Refined topological recursion free energy for hypergeometric type curves

"Noncommutative Geometry Meets Topological Recursion"

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Joint w/ K. Osuga

1. Introduction

2. Topological recursion

3. Refinement

4. Free energy

1 Introduction

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History/motivation

Topological recursion (TR) [Eynard-Orantin, Chekhov-Eynard-Orantin]:

- Matrix models (loop equations)
- Enumerative geometry (Kontsevich-Witten, Gromov-Witten, Hurwitz, Mirzakhani-Weil-Petersson...)
- Differential equations, WKB analysis

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Refined / β-deformed TR [Perm{Chekhov,Eynard,Marchal}, Brini-Marino-Stevan, Manabe-Sulkowski, K-Osuga]:

- β-ensemble analogue to topological recursion
- Several approaches (matrix models, noncommutative spectral curve,...)
- Pure geometric theory formulated and proved in special case by [K-Osuga]

History/motivation

- Motivation: generalize [Iwaki-K] free energy formula involving Donaldson-Thomas invariants to refined setting.

Setting and aim

- Consider spectral curves related to "hypergeometric" curve

$$y^2 = \frac{{m_\infty}^2 x^2 - ({m_\infty}^2 + {m_0}^2 - {m_1}^2)x + {m_0}^2}{x^2(x-1)^2}$$

  + 8 other examples arising from limits/confluence.
- In particular,
  - Compute refined topological recursion free energy (proof when no 2nd order poles - conjecture for others)
  - Interpret in terms of refined BPS structure

Spectral curves of hypergeometric type

All are genus 0, degree two curves,

$$y^2 = Q(x)$$

Name	$Q(x)$	Assumption
Gauss (HG)	$\dfrac{m_\infty{}^2x^2 - (m_\infty{}^2 + m_0{}^2 - m_1{}^2)x + m_0{}^2}{x^2(x-1)^2}$	$m_0, m_1, m_\infty \neq 0$, $m_0 \pm m_1 \pm m_\infty \neq 0$.
Degenerate Gauss (dHG)	$\dfrac{m_\infty{}^2x + m_1{}^2 - m_\infty{}^2}{x(x-1)^2}$	$m_1, m_\infty \neq 0$, $m_1 \pm m_\infty \neq 0$.
Kummer (Kum)	$\dfrac{x^2 + 4m_\infty x + 4m_0{}^2}{4x^2}$	$m_0 \neq 0$, $m_0 \pm m_\infty \neq 0$.
Legendre (Leg)	$\dfrac{m_\infty^2}{x^2-1}$	$m_\infty \neq 0$.
Bessel (Bes)	$\dfrac{x+4m^2}{4x^2}$	$m \neq 0$.
Whittaker (Whi)	$\dfrac{x-4m}{4x}$	$m \neq 0$.
Weber (Web)	$\dfrac{1}{4}x^2 - m$	$m \neq 0$.
Degenerate Bessel (dBes)	$\dfrac{1}{x}$	–
Airy (Ai)	x	–

Spectral curves of hypergeometric type

We focus on:

$$\text{Weber: } y^2 = \frac{x^2}{4} - m$$

$$\text{Whittaker: } y^2 = \frac{x - 4m}{4x}$$

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where we assume $m \neq 0$.

Also (degenerate) Bessel $y^2 = 1/x$ and Airy $y^2 = x$, but they are easy.

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- Bidifferential: meromorphic section

$$B(z_1, z_2) \in p_1^*(T^*\mathcal{C}) \otimes p_2^*(T^*\mathcal{C})$$

with some properties ($p_i : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ projection). For us, $\mathcal{C} = \mathbb{P}^1$ so there is a canonical B,

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Ramification points of x viewed as a branched cover, denoted $r \in \mathcal{R}$

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e.g. for Whittaker $y^2 = \frac{x-4m}{4x}$ take $\mathcal{C} = \mathbb{P}^1$,

$$x(\zeta) = 2m\left(\frac{1}{\zeta-1} - \frac{1}{\zeta+1}\right), \quad y(\zeta) = -\frac{1}{2}\zeta$$

giving $\mathcal{R} = \{0, \infty\}$. There is a global involution $\sigma(\zeta) = -\sigma(\zeta)$ fixing x and sending $y \mapsto -y$.

Topological recursion

Start with $\omega_{0,1}(p_0) := y(p_0)dx(p_0)$, $\omega_{0,2}(p_0, p_1) = B(p_0, p_1)$.

