ is a bounded set}},\ \text{or}\ \textbf{(b)}\ \underline{\Omega_\infty = \mathbb{R}^N}.$$

   *Moreover,*
   *in case **(a)**, vanishing happens: $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty(\Omega(t))} = 0$;*
   *in case **(b)**, spreading happens: $\lim_{t\to\infty} u(t,x) = 1\ \forall x \in \mathbb{R}^N$.*
   *Furthermore, in case **(b)**, for all large $t$, $\Gamma(t)$ is a smooth closed hypersurface contained in the spherical shell*

   $$\left\{x \in \mathbb{R}^N : 0 \le |x| - R(t) \le \frac{\pi}{2}\mathrm{diam}(\Omega_0)\right\},$$

   *where $R(t)$ is a continuous function satisfying*

   $$\lim_{t\to\infty} \frac{R(t)}{t} = c_0^* > 0.$$

The number $c_0^*$ is called the spreading speed of (3), and it is determined by the following result.

**Theorem 2.** ([Y. Du and B. Lou, *J. Eur. Math. Soc.* **2015**])
*For any $\mu > 0$ there exists a unique pair $(c, q) = (c_0^*, q_{c_0^*})$ solving the system*

$$\text{(4)} \qquad \begin{cases} dq'' - cq' + f(q) = 0, \ q > 0 \ \text{ in } (0, \infty), \\ q(0) = 0, \ q(\infty) = 1, \ \mu q'(0) = c. \end{cases}$$

We call $q_{c_0^*}$ a semi-wave with speed $c_0^*$.

**Remark:** The spreading-vanishing dichotomy was first observed in [Y. Du and Z. Lin, *SIAM J. Math. Anal.* **2010**] in the case $N = 1$, where the free boundary model was first proposed.

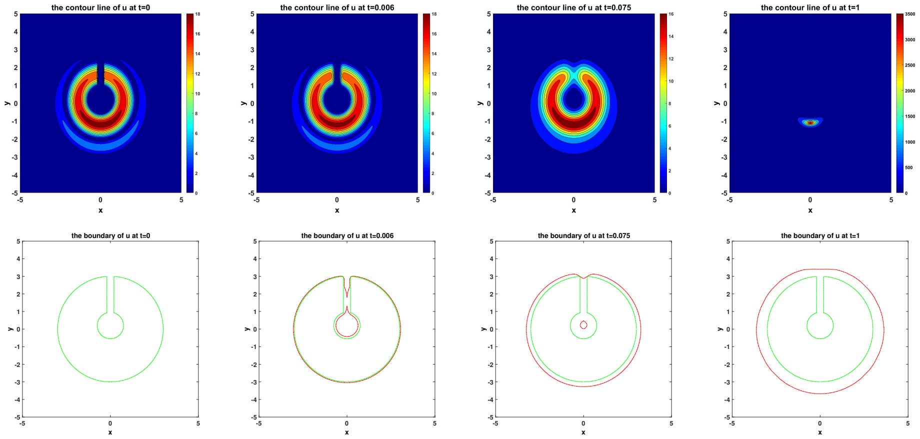
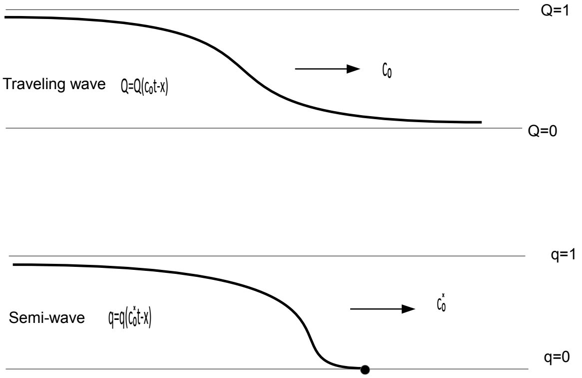



Figure: Annulus with a cut[2]

