Topological Recursion Revised

(on a series of joint papers with A.Alexandrov, B.Bychkov, P.Dunin-Barkowski, and S.Shadrin)

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Topological recursion: an overview

Chekhov-Eynard-Orantin '06
Eynard-Orantin '07

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Goal: to compute a system of quantities (*correlators*)

$$\left\{ f^{(g)}_{k_1,\dots,k_n} \right\}, \quad g \geq 0, \quad (k_1,\dots,k_n) \vdash d = \sum k_i$$

Examples:

- Hurwitz numbers (simple, double, monotone, weighted etc.);
- enumeration of maps (hypermaps, fully simple, weighted etc.);
- correlators of matrix models;
- correlators of CohFT's (GW invariants);
- WP volumes, MV volumes, etc.

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Potential (free energy) F; partition function (tau function) $Z=e^F$:

$$F(p_1,p_2,\dots;\hbar)=\sum_{g\ge 0,\ n\ge 1}\frac{\hbar^{2g-2+n}}{n!}\sum_{k_1,\dots,k_n\ge 1}f^{(g)}_{k_1,\dots,k_n}p_{k_1}\dots p_{k_n}$$

n-point function:

$$H^{(g)}_n(w_1,\dots,w_n)=\sum_{k_1,\dots,k_n}f^{(g)}_{k_1,\dots,k_n}\,w_1^{k_1}\dots w_n^{k_n},\quad n=1,2,\dots.$$

Topological recursion computes $H^{(g)}_n$ in a closed form inductively in g and n

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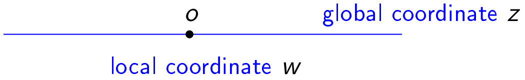
Spectral curve: $\Sigma = \mathbb{C}P^1$



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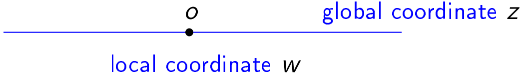

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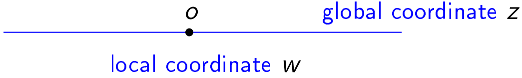

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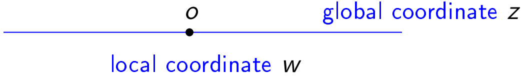

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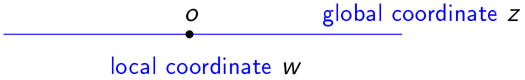

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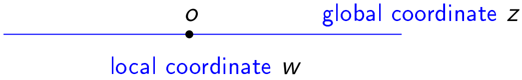

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Example

Hurwitz numbers: $d=\sum_{i=1}^{n}k_i,\ m=2g-2+n+d,$

$$f^{(g)}_{k_1,\dots,k_n}=\frac{|\mathrm{Aut}(k_1,\dots,k_n)|}{m!d!}\#\left\{(\tau_1,\dots,\tau_m)\;\middle|\;\begin{array}{l}\text{1) } \tau_i\in S(d) \text{ a transposition}\\ \text{2) } \tau_1\circ\dots\circ\tau_m \text{ has cyclic type } (k_1,\dots,k_n)\\ \text{3) connectness condition}\end{array}\right\}$$

$$\omega^{(g)}_n=\sum_{k_1,\dots,k_n}f^{(g)}_{k_1,\dots,k_n}\prod_{i=1}^{n}k_iw_i^{k_i-1}dw_i+\delta_{g,0}\delta_{n,2}\frac{dw_1dw_2}{(w_1-w_2)^2}$$

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$$\begin{aligned} w(z)&=z\,e^{-z}\\ &=z-z^2+\frac{z^3}{2}-\dots \end{aligned}\quad\Rightarrow\quad \begin{aligned}\omega^{(0)}_3&=\tfrac{dz_1\,dz_2\,dz_3}{(1-z_1)^2(1-z_2)^2(1-z_3)^2},\\ \omega^{(1)}_1&=\tfrac{(4-z_1)\,z_1}{24\,(1-z_1)^4}\,dz_1,\end{aligned}\qquad \mathcal{P}=\{z=1\},$$

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$$y(x)=z,\qquad x(z)=\log(w(z))=\log z-z,$$

$$\omega^{(0)}_1=y_1\;dx_1=(1-z_1^2)\,dz_1,\qquad \omega^{(0)}_2=\frac{dz_1\,dz_2}{(z_1-z_2)^2}.$$

Topological recursion: initial data

Initial data: $(\Sigma, dx, dy, B, \mathcal{P}) \quad \overset{\text{CEO TR}}{\rightsquigarrow} \quad \{\omega_n^{(g)}\}_{g\geq 0, n\geq 1}$

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- $\Sigma = \mathbb{C}P^1$ (generalization: a smooth algebraic complex curve);
- $B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$ (generalization: a symmetric bidifferential on Σ^2 with similar singularity on the diagonal and no other poles)
- dx, dy meromorphic differentials on Σ
- $\mathcal{P} = \{q_1, \dots, q_N\}$ a set of *simple* zeroes of dx such that $dy|_{q_j} \neq 0$

Initial differentials:

$$\omega_1^{(0)}(z_1) = y(z_1)\, dx(z_1), \qquad \omega_2^{(0)}(z_1, z_2) = B(z_1, z_2)$$

The higher ω-differentials are computed by a recursive procedure inductively in g and n

Topological recursion: two step induction

2g − 2 + n > 0: $K = \{2, \dots, n\}$, $z_K = (z_2, \dots, z_n)$,

First Step: preliminary version of $\omega_n^{(g)}(z, z_K)$ at $z \approx q_j \in \mathcal{P}$; $x(z) = x(\sigma(z))$

$$\tilde{\omega}_n^{(g)}(z, z_K) = \frac{\omega_{n+1}^{(g-1)}(z,\sigma(z),z_K) + \sum\limits_{\substack{g_1+g_2=g,\ J_1\sqcup J_2=K\\ (g_i,|J_i|+1)\neq(0,1)}} \omega_{|J_1|+1}^{(g_1)}(z,z_{J_1})\omega_{|J_2|+1}^{(g_2)}(\sigma(z),z_{J_2})}{(y(z)-y(\sigma(z)))\,dx(z)}.$$

Second Step: final computation of $\omega_n^{(g)}(z, z_K)$

$$\omega_n^{(g)}(z, z_K) = \tilde{\omega}_n^{(g)}(z, z_K) + (\text{holomorphic in } z), \quad z \to q_j, \quad j = 1, \dots, N.$$

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Equivalently,

$$\omega_n^{(g)}(z_1,z_K)=\sum_{j=1}^{N}\operatorname*{res}_{z=q_j}\tilde\omega_n^{(g)}(z,z_K)\int^{z}B(\cdot,z_1).$$

Degenerate spectral curve data

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Requirement: compatibility with limits under degenerations of the spectral curve data

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$$y_\epsilon = z^2, \qquad \begin{matrix} x_\epsilon = z^2 + \epsilon\, z \\ x_\epsilon = z^2 + \epsilon \log z \Rightarrow dx_\epsilon = \frac{2z^2+\epsilon}{z} dz \quad \text{(2 critical points)} \end{matrix}$$

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$$x_\epsilon(z) = z^5 + \epsilon\, z, \qquad \begin{array}{l} \text{(a) } y_\epsilon = \frac{1}{z^3} \\ \text{(b) } y_\epsilon = \frac{1}{z^3+\epsilon} \\ \text{(c) } y_\epsilon = \frac{z^2}{x_\epsilon} = \frac{z}{z^4+\epsilon} \end{array}$$

How CEO TR differentials of these families behave as $\epsilon \to 0$? (Try to guess!)