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$$\omega_{g,n+1}(p_0, p_1, \cdots, p_n) := \sum_{r \in \mathcal{R}} \operatorname*{Res}_{p=r} K_r(p_0, p) \Big[\omega_{g-1,n+2}(p, \sigma(p), p_1, \cdots, p_n) + \sideset{}{'}\sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{1,2,\cdots,n\}}} \omega_{g_1,|I_1|+1}(p, p_{I_1}) \omega_{g_2,|I_2|+1}(\sigma(p), p_{I_2}) \Big]$$

for $2g + n \geq 2$,

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for $2g + n \geq 2$,where

$$K_r(p_0, p_1) = \frac{1}{(y - \sigma(y))dx} \int_{\zeta=\sigma(p)}^{\zeta=p} B(p_0, \zeta)$$

σ is "local conjugation" near ramification point r.

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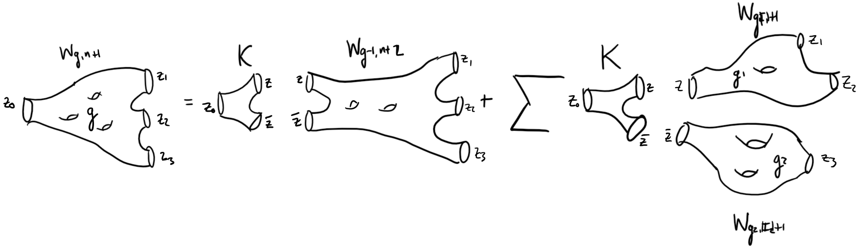

Topological recursion

Definition. Let Φ be any primitive of ydx. The *gth free energy* $(g > 1)$ is

$$F_g = \frac{1}{2-2g} \sum_{r \in \mathcal{R}} \operatorname*{Res}_{p=r} [\Phi(p)\omega_{g,1}(p)]$$

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[Iwaki-Koike-Takei] showed (for example):

$$F_g^{\mathrm{Web}}(\boldsymbol{m}) = \frac{B_{2g}}{2g(2g-2)} \frac{1}{m^{2g-2}}$$
$$F_g^{\mathrm{Whi}}(\boldsymbol{m}) = \frac{B_{2g}}{2g(2g-2)} \frac{2}{m^{2g-2}}$$

when $g > 1$ (and formulas for the other 7 examples)

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Fix $\beta \in \mathbb{C}^*$, and set $\mathcal{Q} = \beta^{\frac{1}{2}} - \beta^{-\frac{1}{2}}$.

Let $\mathcal{P}' \subset \mathcal{P}$ denote poles and zeroes of ydx excluding $\mathcal{R}$.

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$$D(\boldsymbol{\mu}) := \sum_{p \in \mathcal{P'}_+} \mu_p [p].$$

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- For any two distinct ramification points r_1, r_2, $x(r_1) \neq x(r_2)$,
- ydx does not vanish anywhere on $\Sigma \setminus \mathcal{R}$ and has at most a double zero at each ramification point.

New initial data

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This time, we also define:

$$\omega_{\frac{1}{2},1}(p_0) = \frac{\mathcal{Q}}{2}\left(-\frac{dy(p_0)}{y(p_0)} + \sum_{p\in\mathcal{P}'_+} \mu_p\,\eta_p(p_0)\right)$$

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where η_p the unique mero differential on $\mathbb{P}^1$ with residue $+1$ at p and -1 at $\sigma(p)$.

Refined TR

Let $g \in \frac{1}{2}\mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{>0}$. Then $\omega_{g,n} \in \pi_1^*(T^*\Sigma) \otimes \ldots \otimes \pi_n^*(T^*\Sigma)$, are defined by

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$$\omega_{g,n+1}(p_0, p_J) := -2 \left(\sum_{r \in \mathcal{R}} \operatorname*{Res}_{p=r} + \sum_{r \in \sigma(p_{J_0})} \operatorname*{Res}_{p=r} + \sum_{r \in \mathcal{P}'_+} \operatorname*{Res}_{p=r} \right) \mathrm{K}(p_0, p) \cdot \mathrm{Rec}^{\mathcal{Q}}_{g,n+1}(p, p_J)$$

where $p_J = (p_1, \ldots, p_n)$, $p_{J_0} = (p_0, \ldots, p_n)$.