[2] S Liu, Y Du, X Liu International J. Comp. Math., 97(2020), 959-979


Q=1
Traveling wave Q=Q(c0t-x)
c0
Q=0
q=1
Semi-wave q=q(c0*t-x)
c0*
q=0

**Theorem 3.** (Logarithmic shift [Y. Du, H. Matsuzawa and M. Zhou, *JMPA* **2015**])
*Suppose $u_0$ and $\Omega_0$ are radially symmetric in (3), so that*

$$u = u(t, |x|),\ \Omega(t) = \Big\{x \in \mathbb{R}^N : |x| < R(t)\Big\}.$$

*If spreading happens, then, as $t \to \infty$,*

$$(5) \qquad \begin{cases} u(t, |x|) - q_{c_0^*}(R(t) - |x|) \to 0 \text{ uniformly in } x, \\ R(t) - [c_0^* t - (N-1)c_1^* d \log t] \to C = C(u_0) \in \mathbb{R}^1, \end{cases}$$

*where $c_0^*$ is given in Theorem 2, and $c_1^* > 0$ is given by*

$$c_1^* = \frac{1}{\zeta\, c_0^*},\ \ \zeta = 1 + \frac{c_0^*}{\mu^2 \int_0^\infty q_{c_0^*}'(z)^2 e^{-c_0^* z} dz}.$$

**Remark:** By Theorem 3, in case (b) of Theorem 1 (without radial symmetry), there exist $C_1 \le C_2$ such that, for all large $t$,

$$\Gamma(t) \subset \Big\{C_1 \le |x| - \big[c_0^* t - (N-1)c_1^* d \log t\big] \le C_2\Big\}.$$

**Theorem 4.** (Limiting problem as $\mu \to \infty$)

(a) ([Y. Du and Z.M. Guo, *J. Diff. Eqns.* **2012**])
*If* $u$ *and* $\Omega(t)$ *in* (3) *are denoted by* $u_\mu$ *and* $\Omega_\mu(t)$*, respectively, then as* $\mu \to \infty$,

$$\Omega_\mu(t) \to \mathbb{R}^N (\forall t > 0),\ u_\mu \to U \text{ in } C^{1,2}_{loc}((0,\infty) \times \mathbb{R}^N),$$

*where* $U$ *is the unique solution of* (1) *with* $U_0 = u_0$.

(b) ([Y. Du and B. Lou, *J. Eur. Math. Soc.* **2015**])
$c_0^* = c_0^*(\mu)$ *increases to* $c^*$ *as* $\mu \to \infty$.

**Remark:** Theorem 4 indicates that (1) is the limiting problem of (3) as $\mu \to \infty$.

For both (1) and (3), further results have been obtained by many people, including successful extensions to

- more general $f(u)$,
- systems of equations,
- various heterogeneous environments.

**Yet**, in both (1) and (3), using $d\Delta u$ (local diffusion) to describe the spatial dispersal of a population is not ideal in many situations, and replacing it by a nonlocal diffusion operator is sometimes more realistic.

The nonlocal version of (1) has been extensively investigated in the past 10-20 years, and fast progress is still being made. Research on the nonlocal version of (3) has just started.

## 2. The classical Fisher-KPP model with nonlocal diffusion

At any time $t > 0$, an individual at location $x$ can jump to any other location $y$ with probability $J(x-y)$. Under this assumption, the term $du_{xx}$ should be replaced by

$$d\int_{\mathbb{R}} J(x-y)\big[u(t,y)-u(t,x)\big]\,dy. \quad \text{(nonlocal diffusion operator)}$$

Thus a widely used nonlocal version of (1) is

$$(6)\quad \begin{cases} u_t = d\displaystyle\int_{\mathbb{R}} J(x-y)\big[u(t,y)-u(t,x)\big]\,dy + f(u), x \in \mathbb{R}, t > 0, \\ u(0,x) = u_0(x) \text{ for } x \in \mathbb{R}. \end{cases}$$

**(i) Usual assumptions on the convolution kernel $J(x)$:**

$$J \in C(\mathbb{R}), \text{ is nonnegative and even, } \quad \int_{\mathbb{R}} J(x)dx = 1.$$

**(ii) Thin-tailed and fat-tailed convolution kernel:**

$J(x)$ is "thin-tailed" if

$$(\mathbf{J}_{\rm thin}): \quad \int_0^\infty J(x)e^{\lambda x}dx < \infty \text{ for some } \lambda > 0.$$

Otherwise it is called "fat-tailed".