Two examples

Answer:

- (1), $k \leq 3$; (2b): NO LIMIT!
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  - The limit does exist and is govern by GenTR
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- (2c):
  - The limit does exist but it is different from that one of the case (2a)
  - It is govern by BE recursion
  - A closed formula for the TR differentials of this family is also available

Closed expression for GetTR differentials

$$(1): \quad x = z^2 + \frac{\epsilon}{z^{k-2}}, \quad y = z^2, \quad k \geq 4$$

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Input data: two functions $x(z)$, $y(z)$ such that dx and dy are meromorphic

$$\hat{z}(z,v) = e^{\frac{v\hbar}{2}\partial_y} z, \quad \hat{z}_i^{\pm} = \hat{z}(z_i, \pm v_i), \qquad \mathcal{S}(u) = \frac{e^{u/2} - e^{-u/2}}{u},$$

$$\mathbb{W}_n^{\vee}(z_1, v_1, \dots, z_n, v_n) = \sum_{g \geq 0} \hbar^{2g-2+n} \mathbb{W}_n^{\vee,(g)}$$

$$(*) \qquad = \prod_{i=1}^{n} \left(e^{v_i \mathcal{S}(v_i \hbar \partial_{y_i}) x_i} \sqrt{\frac{d\hat{z}_i^+}{dz_i} \frac{d\hat{z}_i^-}{dz_i}}\, dz_i \right) (-1)^{n-1} \sum_{\sigma \in \mathrm{cycl}(n)} \prod_{i=1}^{n} \frac{1}{\hat{z}_i^+ - \hat{z}_{\sigma(i)}^-}$$

$$\frac{(-1)^n \omega_n^{(g)}}{\prod_{i=1}^n dx_i} = \sum_{k_1, \dots, k_n \geq 0} (-\partial_{x_1})^{k_1} \dots (-\partial_{x_n})^{k_n} [v_1^{k_1} \dots v_n^{k_n}] \left(\prod_{i=1}^n \frac{e^{-v_i x_i}}{dx_i} \right) \mathbb{W}_n^{\vee,(g)}$$

Generalized TR: overview

CEO TR: an explicit formula for $\tilde{\omega}_n^{(g)}(z, z_2, \dots, z_n)$

Then, $\omega_n^{(g)}(z_1, z_2, \dots, z_n) = \sum_{q_j \in \mathcal{P}} \operatorname*{res}_{z=q_j} \tilde{\omega}_n^{(g)}(z, z_2, \dots, z_n) \int^{z} B(\cdot, z_1)$.

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Example: $(g, n) = (0, 3)$

CEO TR: $$\tilde{\omega}_3^{(0)}(z, z_2, z_3) = \frac{B(z, z_2)B(\sigma(z), z_3) + B(z, z_3)B(\sigma(z), z_2)}{(y(z) - y(\sigma(z)))\, dx(z)}$$

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GenTR: $$\tilde{\omega}_3^{(0)}(z, z_2, z_3) = d_z \frac{B(z, z_2)B(z, z_3)}{dx(z)\, dy(z)}$$

Generalized TR: overview

$o \in \Sigma$, z local coordinate $\quad \begin{aligned} x &= a\, z^r + \text{h.o.t}, \\ y &= b\, z^s + \text{h.o.t}, \end{aligned} \quad a\, b \neq 0,\ r, s \in \mathbb{Z}$

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The point $o \in \Sigma$ is called
special, if $r + s > 0$ and $(r, s) \neq (1, 1)$,
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Initial data of GenTR: $(\Sigma, dx, dy, B, \mathcal{P})$;

- Σ, B the same as for CEO TR
- dx, dy arbitrary meromorphic differentials (with no restriction on zeroes and poles)
- $\mathcal{P}$ is an arbitrary subset in the set of special points

Generalized TR: basic properties

- $\omega_1^{(0)} = y\,dx$, $\omega_2^{(0)} = B$
- $2g-2+n>0$: $\omega_n^{(g)}$ is global meromorphic, symmetris, and has poles at $z_i = q_j$, $q_j \in \mathcal{P}$.
- Two-step recursion for $\omega_n^{(g)}$:
  - $\tilde\omega_n^{(g)}(z, z_2, \dots, z_n)$ is given by an *explicit formula* (below)
  - It is global meromorphic in z
  - its poles in z are at special points and also at $z_2, \dots, z_n$
  - $\omega_n^{(g)}$ selects those poles of $\tilde\omega_n^{(g)}$ which are *key-special*

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Theorem (Compatibility with known versions of TR)

- $(r, s) = (2, 1) \Leftrightarrow$ *CEO*
- $(r, s) = (2, -1) \Leftrightarrow$ *Chekhov-Norbury irregular recursion*
- $r > 0$, $s = \pm 1 \Leftrightarrow$ *BE recursion*
- $(r, s) = (1, 0) \Leftrightarrow$ *LogTR of* [ABDKS23]

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- $r>0$, $s=\pm 1 \Leftrightarrow$ *BE recursion*
- $(r,s) = (1,0) \Leftrightarrow$ *LogTR of* [ABDKS23]

Remark. GenTR is not compatible with BE TR if $s \neq \pm 1$

Generalized TR: compatibility with limits

Theorem (Compatibility with limits)

GenTR is compatible with limits of the spectral curve data as long as key-special points and key$^\vee$-special points do not collapse together

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Recall: if $\begin{cases} x = z^r + \text{h.o.t}, \\ y = z^s + \text{h.o.t} \end{cases}$ and $s \le -r$, then the point $z = 0$ is not special

Example (1), $k \ge 4$, Example (2a): $\mathcal{P} = \{dx = 0\}$, $\mathcal{P}^\vee = \emptyset$. By Theorem, the limit does exist.

$\left.\begin{array}{l}\text{Example (1), } k < 4\text{: } \mathcal{P} = \{z = 0\} \ne \emptyset \\ \text{Example (2b), (2c): } \mathcal{P} = \{dy = 0\} \ne \emptyset\end{array}\right\}$ Theorem, is not applicable

In the case (2c), the limit does exist, but this convergency is not covered by Theorem

Generalized TR: compatibility with limits

Theorem (Compatibility with limits)

GenTR is compatible with limits of the spectral curve data as long as key-special points and key$^\vee$-special points do not collapse together

Example

$$x = z^2, \quad y = \frac{1}{z+s}, \quad \mathcal{P} = \{0\}, \quad \mathcal{P}^\vee = \emptyset$$

This TR is compatible with the limit as $s \to 0$ (the pole of y is not special)

$s \neq 0$: CEO TR $\rightsquigarrow$ KW potential (with properly rescaled times)

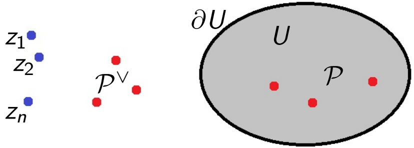
$s = 0$: CN irregular TR $\rightsquigarrow$ BGW potential

Generalized TR: compatibility with limits

Theorem (Compatibility with limits)

GenTR is compatible with limits of the spectral curve data as long as key-special points and key$^\vee$-special points do not collapse together

Proof



$$\omega_n^{(g)}(z_1, z_K) = \frac{1}{2\pi} \int_{z\in\partial U} \left(\tilde{\omega}_n^{(g)}(z, z_K) \int^z B(\cdot, z_1) \right), \qquad z_1, \dots, z_n \in \Sigma \setminus U.$$

This integral depends smoothly on parameters □

xy duality transformation: an explicit closed formula $\{\omega_n^{(g)}\} \longleftrightarrow \{\omega_n^{\vee,(g)}\}$

Theorem (Compatibility with xy swap)

$$
\begin{array}{ccc}
(\Sigma, dx, dy, B, \mathcal{P}) & & (\Sigma, dy, dx, B, \mathcal{P}^\vee) \\
\Big\downarrow {\scriptstyle GenTR} & & \Big\downarrow {\scriptstyle GenTR} \\
\{\omega_n^{(g)}\} & \xleftrightarrow{\;xy\ swap\;} & \{\omega_n^{\vee,(g)}\}
\end{array}
$$