Refined TR

where for $2g + n \geq 3$

$$\mathrm{Rec}^{\mathcal{Q}}_{g,n+1}(p, p_J) = \sum_{i=1}^{n} \Delta\omega_{0,2}(p, p_i) \cdot \omega_{g,n}(p, p_{\widehat{J_i}}) + \omega_{g-1,n+2}(p, p, p_J)$$
$$+ \sum^{**}_{\substack{g_1+g_2=g \\ J_1 \sqcup J_2 = J}} \omega_{g_1,n_1+1}(p, p_{J_1}) \cdot \omega_{g_2,n_2+1}(p, p_{J_2}) + \mathcal{Q}\, dx(p) \cdot d_p \left(\frac{\omega_{g-\frac{1}{2},n+1}(p, p_J)}{dx(p)} \right)$$

where $**$ denotes removal of $\omega_{0,1}, \omega_{0,2}$ terms and

$$\Delta\omega_{0,2}(z, z_i) := \omega_{0,2}(z, z_i) - \omega_{0,2}(\sigma(z), z_i).$$

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(slight complication for low $2g + n$, but same idea)

Properties

Fact: $\omega_{g,n}|_{\mathscr{Q}=0}$ reproduce the unrefined $\omega_{g,n}$ (with the understanding $\omega_{g,n}=0$ for $g \in \frac{1}{2}+\mathbb{Z}_{\geq 0}$).

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Furthermore, analogous properties to the unrefined case hold, though with with more interesting pole structure.

Properties

Theorem [K-Osuga Adv. Math. 2023]. For nice degree two genus zero refined spectral curve, $\omega_{g,n+1}$ satisfy:

1. The multidifferential $\omega_{g,n+1}$ is symmetric;
2. All poles of $\omega_{g,n+1}$ (in any variable) lie in $\mathcal{R}^* \cup \sigma(p_{J_0})$;
3. At any $o \in \Sigma$, $\omega_{g,n+1}$ is residue-free in the first (thus, any) variable:
$$\operatorname*{Res}_{p=o} \omega_{g,n+1}(p, p_J) = 0;$$
4. For (g, n) with $n > 0$, we have
$$(2g+n-2)\,\omega_{g,n}(J) = -\left(\sum_{r\in\mathcal{R}^*} \operatorname*{Res}_{p=r} + \sum_{r\in\sigma(p_J)} \operatorname*{Res}_{p=r}\right) \Phi(p)\cdot\omega_{g,n+1}(p, p_J).$$

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Definition

Definition is exactly the same as before:

$$F_g = \frac{1}{2-2g} \sum_{r \in \mathcal{R}} \operatorname*{Res}_{z=r} [\Phi(z) \omega_{g,1}(z)]$$

for Φ a primitive of ydx, where now $g \in \frac{1}{2}\mathbb{Z}_{>1}$

Result

To state result, let $a_1, \ldots, a_n \in \mathbb{C}^*$. The multiple Bernoulli polynomial $B_{N,k}(x \,|\, a_1, \ldots, a_N)$ is defined by

$$\frac{t^N e^{xt}}{(e^{a_1 t} - 1) \ldots (e^{a_N t} - 1)} = \sum_{k \geq 0} B_{N,k}(x \,|\, a_1, \ldots, a_N) \cdot \frac{t^k}{k!}.$$

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We care about $N = 2$, $a_1 = -\beta^{\frac{1}{2}}$, $a_2 = \beta^{-\frac{1}{2}}$.

Write $B_{2,2g}(x) := B_{2,2g}(x| - \beta^{\frac{1}{2}}, \beta^{-\frac{1}{2}})$

Result

Theorem. [K-Osuga]
(i) For Weber refined spectral curve, we have

$$F_g = \frac{(-1)^{2g-2}}{2g(2g-1)(2g-2)} B_{2,2g}\left(\frac{\mu\mathfrak{Q}}{2} - \frac{\mathfrak{Q}}{2}\right)\left(\frac{1}{m}\right)^{2g-2}$$

(ii) For Whittaker,

$$F_g = \frac{(-1)^{2g-2}}{2g(2g-1)(2g-2)}\left(B_{2,2g}\left(\frac{\mu\mathfrak{Q}}{2}\right) + B_{2,2g}\left(\frac{\mu\mathfrak{Q}}{2} - \mathfrak{Q}\right)\right)\left(\frac{1}{m}\right)^{2g-2}$$

(iii) For degenerate Bessel, and Airy $F_g = 0$.

Conjecture. A similar explicit expression holds for all refined spectral curves of hypergeometric type.

Proof

How to prove?