When the convolution kernel in (6) is thin-tailed, much of the basic theory for (1) carries over (by work of P. Bates, J. Coville, P. Fife, W. Shen, X. Wang, H. Weinberger, H. Yagisida, ...).

On the other hand, accelerated spreading happens when the kernel function is fat-tailed:

**Weinberger (SIMA 1982):** Let $u(t,x)$ be the solution of (6). Then $u(t,x)>0$ for $t>0$, $x\in\mathbb{R}$, and $\lim_{t\to\infty}u(t,x)=1$ locally uniformly for $x\in\mathbb{R}$. Moreover, for any given $\delta\in(0,1)$, if $[g_\delta(t),h_\delta(t)]$ is the smallest interval containing $\Omega_\delta(t):=\{x: u(t,x)>\delta\}$, then

$$\lim_{t\to\infty}\frac{h_\delta(t)}{t}=\lim_{t\to\infty}\frac{g_\delta(t)}{-t}=\begin{cases} c_*\in(0,\infty) & \text{if } J \text{ is thin-tailed},\\ \infty & \text{if } J \text{ is fat-tailed}^3.\end{cases}$$

**J. Garnier (SIMA 2011):** Examples of $J$ are given such that $h_\delta(t)$ and $-g_\delta(t)$ behave like

$$\begin{cases} e^{\alpha t}\ (\alpha>0) & \text{when } J(x)\sim|x|^\sigma\ (\sigma<-2),\\ t^\beta\ (\beta>1) & \text{when } J(x)\sim e^{-|x|^{1/\beta}}.\end{cases}$$

And many further results along this line appeared in recent years.

[3] Weinberger & X.-Q. Zhao (Math. Bios. 2010)

**(iii) The fractional Laplacian $(-\Delta)^s$ $(0 < s < 1)$:**

- Convolution kernel of $(-\Delta)^s$:

$$k(|z|) = c_{N,s}|z|^{-(N+2s)}, \quad c_{N,s} := \frac{4^s\Gamma(\frac{N}{2}+s)}{\pi^{N/2}|\Gamma(s)|},$$

which is singluar at 0 and $\int_{\mathbb{R}^N} k(|z|)dz = \infty$. The convolution operator is understood as

$$\int_{\mathbb{R}^N} \frac{u(t,y)-u(t,x)}{|x-y|^{N+2s}}dy := \lim_{\epsilon\to 0}\int_{\mathbb{R}^N\setminus B_\epsilon(0)} \frac{u(t,y)-u(t,x)}{|x-y|^{N+2s}}dy.$$

- Accelerated spreading speed [**Cabré-Roquejoffre**(CMP 2013)]:
*If* $-d\Delta u$ *in* (1) *is replaced by* $(-\Delta u)^s$ *with* $s \in (0,1)$, *then as* $t \to \infty$, *for any small* $\epsilon > 0$,

$$\begin{cases} u(t,x) \to 0 \text{ uniformly in } \left\{|x| \geq e^{(\sigma_* + \epsilon)t}\right\}; \\ u(t,x) \to 1 \text{ uniformly in } \left\{|x| \leq e^{(\sigma_* - \epsilon)t}\right\}, \end{cases}$$

*where*

$$\sigma_* := \frac{1}{N+2s}.$$

So the spreading front propagates exponentially in time.

## 3. The 1-D free boundary model with nonlocal diffusion

$$(7)\quad \begin{cases} u_t = d\displaystyle\int_{\mathbb{R}} J(x-y)\big[u(t,y)-u(t,x)\big]\,dy + f(u), \ g(t)<x<h(t), \\ u(t,g(t)) = u(t,h(t)) = 0, \\ h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{+\infty} J(x-y)u(t,x)dydx, \\ g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J(x-y)u(t,x)dydx, \\ u(0,x) = u_0(x),\ \ h(0) = -g(0) = h_0, \qquad x \in [-h_0, h_0], \end{cases}$$

where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t,x)$, which is always assumed to be identically 0 for $x \in \mathbb{R} \setminus [g(t), h(t)]$.

The initial function $u_0(x)$ satisfies $u_0 \in C([-h_0, h_0])$, and

$$u_0(-h_0) = u_0(h_0) = 0 \ \text{ and } \ u_0(x) > 0 \ \text{ in } \ (-h_0, h_0),$$

so $[-h_0, h_0]$ represents the initial population range of the species.