Generalized TR: xy swap duality

xy duality transformation: an explicit closed formula $\{\omega_n^{(g)}\} \longleftrightarrow \{\omega_n^{\vee,(g)}\}$

Theorem (Compatibility with xy swap)

$$\begin{array}{ccc} (\Sigma, dx, dy, B, \mathcal{P}) & & (\Sigma, dy, dx, B, \mathcal{P}^\vee) \\ \downarrow{\scriptstyle GenTR} & & \downarrow{\scriptstyle GenTR} \\ \{\omega_n^{(g)}\} & \xleftrightarrow{xy\ swap} & \{\omega_n^{\vee,(g)}\} \end{array}$$

Remark. The actual definition of GenTR is motivated by the validity of this theorem

GenTR relation for $\omega_n^{(g)}$ $\Leftrightarrow$ $\omega_n^{\vee,(g)}$ is regular at $\mathcal{P}$

Generalized TR: xy swap duality

Corollary

$\Sigma = \mathbb{C}P^1$, $\mathcal{P}^\vee = \emptyset$ (*all special points are treated as key-special*). *Then,* $\omega_n^{\vee,(g)} = 0$ *for* $2g - 2 + n > 0$ *and an explicit formula* $(*)$ *for* $\omega_n^{(g)}$ *holds.*

Generalized TR: xy swap duality

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$$\hat z(z,v) = e^{\frac{v\hbar}{2}\partial_y} z, \quad \hat z_i^\pm = \hat z(z_i, \pm v_i), \qquad \mathcal{S}(u) = \frac{e^{u/2}-e^{-u/2}}{u},$$

$$\mathbb{W}_n^\vee(z_1,v_1,\dots,z_n,v_n) = \sum_{g\ge 0} \hbar^{2g-2+n}\mathbb{W}_n^{\vee,(g)}$$

$$(*) \qquad = \prod_{i=1}^n \left(e^{v_i \mathcal{S}(v_i\hbar\partial_{y_i})x_i} \sqrt{\frac{d\hat z_i^+}{dz_i}\frac{d\hat z_i^-}{dz_i}}\, dz_i \right) (-1)^{n-1} \sum_{\sigma\in\mathrm{cycl}(n)} \prod_{i=1}^n \frac{1}{\hat z_i^+ - \hat z_{\sigma(i)}^-}$$

$$\frac{(-1)^n \omega_n^{(g)}}{\prod_{i=1}^n dx_i} = \sum_{k_1,\dots,k_n\ge 0} (-\partial_{x_1})^{k_1}\dots(-\partial_{x_n})^{k_n} \, [v_1^{k_1}\dots v_n^{k_n}] \left(\prod_{i=1}^n \frac{e^{-v_i x_i}}{dx_i}\right) \mathbb{W}_n^{\vee,(g)}$$

Generalized TR: KP integrability

Theorem (KP integrability)

If $\Sigma = \mathbb{C}P^1$, then GenTR differentials are KP integrable

(see the talk of Sasha Alexandrov for details)

Corollary

GenTR potential for $\begin{cases} dx = z^{r-1}dz \\ dy = z^{s-1}dz \end{cases}$ *is a solution of KP hierarchy for any* (r, s), $r + s > 0$

Example: $(r, s) = (1, 2)$

Example

$$\begin{cases} x = z, \\ y = z^2. \end{cases}$$

Special points $= \{0\}$

$\mathcal{P}$	$\mathcal{P}^\vee$	GenTR	GenTR$^\vee$
$\emptyset$	$\{0\}$	trivial	KW
$\{0\}$	$\emptyset$	new!	trivial

Expansion point: $z = \infty$, expansion local coordinate: $1/z$

$$F = -\tfrac{1}{48} p_2 \hbar + (\tfrac{1}{96} p_1^4 - \tfrac{1}{96} p_2^2)\hbar^2 + (\tfrac{1}{48} p_2 p_1^4 + \tfrac{1}{24} p_4 p_1^2 - \tfrac{1}{144} p_2^3 - \tfrac{9}{1280} p_6)\hbar^3 + (\tfrac{9}{640} p_3 p_1^5 + \tfrac{1}{32} p_2^2 p_1^4 + \tfrac{125}{1152} p_5 p_1^3 + \tfrac{9}{256} p_3^2 p_1^2 + \tfrac{1}{8} p_2 p_4 p_1^2 + \tfrac{343}{2880} p_7 p_1 + \tfrac{29}{2880} p_4^2 - \tfrac{1}{192} p_2^4 - \tfrac{27}{1280} p_2 p_6)\hbar^4 + O(\hbar^5)$$

This potential is a solution of KP hierarchy.
No enumerative meaning of its coefficients is known!

Example: $(r,s)=(1,2)$

Example

$$\begin{cases} x = z, \\ y = z^2. \end{cases}$$

Special points $=\{0\}$

$\mathcal{P}$	$\mathcal{P}^\vee$	GenTR	GenTR$^\vee$
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CEO TR deformation: $\begin{cases} x = z + \frac{\epsilon}{z^2}, \\ y = z^2 \end{cases}$

To be explained

Still missing:

- GenTR recursion formula for $\tilde{\omega}_n^{(g)}$
- compatibility with CEO and other versions of TR
- xy swap formula
- symplectic duality as a generalization of xy duality, and closed formulas for BE TR differentials as a special case of symplectic duality (not this time. . .)
- definition of KP integrability (the talk of A. Alexandrov)

xy swap transformation at a nice point

$$\{\omega_n^{(g)}\} \longleftrightarrow \{\omega_n^{\vee,(g)}\}$$

$$\omega_1^{(0)}(z) = y\,dx \qquad \omega_1^{\vee,(0)}(z) = x\,dy$$

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Definition

A point $o \in \Sigma$ is called *nice* if $x = \log z + O(z)$, $y = \log z + O(z)$

$$dx = \frac{dz}{z} + (\text{holomorphic}), \quad dy = \frac{dz}{z} + (\text{holomorphic})$$

Then, $X = e^x$ and $Y = e^y$ can serve as local coordinates

$$dx = \frac{dX}{X}, \quad dy = \frac{dY}{Y}, \quad \partial_x = X\partial_X, \quad \partial_y = Y\partial_Y.$$

xy swap transformation at a nice point

$$\{\omega_n^{(g)}\} \stackrel{(o,X)}{\longleftrightarrow} F \longleftrightarrow e^F \xrightarrow{e^{-\hbar Q}} e^{-\hbar Q} e^F = e^{F^\vee} \longleftrightarrow F^\vee \stackrel{(o,Y)}{\longleftrightarrow} \{(-1)^n \omega_n^{\vee,(g)}\}$$

$$Q = \frac{1}{2} \sum_{i,j} \Big((i+j) p_i p_j \tfrac{\partial}{\partial p_{i+j}} + i\, j\, p_{i+j} \tfrac{\partial^2}{\partial p_i \partial p_j} \Big)$$

$$\omega_n = \sum_{g \geq 0} \hbar^{2g-2+n} \omega_n^{(g)} \qquad \omega_n^\vee = \sum_{g \geq 0} \hbar^{2g-2+n} \omega_n^{(g)}$$

$$\omega_n - \delta_{n,2} \tfrac{dX_1 dX_2}{(X_1 - X_2)^2} - \delta_{n,1} \hbar^{-1} x_1 dx_1 = \sum_{k_1,\dots,k_n \geq 1} \frac{\partial^n F}{\partial p_{k_1} \dots \partial p_{k_n}} \Big|_{p=0} \textstyle\prod_{i=1}^n d(X_i^{k_i})$$

$$(-1)^n \omega_n^\vee - \delta_{n,2} \tfrac{dY_1 dY_2}{(Y_1 - Y_2)^2} + \delta_{n,1} \hbar^{-1} y_1 dy_1 = \sum_{k_1,\dots,k_n \geq 1} \frac{\partial^n F^\vee}{\partial p_{k_1} \dots \partial p_{k_n}} \Big|_{p=0} \textstyle\prod_{i=1}^n d(Y_i^{k_i})$$

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Theorem

The composition $\{\omega_n^{(g)}\} \longmapsto \{\omega_n^{\vee,(g)}\}$ is given by a closed finite expression that extends to a transformation of global meromorphic differentials and does not involve any information on a chosen expansion point o (and neither requires a very existence of a nice point).