[1] of the corresponding quantum curve [K-Osuga]

Proof

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Similar to Iwake-Koike-Takei, we consider Voros coefficient[1]:

$$V = \sum_{k=1}^{\infty} \hbar^k \int_{\infty_-}^{\infty_+} \left(\sum_{\substack{2g-2+n=k \\ g\geq 0, n\geq 1}} \frac{1}{\beta^{n/2} n!} \frac{d}{dz} \int_{D(z;\boldsymbol{\nu})} \cdots \int_{D(z;\boldsymbol{\nu})} \omega_{g,n}(\zeta_1, \ldots, \zeta_n) \right)$$

where

$$D(z;\boldsymbol{\nu}) := [z] - \sum_{p\in\mathcal{P}'} \nu_p[p], \qquad \sum_{p\in\mathcal{P}'} \nu_p = 1,$$

and $\infty_\pm$ denotes the preimages under x of ∞

[1] of the corresponding quantum curve [K-Osuga]

Proof

We rely on the relation (as f.p.s. in $\hbar$) between V and F:

$$V = F\left(\widehat{m} + \frac{\hbar}{2\beta^{\frac{1}{2}}}\right) - F\left(\widehat{m} - \frac{\hbar}{2\beta^{\frac{1}{2}}}\right) + \text{l.o.}$$

where

$$\widehat{m} = m - \frac{\nu\hbar}{2\beta^{\frac{1}{2}}}.$$

and $F := \sum_g \hbar^{2g-2} F_g$.

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<u>Problem</u>: It is false for the more complicated HG type curves (whenever $Q(x)dx^2$ has a 2nd order pole), and for β case less trivial to prove even when it is true.

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Why? V is like a limit of (log of) wavefunction ψ but the expression for ψ is not continuous at the limiting points!

$$\lim_{p\to\sigma(q_0)} \int_{q_0}^{p} \cdots \int_{q_n}^{p} \omega_{g,n+1} \neq \int_{q_0}^{\sigma(q_0)} \cdots \int_{q_n}^{(q_n)} \omega_{g,n+1}$$

can occur if $q_i \in \mathcal{P}$.

Toy example

Consider integrating

$$\omega(\zeta_0, \zeta_1) = \frac{\zeta_0^2 + \zeta_1^2 + \zeta_0^2\zeta_1^2 + 3\zeta_0\zeta_1}{\zeta_0^2\zeta_1^2(\zeta_0+\zeta_1)^3} d\zeta_0 d\zeta_1$$

from $q_0 = q_1 = -1$, $\sigma(\zeta) = -\zeta$. Then

$$\int_{-1}^{1}\int_{-1}^{1} \omega = 0,$$

and even $\int_a^1 \int_{-1}^1 \omega = 0$ for any a. Yet

$$\int_{-1}^{z}\int_{-1}^{z} \omega = -\frac{(1+z)^2(2z-1)}{4z^3} \xrightarrow[z\to 1]{} -1,$$

In fact,

$$\int_a^z \int_{-1}^z \omega = \frac{(1-z)(1+z)}{2z\zeta_0(z+\zeta_0)}\Bigg|_{\zeta_0=a}^{\zeta_0=z} = \frac{(1-z)(1+z)}{4z^3} - \frac{(1-z)(1+z)}{2za(z+a)}.$$

Proof

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Then use "contiguity relations" relating Voros coeffs for classical ODEs, get difference eqn for F.

Proof

Let

$$\Delta_{\epsilon_1,\epsilon_2} \cdot f(m;\hbar)$$
$$= -f\left(m - \frac{\epsilon_1}{2} - \frac{\epsilon_2}{2};\hbar\right) + f\left(m - \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2};\hbar\right) + f\left(m + \frac{\epsilon_1}{2} - \frac{\epsilon_2}{2};\hbar\right) - f\left(m + \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2};\hbar\right)$$

Then for Weber,

$$\Delta_{\epsilon_1,\epsilon_2} \cdot F = \log\left(m + \frac{\mu}{2}\mathfrak{Q}\hbar\right)$$

and for Whittaker,

$$\Delta_{\epsilon_1,\epsilon_2} \cdot F = \log\left(m + \frac{\mu}{2}\mathfrak{Q}\hbar + \frac{\mathfrak{Q}\hbar}{2}\right) + \log\left(m + \frac{\mu}{2}\mathfrak{Q}\hbar - \frac{\mathfrak{Q}\hbar}{2}\right).$$

Proof

Finally, use definition of double Bernoulli polynomials to solve this difference equation.