We assume that the kernel function $J : \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative, and has the properties

$$\textbf{(J):} \qquad J(0) > 0, \ \int_{\mathbb{R}} J(x)dx = 1, \ J(x) = J(-x), \ \sup_{\mathbb{R}} J < \infty.$$

As before, for simplicity, we take the special Fisher-KPP type nonlinearity

$$f(u) = u(1-u).$$

Note that

$$d\int_{\mathbb{R}} J(x-y)\big[u(t,y)-u(t,x)\big]\,dy = d\int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - du(t,x).$$

## Meaning of the free boundary conditions

The total population mass moved out of the range $[g(t), h(t)]$ at time $t$ through its right boundary $x = h(t)$ per unit time is given by

$$d \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y)u(t,x)dydx.$$

As we assume that $u(t,x) = 0$ for $x \notin [g(t), h(t)]$, this quantity of mass is lost in the spreading process of the species. We may call this quantity the outward flux at $x = h(t)$ and denote it by $J_h(t)$. Similarly we can define the outward flux at $x = g(t)$ by

$$J_g(t) := d \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t,x)dydx.$$

Then the free boundary conditions in (7) can be interpreted as assuming that the expanding rate of the front is proportional to the outward flux (by a factor $\mu/d$):

$$g'(t) = -\mu J_g(t),\ h'(t) = \mu J_h(t).$$

**Remarks:**

- Problem (7) was first proposed in
  - (i) Jiafeng Cao, Y. Du, Fang Li and Wan-Tong Li, **JFA 2019**,
  - (ii) C. Cortázar, F. Quirós and N. Wolanski, **Interfaces Free Bound. 2019**.

  In (ii) the case $f(u) \equiv 0$ was considered. The free boundary conditions were proposed independently in these two papers.
- For a plant species, seeds carried across the range boundary may fail to establish due to numerous reasons, such as isolation from other members of the species causing poor or no pollination, or causing overwhelming attacks from enemy species. However, some of those not very far from the range boundary may survive, which results in the expansion of the population range. The free boundary condition here assumes that this survival rate is roughly a constant for a given species. For an animal species, a similar consideration can be applied.

## Main results

### (a) Spreading-vanishing dichotomy and criteria [Cao-Du-Li-Li, JFA 2019]

**Theorem 5** (Existence and uniqueness:) Problem (7) has a unique solution $(u, g, h)$ defined for all $t > 0$.

**Theorem 6** (Spreading-vanishing dichotomy): Let $(u, g, h)$ be the unique solution of (7). Then one of the following happens:

- Spreading $\begin{cases} \lim_{t\to+\infty}(g(t), h(t)) = \mathbb{R}, \\ \lim_{t\to+\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}, \end{cases}$
- Vanishing $\begin{cases} \lim_{t\to+\infty}(g(t), h(t)) = (g_\infty, h_\infty) \text{ is a finite interval}, \\ \lim_{t\to+\infty} \|u(t,\cdot)\|_{L^\infty([g(t),h(t)])} = 0. \end{cases}$

**Theorem 7** (Spreading-vanishing criteria):

$(\alpha)$ If $d \le f'(0) = 1$, then spreading always happens.

$(\beta)$ If $d > f'(0) = 1$, then there exists a unique $\ell^* > 0$ such that spreading always happens if $h_0 \ge \ell^*$; and for $h_0 \in (0, \ell^*)$, there exists a unique $\mu^* > 0$ so that spreading happens exactly when $\mu > \mu^*$.

## (b) Spreading speed [Y. Du, Fang Li and Maolin Zhou, JMPA 2021]

We need to introduce a key condition on the kernel function, namely

$$\textbf{(J1):} \quad \int_0^\infty xJ(x)\,dx < +\infty.$$

**Theorem 8** (Spreading speed): Suppose **(J)** is satisfied, and spreading happens to the unique solution $(u, g, h)$ of (7). Then the following conclusions hold.