The obtained transformation is called the *xy swap duality*

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Remark. The potentials F and $F^\vee$ do depend on the expansion point and a choice of local coordinates. The treatment of xy duality as the action of $e^{-\hbar Q}$ on the corresponding partition function is valid for a *nice point* only

xy swap: the formula

$$W_n(z_1,u_1,\dots,z_n,u_n) = \Big(\prod_{i=1}^n u_i\hbar\mathcal{S}(u_i\hbar\partial_{x_i})\Big)\frac{\omega_n}{\prod_{i=1}^n dx_i},\qquad \mathcal{S}(u)=\frac{e^{u/2}-e^{-u/2}}{u}$$

$$\mathbb{W}_n(z_1,u_1,\dots,z_n,u_n) = \prod_{i=1}^n \frac{dx_i}{u_i\hbar}\sum_{\gamma\in\Gamma_n}\frac{1}{|\mathrm{Aut}(\gamma)|}\prod_{e\in E(\gamma)} W_{|e|}(z_{e_1},u_{e_1},\dots,z_{e_{|e|}},u_{e_{|e|}})$$

$$\frac{(-1)^n\omega_n^{\vee,(g)}}{\prod_{i=1}^n dy_i} = \sum_{k_1,\dots,k_n\geq 0}(-\partial_{y_1})^{k_1}\dots(-\partial_{y_n})^{k_n}\,[u_1^{k_1}\dots u_n^{k_n}]\Big(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\Big)\,\mathbb{W}_n^{(g)}$$

Γ_n is the set of *hypergraphs* (graphs with hyperedges) with n marked vertices

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Corrections and details. 1. The dependence of $\Big(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\Big)\mathbb{W}_n^{(g)} = [\hbar^{2g-2+n}]\Big(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\Big)\mathbb{W}_n$ in u-variables is polynomial.

2. If $|e|=2$ and $e(1)=e(2)$, use the regularized differential $\omega_2(\tilde z_1,\tilde z_2)-\frac{d\tilde x_1 d\tilde x_2}{(\tilde x_1-\tilde x_2)^2}$ instead in the definition of the edge contribution $W_{|e|}$.

xy swap: basic properties

1. $\omega_1^{\vee,(0)} = x\,dy$, $\omega_2^{\vee,(0)} = \omega_2^{(0)}$
2. $2g-2+n>0$: $\omega_n^{\vee,(g)}$ is globally defined and meromorphic
3. Moreover, it is *regular on diagonals*
4. The *inverse transformation* is given by the same formulas with x and y swapped

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3. Moreover, it is *regular on diagonals*
4. The *inverse transformation* is given by the same formulas with x and y swapped

Remark. We aware of no direct combinatorial proof of the last two properties. The arguments we are using involve computation in the space of power expansions at a (nice) point

Example

$$\omega_3^{(0)} + \omega_3^{\vee,(0)} = d_1 \frac{B(z_1,z_2)B(z_1,z_3)}{dx_1\,dy_1} + d_2 \frac{B(z_2,z_3)B(z_2,z_1)}{dx_2\,dy_2} + d_3 \frac{B(z_3,z_1)B(z_3,z_2)}{dx_3\,dy_3}$$

Derivation of xy swap formula, Step 1: inclusion/exclusion

$$\{\omega_n^{(g)}\} \xleftrightarrow{(o,X)} F \longleftrightarrow e^F \xrightarrow{e^{-\hbar Q}} e^{-\hbar Q}e^F = e^{F^\vee} \longleftrightarrow F^\vee \xleftrightarrow{(o,Y)} \{(-1)^n\omega_n^{\vee,(g)}\}$$

Notation: $\langle F\rangle = F\big|_{p=0}$, 'taking the free term of a series',

$$J^+(X) = \sum_{k=1}^{\infty} kX^k\partial_{p_k}$$

Then,

$$\frac{\omega_n}{\prod_{i=1}^n dx_i} - \delta_{n,2}\frac{X_1X_2}{(X_1-X_2)^2} = \sum_{k_1,\dots,k_n} \frac{\partial^n F}{\partial p_{k_1}\dots\partial p_{k_n}}\Big|_{p=0} \prod_{i=1}^n k_iX_i^{k_i}$$
$$= \langle J^+(X_1)\dots J^+(X_n)F\rangle$$

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$$\begin{aligned}\frac{\omega_n}{\prod_{i=1}^n dx_i} - \delta_{n,2}\frac{X_1X_2}{(X_1-X_2)^2} &= \sum_{k_1,\dots,k_n} \frac{\partial^n F}{\partial p_{k_1}\dots\partial p_{k_n}}\Big|_{p=0} \prod_{i=1}^n k_i X_i^{k_i} \\ &= \langle J^+(X_1)\dots J^+(X_n)F\rangle \\ &= \langle J^+(X_1)\dots J^+(X_n)e^F\rangle^\circ\end{aligned}$$

Derivation of xy swap formula, Step 1: inclusion/exclusion

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Then,

$$\frac{\omega_n}{\prod_{i=1}^n dx_i} - \delta_{n,2} \frac{X_1 X_2}{(X_1 - X_2)^2} = \big\langle J^+(X_1) \dots J^+(X_n) e^F \big\rangle^\circ$$

where the 'connected' correlators are defined through inclusion/exclusion

$$\big\langle J^+(X) e^F \big\rangle = \big\langle J^+(X) e^F \big\rangle^\circ$$

$$\big\langle J^+(X_1) J^+(X_2) e^F \big\rangle = \big\langle J^+(X_1) J^+(X_2) e^F \big\rangle^\circ + \big\langle J^+(X_1) e^F \big\rangle^\circ \big\langle J^+(X_2) e^F \big\rangle^\circ$$

$$\dots$$

$$\big\langle J^+(X_1) \dots J^+(X_n) e^F \big\rangle = \sum_{\sqcup I_\alpha = \{1,\dots,n\}} \prod_\alpha \big\langle \prod_{i \in I_\alpha} J^+(X_i) e^F \big\rangle^\circ$$

Derivation of xy swap formula, Step 1: inclusion/exclusion

$$\{\omega_n^{(g)}\} \overset{(o,X)}{\longleftrightarrow} F \longleftrightarrow e^F \xrightarrow{e^{-\hbar Q}} e^{-\hbar Q} e^F = e^{F^\vee} \longleftrightarrow F^\vee \overset{(o,Y)}{\longleftrightarrow} \{(-1)^n \omega_n^{\vee,(g)}\}$$

Notation: $\langle F \rangle = F\,\big|_{p=0}$, 'taking the free term of a series',

$$J^+(X) = \sum_{k=1}^{\infty} k X^k \partial_{p_k}$$

Then,

$$\frac{\omega_n}{\prod_{i=1}^n dx_i} - \delta_{n,2} \frac{X_1 X_2}{(X_1 - X_2)^2} = \langle J^+(X_1) \dots J^+(X_n) e^F \rangle^\circ$$

Moreover, define

$$J(X) = \sum_{k=-\infty}^{\infty} X^k J_k, \qquad J_k = \begin{cases} k\,\partial_{p_k}, & k > 0, \\ 0, & k = 0, \\ p_{-k}, & k < 0 \end{cases}$$

Then,

$$\frac{\omega_n}{\prod_{i=1}^n dx_i} = \langle J(X_1) \dots J(X_n) e^F \rangle^\circ$$