- If **(J1)** is satisfied, then there exists a unique $c_0 > 0$ such that

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{g(t)}{-t} = c_0. \quad \text{(linear spreading)}$$

- If **(J1)** does not hold, then

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{g(t)}{-t} = \infty. \quad \text{(accelerated spreading)}$$

The spreading speed $c_0$ is determined by semi-wave solutions to (7). These are pairs $(c, \phi)$ determined by the following system:

$$(8) \quad \begin{cases} d\int_{-\infty}^{0} J(x-y)\phi(y)dy - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, \ x < 0, \\ \phi(-\infty) = 1, \quad \phi(0) = 0, \\ c = \mu \int_{-\infty}^{0} \int_{0}^{+\infty} J(x-y)\phi(x)dydx. \end{cases}$$

**Theorem 9** (Semi-wave): Suppose **(J)** holds. Then (8) has a solution pair $(c, \phi) = (c_0, \phi_0)$ with $\phi_0 \in C^1((-\infty, 0])$ monotone if and only if **(J1)** holds. Moreover, when **(J1)** holds, there exists a unique solution pair, and $c_0 > 0$, $\phi_0'(x) < 0$.

## (c) Sharper estimates of the spreading rate

[Y. Du, Wenjie Ni, J. Eur. Math. Soc., 2023 (online)]

We will use the notation

$$\eta(t) \sim \xi(t) \quad \text{if and only if } c_1\xi(t) \le \eta(t) \le c_2\xi(t)$$

for some positive constants $c_1 \le c_2$.

**Theorem 10** (Sharper estimates): Suppose **(J)** is satisfied, and spreading happens to the unique solution $(u, g, h)$ of (7). If additionally

$$J(x) \sim |x|^{-\alpha} \text{ for } |x| \gg 1, \quad \left( \text{and so} \begin{cases} \textbf{(J)} \iff \{\alpha > 1\} \\ \textbf{(J1)} \iff \{\alpha > 2\} \end{cases} \right)$$

then

$$c_0 t + g(t),\ c_0 t - h(t) \sim 1 \qquad \text{if } \alpha > 3,$$

$$c_0 t + g(t),\ c_0 t - h(t) \sim \begin{cases} \ln t & \text{if } \alpha = 3, \\ t^{3-\alpha} & \text{if } 3 > \alpha > 2, \end{cases} \qquad \text{(shifts)}$$

$$-g(t),\ h(t) \sim \begin{cases} t \ln t & \text{if } \alpha = 2, \\ t^{\frac{1}{\alpha-1}} & \text{if } 2 > \alpha > 1. \end{cases} \qquad \text{(acceleration)}$$

## Comparisons

- Under condition **(J)**,

$$\Big\{\text{accelerated spreading for (6)}\Big\} \Longleftrightarrow \Big\{J \text{ does not satisfy } (\mathbf{J}_{\rm thin})\Big\}.$$

$$\Big\{\text{accelerated spreading for (7)}\Big\} \Longleftrightarrow \Big\{J \text{ does not satisfy } \textbf{(J1)}\Big\}.$$

For the corresponding <u>local diffusion</u> problems (1) and (3), accelerated spreading never happens.

- Under condition **(J)**,

$$(\mathbf{J}_{\rm thin}) \Longrightarrow \textbf{(J1)} \left( \int_0^\infty J(x)e^{\lambda x}dx < \infty \;\; \Longrightarrow \int_0^\infty J(x)xdx < \infty \right)$$

$$\textbf{(J1)} \not\Longrightarrow (\mathbf{J}_{\rm thin}).$$

Therefore, accelerated spreading happens less often in the nonlocal free boundary problem (7) than in the corresponding problem (6).

## 4. The nonlocal free boundary problem in $\mathbb{R}^N$ with radial symmetry [Y. Du and Wenjie Ni, SIMA 2022 & JFA (to appear)]

The radially symmetric version of (7) in $\mathbb{R}^N$ ($N \geq 2$) is

$$
(9) \quad \begin{cases}
u_t = d\int_{B_{h(t)}} J(|x-y|)u(t,|y|)dy - du + f(u), & t>0,\ x\in B_{h(t)},\\
u = 0, & t>0,\ x\in \partial B_{h(t)},\\
h'(t) = \dfrac{\mu}{|\partial B_{h(t)}|}\int_{B_{h(t)}}\int_{\mathbb{R}^N\setminus B_{h(t)}} J(|x-y|)u(t,|x|)dydx, & t>0,\\
h(0) = h_0,\ u(0,|x|) = u_0(|x|), & x\in \overline{B}_{h_0},
\end{cases}
$$

where $B_{h(t)} = \{x \in \mathbb{R}^N : |x| < h(t)\}$, and $u = u(t,|x|)$ is radially symmetric. The initial function $u_0$ satisfies

$$
\begin{cases}
u_0 \text{ is radial and continuous in } \overline{B}_{h_0},\\
u_0 > 0 \text{ in } B_{h_0},\ u_0 = 0 \text{ on } \partial B_{h_0}.
\end{cases}
$$

As before, for simplicity

$$
f(u) = u(1-u).
$$

For (9), our basic assumptions on the kernel function $J(|x|)$ are

**(J):** $J \in C(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, $J \geq 0$, $J(0) > 0$, $\displaystyle\int_{\mathbb{R}^N} J(|x|)dx = 1$.

For $r := |x|$ with $x \in \mathbb{R}^N$ and $\rho > 0$, define

$$\tilde{J}(r,\rho) = \tilde{J}(|x|,\rho) := \int_{\partial B_\rho} J(|x-y|)dS_y.$$

Then (9) can be rewritten into the equivalent form

$$(10) \quad \begin{cases} u_t(t,r) = d\displaystyle\int_0^{h(t)} \tilde{J}(r,\rho)u(t,\rho)d\rho - du + f(u), & t>0, r\in[0,h(t)), \\ u(t,h(t)) = 0, & t>0, \\ h'(t) = \dfrac{\mu}{h^{N-1}(t)}\displaystyle\int_0^{h(t)}\int_{h(t)}^{+\infty} \tilde{J}(r,\rho)r^{N-1}u(t,r)d\rho dr, & t>0, \\ h(0) = h_0,\ u(0,r) = u_0(r), & r \in [0,h_0]. \end{cases}$$

(Here a universal constant is absorbed by $\mu$.)

**Theorem 11** (Existence and uniqueness): Suppose $(\mathbf{J})$ is satisfied. Then problem (9), or equivalently (10), admits a unique positive solution $(u, h)$ defined for all $t > 0$.

**Theorem 12** (Spreading-vanishing dichotomy): Suppose $(\mathbf{J})$ is satisfied. Let $(u, h)$ be the solution of (9). Then one of the following alternatives must occur :

(i) Spreading: $\lim_{t\to\infty} h(t) = \infty$ and

$$\lim_{t\to\infty} u(t, |x|) = 1 \text{ locally uniformly in } \mathbb{R}^N,$$

(ii) Vanishing: $\lim_{t\to\infty} h(t) = h_\infty < \infty$ and

$$\lim_{t\to\infty} u(t, |x|) = 0 \text{ uniformly for } x \in B_{h(t)}.$$

**Theorem 13** (Spreading-vanishing criteria): In Theorem 12,

(1) if $d \le f'(0) = 1$, then spreading always happens,

(2) if $d > f'(0) = 1$ then there exists $L_* > 0$ such that

  (i) for $h_0 \ge L_*$, spreading always happens,

  (ii) for $0 < h_0 < L_*$, there is $\mu_* > 0$ such that spreading happens if and only if $\mu > \mu_*$.

Here $L_*$ is independent of $u_0$, but $\mu_*$ depends on $u_0$.

## Spreading speed of (9)

We need to introduce the following function, which will determine the spreading speed. For any $\xi \in \mathbb{R}$, define

$$(11) \qquad J_*(\xi) := \int_{\mathbb{R}^{N-1}} J(|(\xi, x')|)dx',$$

where $x' = (x_2, ..., x_N) \in \mathbb{R}^{N-1}$.

Condition **(J)** implies

$$\begin{cases} J_* \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ is nonnegative, even, } J_*(0) > 0, \\ \int_{\mathbb{R}} J_*(\xi)d\xi = \int_{\mathbb{R}^N} J(|x|)dx = 1. \end{cases}$$

Moreover,

$$J_*(\xi) = \omega_{N-1} \int_{|\xi|}^\infty J(r)r(r^2 - \xi^2)^{(N-3)/2}dr,$$

$$\int_0^\infty J_*(\xi)\xi d\xi = \frac{\omega_{N-1}}{N-1} \int_0^\infty J(r)r^N dr.$$

where $\omega_k$ denotes the area of the unit sphere in $\mathbb{R}^k$.