(with the singular $(0,2)$ correction taken into account automatically)

Derivation of xy swap formula, Step 1: inclusion/exclusion

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$$Q = \frac{1}{2} \sum_{i,j} \Big((i+j) p_i p_j \tfrac{\partial}{\partial p_{i+j}} + i\, j\, p_{i+j} \tfrac{\partial^2}{\partial p_i \partial p_j} \Big)$$

Similarly,

$$\frac{(-1)^n \omega_n^\vee}{\prod_{i=1}^n dy_i} = \big\langle J(Y_1) \dots J(Y_n)\, e^{F^\vee} \big\rangle^\circ$$

Derivation of xy swap formula, Step 1: inclusion/exclusion

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Similarly,

$$\begin{aligned}
\frac{(-1)^n \omega_n^\vee}{\prod_{i=1}^n dy_i} &= \big\langle J(Y_1)\dots J(Y_n)\; e^{F^\vee}\big\rangle^\circ \\
&= \big\langle J(Y_1)\dots J(Y_n)\; e^{-\hbar Q} e^F\big\rangle^\circ \\
&= \big\langle \mathbb{J}(Y_1)\dots \mathbb{J}(Y_n)\; e^{F}\big\rangle^\circ, \qquad \mathbb{J}(Y) = e^{\hbar Q} J(Y) e^{-\hbar Q}
\end{aligned}$$

The next step: to compute the operator $\mathbb{J}(Y) = e^{\hbar Q} J(Y) e^{-\hbar Q}$ acting on $\mathbb{C}[[p_1, p_2, \dots]]$

Derivation of xy swap formula, Step 2: computation of $e^{\hbar Q} J(Y) e^{-\hbar Q}$

Main tool: *bosonic representation of $\widehat{\mathfrak{gl}}(\infty)$* on $\mathbb{C}[[p_1, p_2, \dots]]$

$$\sum_{i,j\in\mathbb{Z}} z_1^j z_2^{-i-1} E_{i,j} = \frac{e^{\sum_{i<0} \frac{z_1^i - z_2^i}{i} J_i} e^{\sum_{i>0} \frac{z_1^i - z_2^i}{i} J_i} - 1}{z_1 - z_2}$$

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Universal bosonic operator:

$z_1 = Xe^{u/2}$, $z_2 = Xe^{-u/2}$, $\partial_x = X\partial_X$,

$$\mathcal{E}(X,u) = \sum_{k,m\in\mathbb{Z}} X^m e^{u(k+\frac{1-m}{2})} E_{k-m,m} + \frac{1}{u\mathcal{S}(u)} = \frac{e^{\sum_{i<0} u\mathcal{S}(ui)X^i J_i} e^{\sum_{i>0} u\mathcal{S}(ui)X^i J_i}}{u\mathcal{S}(u)}$$

$$= \frac{e^{u\mathcal{S}(u\partial_x)\sum_{i<0} X^i J_i} e^{u\mathcal{S}(u\partial_x)\sum_{i>0} X^i J_i}}{u\mathcal{S}(u)}$$

Derivation of xy swap formula, Step 2: computation of $e^{\hbar Q}J(Y)e^{-\hbar Q}$

Main tool: *bosonic representation of* $\widehat{\mathfrak{gl}}(\infty)$ on $\mathbb{C}[[p_1, p_2, \dots]]$

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All operators involved belong to $\widehat{\mathfrak{gl}}(\infty)$:

$$J_m = [X^m u^0]\mathcal{E}(X,u) = \sum_{k\in\mathbb{Z}} E_{k-m,k}, \qquad J(X) = [u^0]\mathcal{E}(X,u),$$

$$Q = [X^0 u^2]\mathcal{E}(X,u) = \frac{1}{2}\sum_{k\in\mathbb{Z}} \left(k+\tfrac{1}{2}\right)^2 E_{k,k}$$

$$\mathbb{J}(Y) = e^{\hbar Q} J(Y) e^{-\hbar Q} = \sum_{k,m\in\mathbb{Z}} \frac{e^{-\frac{\hbar}{2}\left(k+\frac{1}{2}\right)^2}}{e^{-\frac{\hbar}{2}\left(k-m+\frac{1}{2}\right)^2}} Y^m E_{k-m,k}$$

Lemma

$$\mathbb{J}(Y)=e^{\hbar Q}J(Y)e^{-\hbar Q}=\sum_{j=0}^{\infty}(-\partial_y)^j[u^j]e^{-u(y-x)}\frac{dx}{dy}\mathcal{E}(X,u\hbar)$$

Or, taking the coefficient of $E_{k-m,k}$,

$$\frac{e^{-\frac{\hbar}{2}\left(k+\frac{1}{2}\right)^2}}{e^{-\frac{\hbar}{2}\left(k-m+\frac{1}{2}\right)^2}}Y^m=\sum_{j=0}^{\infty}(-Y\partial_Y)^j[u^j]\left(\tfrac{X}{Y}\right)^u e^{u\hbar\left(k+\frac{1-m}{2}\right)}\frac{dX}{X}\frac{Y}{dY}X^m$$

Substituting, we obtain

$$\frac{(-1)^n\omega_n^\vee}{\prod_{i=1}^n dy_i}=\Big\langle\prod_{i=1}^n\mathbb{J}(Y_i)\,e^F\Big\rangle^\circ=\sum_{k_1,\dots,k_n\geq 0}(-\partial_{y_1})^{k_1}\dots(-\partial_{y_n})^{k_n}[u_1^{k_1}\dots u_n^{k_n}]\left(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\right)\mathbb{W}_n$$

where $\quad\mathbb{W}_n=\left(\prod_{i=1}^n e^{u_ix_i}dx_i\right)\Big\langle\prod_{i=1}^n\mathcal{E}(X_i,u_i\hbar)\,e^F\Big\rangle^\circ$

Derivation of xy swap formula, Step 3: computation of $\mathbb{W}_n$

$$\mathbb{W}_n = \Big(\prod_{i=1}^{n} e^{u_i x_i} dx_i\Big) \left\langle \mathcal{E}(X_1, u_1\hbar) \dots \mathcal{E}(X_n, u_n\hbar)\, e^F \right\rangle^\circ$$

- Insert $\mathcal{E}(X, u\hbar) = \frac{e^{u\hbar \mathcal{S}(u\hbar\partial_x)\sum_{i<0} X^i J_i} e^{u\hbar \mathcal{S}(u\hbar\partial_x)\sum_{i>0} X^i J_i}}{u\hbar\mathcal{S}(u\hbar)}$,
- expand the exponents,
- apply inclusion/exclusion.

The result is an expression for $\mathbb{W}_n$ in terms of $\omega_{n'}$ via *summation over hypergraphs*

Derivation of xy swap formula, Step 3: computation of $\mathbb{W}_n$

$$\mathbb{W}_n = \Big(\prod_{i=1}^{n} e^{u_i x_i} dx_i\Big) \left\langle \mathcal{E}(X_1, u_1\hbar)\dots\mathcal{E}(X_n, u_n\hbar)\, e^F \right\rangle^\circ$$

- Insert $\mathcal{E}(X, u\hbar) = \dfrac{e^{u\hbar\mathcal{S}(u\hbar\partial_x)\sum_{i<0} x^i J_i}\, e^{u\hbar\mathcal{S}(u\hbar\partial_x)\sum_{i>0} x^i J_i}}{u\hbar\mathcal{S}(u\hbar)}$,
- expand the exponents,
- apply inclusion/exclusion.