**Theorem 14** (Spreading speed): In Theorem 12, if spreading happens, then

$$\lim_{t\to\infty}\frac{h(t)}{t}=\begin{cases}c_0 & \text{if } J_* \text{ satisfies } \textbf{(J1)},\\ \infty & \text{if } J_* \text{ does not satisfy } \textbf{(J1)},\end{cases}$$

where $c_0$ is given by Theorem 9 with $J$ replaced by $J_*$.

**Theorem 15** (Rate of spreading) In Theorem 12, if there exists $\beta > N$ such that $J(r) \sim r^{-\beta}$ for all large $r$, and if spreading happens, then for all large $t$,

$$\begin{cases}h(t) \sim t^{1/(\beta-N)} & \text{if } \beta\in(N,N+1),\\ h(t)\sim t\ln t & \text{if } \beta=N+1,\\ |c_0t-h(t)|=O(t^{N+2-\beta}) & \text{if } \beta\in(N+1,N+2],\\ c_0t-h(t)\sim \ln t & \text{if } \beta>N+2.\end{cases}$$

**Remarks:**

- $|c_0 t - h(t)| = O(t^{N+2-\beta}) \implies c_0 t - h(t) \sim t^{N+2-\beta}$ when $N = 3$ (we believe this is true for all $N \geq 2$).
- In dimension 1, when $\beta > N + 2 = 3$, it holds $c_0 t - h(t) \sim 1$ for $t \gg 1$ (no logarithmic shift!).
- The change of pattern only happens for $\beta > N + 2$. By Theorem 10,

$$\{N = 1\} \implies \begin{cases} h(t) \sim t^{1/(\beta-1)} & \text{if } \beta \in (1, 2), \\ h(t) \sim t \ln t & \text{if } \beta = 2, \\ c_0 t - h(t) \sim t^{3-\beta} & \text{if } \beta \in (2, 3], \\ c_0 t - h(t) \sim 1 & \text{if } \beta > 3. \end{cases}$$

The difficulties in treating the high dimensional case mainly arise from

$$J_*(\xi) = \omega_{N-1} \int_{|\xi|}^{\infty} J(r) r (r^2 - \xi^2)^{(N-3)/2} dr$$

$$\tilde{J}(r, \rho) = \omega_{N-1} \frac{2\rho}{(2r)^{N-2}} \int_{|\rho-r|}^{\rho+r} \Big( \left[(\rho + r)^2 - \eta^2\right] \left[\eta^2 - (\rho - r)^2\right] \Big)^{\frac{N-3}{2}} \eta J(\eta) d\eta.$$

## 5. Other extensions (1-D):

- Epidemic models with nonlocal diffusion

  (Y. Du and Wenjie Ni [Nonlinearity 2020], Meng Zhao, Wan-Tong Li and Y. Du [CPAA 2020], Meng Zhao, Yang Zhang, Wan-Tong Li and Y. Du [JDE 2020], T-Y. Chang and Y. Du [ERA 2021], Rong Wang and Y. Du [JDE 2022], Y. Du, Wenjie Ni and Rong Wang [Nonlinearity 2023])

- Accelerated spreading and sharp estimates of spreading rate for cooperative systems including epidemic models

  (Y. Du, Wan-Tong Li, Wenjie Ni and Meng Zhao [JDDE 2022], Y. Du and Wenjie Ni [JDE 2021], Y. Du and Wenjie Ni [Preprint 2021], ...)

- More precise rate of acceleration and varied free boundary conditions

  (Y. Du and Wenjie Ni [Math. Ann. 2024], Y. Du, Wenjie Ni and Xin Long [Preprint 2024]).

- Lotka-Volterra systems

  (Y. Du, M.X. Wang and Meng Zhao [DCDS-A 2021], Y. Du, Wenjie Ni and Linfei Shi [Preprint, 2024], M.X. Wang and collaborators, ...)

- Approximation by local diffusion models

  (Y. Du and Wenjie Ni [CCM 2022])

- Fisher-KPP with fixed and free boundary

  (Lei Li, W.T. Li, M.X. Wang [JDE 2022])

**Thank You!**