The result is an expression for $\mathbb{W}_n$ in terms of $\omega_{n'}$ via *summation over hypergraphs*

$$W_n(z_1, u_1, \dots, z_n, u_n) = \Big(\prod_{i=1}^{n} u_i\hbar\mathcal{S}(u_i\hbar\partial_{x_i})\Big) \frac{\omega_n}{\prod_{i=1}^{n} dx_i},$$

$$\mathbb{W}_n(z_1, u_1, \dots, z_n, u_n) = \Big(\prod_{i=1}^{n} \frac{dx_i}{u_i\hbar}\Big) \sum_{\gamma\in\Gamma_n} \frac{1}{|\mathrm{Aut}(\gamma)|} \prod_{e\in E(\gamma)} W_{|e|}(z_{e_1}, u_{e_1}, \dots, z_{e_{|e|}}, u_{e_{|e|}})$$

xy swap transformation: summary

Nice point: $x = \log z + O(z)$, $y = \log z + O(z)$

The action on the *partition functions* associated with the local coordinates $X = e^x$, $Y = e^y$:

by the operator $e^{-\hbar Q}$, $\qquad Q = \frac{1}{2}\sum_{i,j}\Big((i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + i\,j\,p_{i+j}\frac{\partial^2}{\partial p_i \partial p_j}\Big)$

$$\{\omega_n^{(g)}\} \overset{(o,X)}{\longleftrightarrow} F \longleftrightarrow e^F \overset{e^{-\hbar Q}}{\longrightarrow} e^{-\hbar Q} e^F {=} e^{F^\vee} \longleftrightarrow F^\vee \overset{(o,Y)}{\longleftrightarrow} \{(-1)^n \omega_n^{\vee,(g)}\}$$

xy swap transformation: summary

Nice point: $x = \log z + O(z)$, $y = \log z + O(z)$

The action on the *power expansions of the differentials* at a nice point:

$$\omega_n \sim \Big\langle \prod J(X_i)\, e^F \Big\rangle^\circ \xrightarrow{\;xy \text{ swap}\;} \omega_n^\vee \sim \Big\langle \prod J(Y_i)\, e^{F^\vee} \Big\rangle^\circ = \Big\langle \prod \mathbb{J}(Y_i)\, e^F \Big\rangle^\circ$$

summation over graphs ($\omega_n \to \mathbb{W}_n$)

$$\mathbb{W}_n \sim \Big\langle \prod \mathcal{E}(X_i, u_i\hbar)\, e^F \Big\rangle^\circ$$

$\prod_{i=1}^{n} \sum_{k\ge 0} \partial_{y_i}^k [u_i^k]$ ($\mathbb{W}_n \to \omega_n^\vee$)

$$J(X) = \sum_{k=-\infty}^{\infty} J_k X^k, \qquad \mathbb{J}(Y) = e^{\hbar Q} J(Y) e^{-\hbar Q},$$

$$\mathcal{E}(X,u) = \frac{e^{u\mathcal{S}(u\partial_x)\sum_{i<0} X^i J_i}\, e^{u\mathcal{S}(u\partial_x)\sum_{i>0} X^i J_i}}{u\mathcal{S}(u)}, \qquad \partial_x = X\partial_X, \quad \partial_y = Y\partial_Y$$

xy swap transformation: summary

The action on *global meromorphic differentials*:

$$W_n(z_1,u_1,\dots,z_n,u_n) = \Big(\prod_{i=1}^n u_i\hbar\mathcal{S}(u_i\hbar\partial_{x_i})\Big)\,\frac{\omega_n}{\prod_{i=1}^n dx_i},\qquad \mathcal{S}(u)=\frac{e^{u/2}-e^{-u/2}}{u}$$

$$\mathbb{W}_n(z_1,u_1,\dots,z_n,u_n) = \Big(\prod_{i=1}^n \frac{dx_i}{u_i\hbar}\Big)\sum_{\gamma\in\Gamma_n}\frac{1}{|\mathrm{Aut}(\gamma)|}\prod_{e\in E(\gamma)} W_{|e|}(z_{e_1},u_{e_1},\dots,z_{e_{|e|}},u_{e_{|e|}})$$

(with a regularization of certain singular $(0,2)$ contributions)

$$\frac{(-1)^n\omega_n^{\vee,(g)}}{\prod_{i=1}^n dy_i} = \sum_{k_1,\dots,k_n\ge 0}(-\partial_{y_1})^{k_1}\dots(-\partial_{y_n})^{k_n}\,[u_1^{k_1}\dots u_n^{k_n}]\,\Big(\prod_{i=1}^n\frac{e^{-u_iy_i}}{dy_i}\Big)\,\mathbb{W}_n^{(g)}$$

More properties of xy swap

- The xy swap transformation produces no singularities apart from the special points: if $\omega_n^{(g)}$ is regular at some non-special point for all (g, n) with $2g - 2 + n > 0$, then the same holds for $\omega_n^{\vee,(g)}$
- This property motivates the definition of GenTR: it is *defined* by the requirement that all xy dual differentials are holomorphic at the key-special points. Then the compatibility GenTR with xy swap becomes a reformulation of the definition:

$$\begin{array}{ccc} (\Sigma, dx, dy, B, \mathcal{P}) & & (\Sigma, dy, dx, B, \mathcal{P}^\vee) \\ \downarrow \text{GenTR} & & \downarrow \text{GenTR} \\ \{\omega_n^{(g)}\} & \xleftrightarrow{xy \text{ swap}} & \{\omega_n^{\vee,(g)}\} \end{array}$$

More concretely, this idea is realized below

Partial xy swap duality and definition of GenTR

In the power expansions at a nice point:

$$\omega_n \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\ J(X)\ e^F \Big\rangle^\circ \longrightarrow \omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\ \mathbb{J}(Y)\ e^F \Big\rangle^\circ$$

combinatorial expression

$$\mathcal{W}_n \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\ \mathcal{E}(X, u\hbar)\ e^F \Big\rangle^\circ$$

$$\sum_{k\geq 0} \partial_y^k [u^k]$$

- ω_n, $\mathcal{W}_n$, $\omega^{(g)}_{n-1,1}$ extend globally on Σ^n
- $\omega^{(g)}_{n-1,1}$ is holomorpfic in $z = z_n$ at $q \in \mathcal{P}$ for $i = 1, \dots, n$ and all (g, n) iff $\omega^{\vee,(g)}_n$ is holomorphic at q in all z_i's for all (g, n)

Definition of GenTR

$$\mathcal{T}_n(z_{[\![n-1]\!]};z,u)=\sum_{k=1}^{\infty}\frac{1}{k!}\prod_{i=1}^{k}\Big(\big\lfloor_{\tilde z_i\to z}\,u\hbar\mathcal{S}\big(u\hbar\tfrac{d}{d\tilde x_i}\big)\tfrac{1}{d\tilde x_i}\Big)\big(\omega_{n-1+k}(z_{[\![n-1]\!]},\tilde z_{[\![k]\!]})-\delta_{n,1}\delta_{k,2}\tfrac{d\tilde x_1 d\tilde x_2}{(\tilde x_1-\tilde x_2)^2}\big),$$

$$\mathcal{W}_n(z_{[\![n-1]\!]};z,u)=\frac{dx}{u\hbar}\,e^{\mathcal{T}_1(z,u)}\sum_{[\![n]\!]=\sqcup_\alpha J_\alpha,\ J_\alpha\neq\emptyset}\ \prod_\alpha \mathcal{T}_{|J_\alpha|+1}(z_{J_\alpha};z,u)$$

$$\omega^{(g)}_{n-1,1}(z_{[\![n-1]\!]},z)=-dy\sum\nolimits_{r\geq 0}(-\partial_y)^r[u^r]e^{-u\,y}\,\frac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]};z,u)}{dy}$$

$$=-\omega^{(g)}_n(z_{[\![n-1]\!]},z)-\underbrace{dy\sum_{r\geq 1}(-\partial_y)^r[u^r]e^{-u\,y}\,\frac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]};z,u)}{dy}}_{\text{involves } \omega^{(g')}_{n'}\text{'s with } 2g'-2+n'<2g-2+n}$$

Definition of GenTR

$$\mathcal{T}_n(z_{[\![n-1]\!]};z,u)=\sum_{k=1}^{\infty}\frac{1}{k!}\prod_{i=1}^{k}\Big(\lfloor_{\tilde z_i\to z}\,u\hbar\mathcal{S}(u\hbar\tfrac{d}{d\tilde x_i})\tfrac{1}{d\tilde x_i}\Big)\big(\omega_{n-1+k}(z_{[\![n-1]\!]},\tilde z_{[\![k]\!]})-\delta_{n,1}\delta_{k,2}\tfrac{d\tilde x_1 d\tilde x_2}{(\tilde x_1-\tilde x_2)^2}\big),$$

$$\mathcal{W}_n(z_{[\![n-1]\!]};z,u)=\frac{dx}{u\hbar}\,e^{\mathcal{T}_1(z,u)}\sum_{[\![n]\!]=\sqcup_\alpha J_\alpha,\ J_\alpha\neq\emptyset}\ \prod_\alpha \mathcal{T}_{|J_\alpha|+1}(z_{J_\alpha};z,u)$$

$$\begin{aligned}\omega^{(g)}_{n-1,1}(z_{[\![n-1]\!]},z)&=-dy\sum\nolimits_{r\geq 0}(-\partial_y)^r[u^r]e^{-u\,y}\tfrac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]};z,u)}{dy}\\&=-\omega^{(g)}_n(z_{[\![n-1]\!]},z)+\tilde\omega^{(g)}_n(z_{[\![n-1]\!]},z)\end{aligned}$$

Definition (Differentials $\tilde\omega^{(g)}_n$ of Generalized Topological Recursion)

$$\tilde\omega^{(g)}_n=-dy\sum_{r\geq 1}(-\partial_y)^r[u^r]e^{-u\,y}\frac{\mathcal{W}^{(g)}_n(z_{[\![n-1]\!]};z,u)}{dy}$$

Compatibility of GenTR with xy duality

In a sense, compatibility of GenTR with xy swap is implicitly implemented to the definition of GenTR:

$$\omega^{(g)}_{n-1,1}(z_{[\![n-1]\!]},z) = -\omega^{(g)}_{n}(z_{[\![n-1]\!]},z) + \tilde{\omega}^{(g)}_{n}(z_{[\![n-1]\!]},z)$$

$\{\omega^{(g)}_n\}$ satisfy GenTR

$$\Longrightarrow \quad \omega^{(g)}_{n-1,1} \text{ is holomorphic in } z \text{ at key-special points}$$

$$\Longrightarrow \quad \omega^{\vee,(g)}_{n} \text{ is holomorphic in } z_i\text{'s at key-special points}$$

By the same reason,

$$\{\omega^{(g)}_n\} \text{ are holomorphic at } q \in \mathcal{P}^\vee \quad \Longrightarrow \quad \{\omega^{\vee,(g)}_n\} \text{ satisfy GenTR}^\vee$$

Example: intersection numbers with r-spin Chiodo classes

$$\begin{cases} x = \log z - z^r \\ y = z^s \end{cases}, \qquad dx = (1 - r\, z^r)\frac{dz}{z}$$

Q. CEO TR for this curve is well defined. But how to apply it in practice for small (g, n)?

A. Apply *xy swap* and *GenTR* on the other side of duality!

(This example demonstrates the benefit from using both concepts)

Example: intersection numbers with r-spin Chiodo classes

$$\begin{cases} x = \log z - z^r \\ y = z^s \end{cases}, \quad dx = (1 - r\,z^r)\frac{dz}{z}, \qquad \begin{cases} x^\vee = z^s \\ y^\vee = \log z + O(z) \end{cases}$$

n-point functions of the dual (generalized) TR for $r = 3$, $s = 2$:

$$H_3^{\vee,(0)} = 0, \quad H_1^{\vee,(1)} = -\frac{1}{48} z_1^{-2}, \quad H_2^{\vee,(1)} = -\frac{1}{48}(z_1^{-1} z_2^{-3} + z_1^{-3} z_2^{-1})$$

Example: intersection numbers with r-spin Chiodo classes

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n-point functions of the original (CEO) TR, by xy duality:

$$H_3^{(0)} = \tfrac{3}{2} \frac{(z_1 + z_2 + z_3)(3z_1 z_2 z_3 + 1) + 3(z_1^2 z_2^2 + z_3^2 z_2^2 + z_1^2 z_3^2)}{(1 - 3z_1^3)(1 - 3z_2^3)(1 - 3z_3^3)}$$

$$H_1^{(1)} = \tfrac{1}{16}(3\,\partial_{x_1} - 1)\frac{z_1}{1 - 3z_1^3}$$

$$H_2^{(1)} = \tfrac{3}{32} \sum_{k_1,k_2,k_3} (-\partial_{y_1})^{k_1}(-\partial_{y_2})^{k_2}(-\partial_{y_3})^{k_3}[u_1^{k_1} u_2^{k_2} u_3^{k_3}]$$

$$\frac{(u_1 + u_2)z_1 z_2 + 3(u_1^2 + u_2 u_1 + u_2^2)(z_1^2 + z_2 z_1 + z_2^2) - 2(z_1^2 + z_2^2)}{(1 - 3z_1^3)(1 - 3z_2^3)}$$

Loop equations

$(\Sigma, dx, dy, B, \mathcal{P})$ initial spectral curve data of CEO TR
$q \in \mathcal{P}$ one of zeroes of dx, $\quad K = (2, \dots, n)$

The **Linear and Quadratic Loop equations** are an equivalent reformulation of CEO defining relations for the principal part of the pole of $\omega_n^{(g)}$ at $z = q$:

The differentials

$$\omega_n^{(g)}(z, z_K) + \omega_n^{(g)}(\sigma(z), z_K)$$
$$\frac{1}{2dx(z)}\Big(\omega_{n+1}^{(g-1)}(z, \sigma(z), z_K) + \sum_{\substack{g_1+g_2=g\\ J_1\sqcup J_2=K}} \omega_{|J_1|+1}^{(g_1)}(z, z_{J_1})\omega_{|J_2|+1}^{(g_2)}(\sigma(z), z_{J_2})\Big)$$

are holomorphic at $z = q$.

Loop equations

$(\Sigma, dx, dy, B, \mathcal{P})$ initial spectral curve data of CEO TR
$q \in \mathcal{P}$ one of zeroes of dx, $\quad K = (2, \dots, n)$

The **Linear and Quadratic Loop equations** are an equivalent reformulation of CEO defining relations for the principal part of the pole of $\omega_n^{(g)}$ at $z = q$:

Equivalently, the differentials

$$\mathcal{W}_n^{(g),0} = \omega_n^{(g)}(z, z_K)$$
$$\mathcal{W}_n^{(g),1} = \frac{1}{2dx(z)}\Big(\omega_{n+1}^{(g-1)}(z, z, z_K) + \sum_{\substack{g_1+g_2=g \\ J_1 \sqcup J_2 = K}} \omega_{|J_1|+1}^{(g_1)}(z, z_{J_1})\omega_{|J_2|+1}^{(g_2)}(z, z_{J_2})\Big)$$

have a pole at $z = q$ with odd principal part with respect to σ

Higher loop equations for CEO TR differentials

Define $K=(2,\dots,n)$,

$$\mathcal{T}_n(z,u;z_K)=\sum_{k=1}^{\infty}\frac{1}{k!}\prod_{i=1}^{k}\Big(\lfloor_{\tilde z_i\to z}\,u\hbar\mathcal{S}(u\hbar\tfrac{d}{d\tilde x_i})\tfrac{1}{d\tilde x_i}\Big)\big(\omega_{n-1+k}(\tilde z_{[\![k]\!]},z_K)-\delta_{n,1}\delta_{k,2}\tfrac{d\tilde x_1 d\tilde x_2}{(\tilde x_1-\tilde x_2)^2}\big),$$

$$\mathcal{W}_n(z,u;z_K)=\frac{dx}{u\hbar}\,e^{\mathcal{T}_1(z,u)}\sum_{K=\sqcup_\alpha J_\alpha,\; J_\alpha\neq\emptyset}\prod_\alpha\mathcal{T}_{|J_\alpha|+1}(z,u;z_{J_\alpha})$$

$$\mathcal{W}_n^{(g),k}=[u^k]\mathcal{W}_n^{(g)}$$

Then, $\mathcal{W}_n^{(g),0}=[u^0]\mathcal{W}_n^{(g)}$, $\mathcal{W}_n^{(g),1}=[u^1]\mathcal{W}_n^{(g)}$ are the same as above

Theorem (Higher Loop Equations for CEO TR differentials)

The pole of $\mathcal{W}_n^{(g),k}$ at $z=q\in\mathcal{P}$ has odd principal part for any $k\geq 0$.

Higher loop equations

$(\Sigma, dx, dy, B, \mathcal{P})$ GenTR spectral curve data

$q \in \mathcal{P}$ a key-special point with exponents (r, s) such that $r \geq 2$ and $s = 1$, that is:

- x has a critical point at q of multiplicity $r - 1$
- dy is holomorphic and nonzero at q

Higher loop equations

$(\Sigma, dx, dy, B, \mathcal{P})$ GenTR spectral curve data
$q \in \mathcal{P}$ a key-special point with exponents (r, s) such that $r \geq 2$ and $s = 1$

Definition

Ξ_q is spanned by differentials $\left(d\frac{1}{dx}\right)^k \alpha$ where $k \geq 0$ and α is holomorphic at q

Theorem (Loop Equations for GenTR differentials)

$\mathcal{W}_n^{(g),k} \in \Xi_q$ for any $k \geq 0$.

Higher loop equations

$(\Sigma, dx, dy, B, \mathcal{P})$ GenTR spectral curve data
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Definition

Ξ_q is spanned by differentials $\left(d\frac{1}{dx}\right)^k \alpha$ where $k \geq 0$ and α is holomorphic at q

Theorem (Loop Equations for GenTR differentials)

$\mathcal{W}_n^{(g),k} \in \Xi_q$ *for any* $k \geq 0$.

Remark. 1. For $r = 2$ these loop equations are equivalent to those discussed above

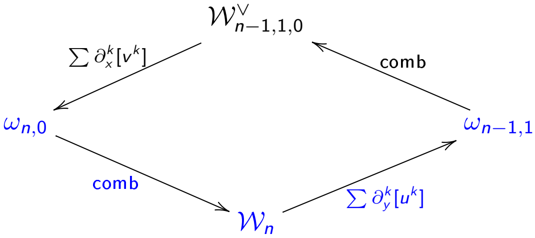
2. $$\mathcal{W}_n^{(g),k} = \frac{y(z)^k}{k!}\omega_n^{(g)}(z, z_K) + \left(\begin{matrix}\text{terms containing } \omega_{n'}^{(g')} \\ \text{with } 2g' - 2 + n' < 2g - 2 + n\end{matrix}\right)$$

The first r loop equations (with $k = 0, 1, \ldots, r - 1$) $\rightsquigarrow$ unique polar part of $\omega_n^{(g)}$ at $z = q$.

This identification of polar part of $\omega_n^{(g)}$ is equivalent to GenTR relations

Then, the higher loop equations (for $k \geq r$) are satisfied automatically

Proof of Loop Equations for GenTR



$$\omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} J(Y) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; \mathcal{E}(X,u)\; e^F \Big\rangle^\circ$$

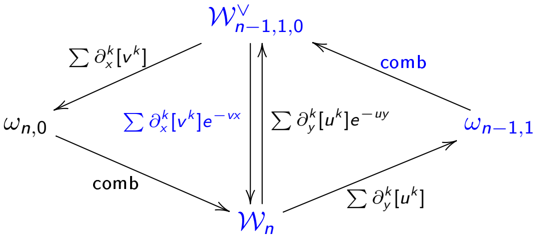
$$\mathcal{W}^\vee_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} \mathcal{E}(Y,v) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

$$\omega_{n-1,1} = -dy \sum_{k\geq 0} (-\partial_y)^k [u^k] e^{-u\,y} \frac{\mathcal{W}_{n-1,1,0}}{dy}$$

$$= -\sum_{k\geq 0} \Big(-d\frac{1}{dy}\Big)^k [u^k] e^{-u\,y} \mathcal{W}_{n-1,1,0}$$

$$\omega_n = -\sum_{k\geq 0} \Big(-d\frac{1}{dx}\Big)^k [v^k] e^{-v\,x} \mathcal{W}^\vee_{n-1,1,0}$$

Proof of Loop Equations for GenTR



$$\omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} J(Y) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; \mathcal{E}(X,u)\; e^F \Big\rangle^\circ$$

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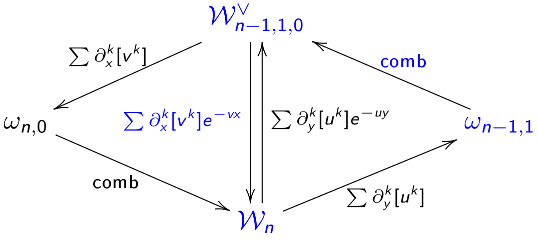
$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} = -\sum_{k\geq 0} \Big(-d\,\frac{1}{dx}\Big)^k e^{u\,y}[v^k] e^{-v\,x} \mathcal{W}^\vee_{n-1,1,0}$$

Proof of Loop Equations for GenTR



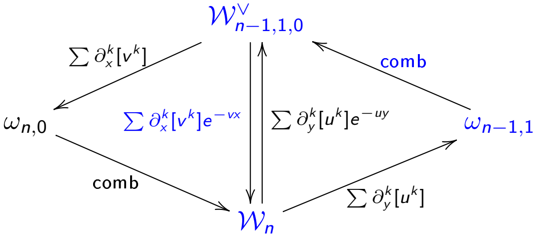
$$\omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} J(Y) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; \mathcal{E}(X,u)\; e^F \Big\rangle^\circ$$

$$\mathcal{W}^{\vee}_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\; e^{\hbar Q} \mathcal{E}(Y,v) e^{-\hbar Q}\; e^F \Big\rangle^\circ$$

$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} = -\sum_{k\ge 0} \Big(-d\,\frac{1}{dx}\Big)^k \underbrace{e^{u\,y}[v^k] e^{-v\,x} \mathcal{W}^{\vee}_{n-1,1,0}}_{\text{holomorphic}}$$

Proof of Loop Equations for GenTR



$$\omega_{n-1,1} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\ e^{\hbar Q} J(Y) e^{-\hbar Q}\ e^F \Big\rangle^{\circ}$$

$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\ \mathcal{E}(X,u)\ e^F \Big\rangle^{\circ}$$

$$\mathcal{W}^{\vee}_{n-1,1,0} \sim \Big\langle \prod_{i=1}^{n-1} J(X_i)\ e^{\hbar Q} \mathcal{E}(Y,v) e^{-\hbar Q}\ e^F \Big\rangle^{\circ}$$

$$\mathcal{W}_n = \mathcal{W}_{n-1,1,0} = -\sum_{k\geq 0} \Big(-d\,\frac{1}{dx}\Big)^k \underbrace{e^{u\,y}[v^k]e^{-v\,x}\mathcal{W}^{\vee}_{n-1,1,0}}_{\text{holomorphic}}$$

Loop equations for GenTR and its compatibility with CEO TR follow from the very existence of such identity. It holds globally, but the only its proof available involves computations in the expansions at a nice point

Thanks for your attention!