Theorem. [K-Osuga]

(i) For Weber refined spectral curve, we have

$$F_g = \frac{(-1)^{2g-2}}{2g(2g-1)(2g-2)} B_{2,2g}\left(\frac{\mu\mathcal{Q}}{2} - \frac{\mathcal{Q}}{2}\right)\left(\frac{1}{m}\right)^{2g-2}$$

(ii) For Whittaker,

$$F_g = \frac{(-1)^{2g-2}}{2g(2g-1)(2g-2)} \left[B_{2,2g}\left(\frac{\mu\mathcal{Q}}{2}\right) + B_{2,2g}\left(\frac{\mu\mathcal{Q}}{2} - \mathcal{Q}\right)\right]\left(\frac{1}{m}\right)^{2g-2}$$

(iii) For degenerate Bessel, and Airy $F_g = 0$.

Further

- Complete the other cases
- Higher genus (progress for hyperelliptic in [Osuga 23])
- Higher degree / no involution
- Relation to Donaldson-Thomas theory? (see [K-Williams 24])
- Quantized BPS Riemann-Hilbert problem (ongoing)
- x-y swap property?
- Refined analogues of enumerative applications
- ...

Definition. A *refined BPS structure* is a tuple (Γ, Z, Ω):

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- $\Omega(\gamma) = \Omega(-\gamma)$
- For some (any) norm $||\cdot||$ on $\Gamma \otimes \mathbb{R}$, there is > 0 s.t.

$$\Omega \neq 0 \implies |Z(\gamma)| > C \cdot ||\gamma||$$

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Note: We often use $q^{\frac{1}{2}} := -\mathbb{L}^{\frac{1}{2}}$

GMN construction

Gaiotto-Moore-Neitzke constructed BPS structures – we consider rank 2 case.

Choose a sufficiently nice meromorphic quadratic differential $\varphi = Q(x)dx^{\otimes 2}$ (say, hypergeometric type).

Let $\widetilde{\Sigma}$ denote Σ with simple poles filled in.

GMN construction

Define:

- $\Gamma := \{\gamma \in H_1(\widetilde{\Sigma}, \mathbb{Z}) \,|\, \sigma_* \gamma = -\gamma\}$, σ the sheet-exchange
- $Z(\gamma) := \oint_\gamma \lambda = \oint_\gamma \sqrt{Q(x)}dx$

(in all our examples, Σ is genus 0, Γ is easy to determine and $Z(\gamma)$ is easily computed as linear combinations of parameters m_i.

Now, to define $\Omega : \Gamma \to \mathbb{Z}$.

GMN construction

Fix $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$. The *foliation of phase* ϑ, $\mathcal{F}_\vartheta(\varphi)$ is given by

$$\mathrm{Im}\, e^{-i\vartheta} \int^{x} \sqrt{Q(x)}dx = \mathrm{const}$$

A *trajectory of phase* ϑ is any maximal leaf of $\mathcal{F}_\vartheta(\varphi)$.

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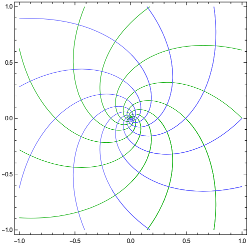


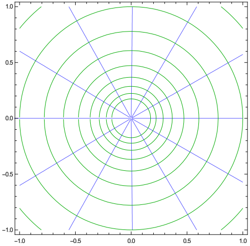
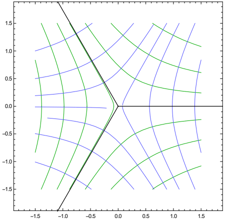
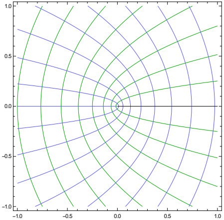


Figure 1: $Q(x) = r/x^2$

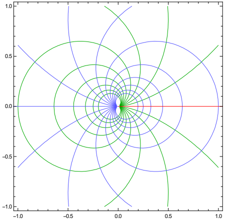
GMN construction



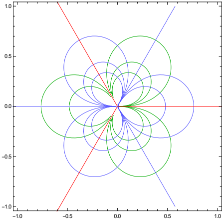
a $Q(x) = x$



b $Q(x) = 1/x$



c $Q(x) = 1/x^3$



d $Q(x) = 1/x^5$

BPS invariants

Fact: Trajectory pentachotomy:

1. saddle
2. separating
3. generic
4. closed
5. recurrent

BPS invariants

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BPS invariants

Fact: Every saddle trajectory or closed trajectory has a *canonical lift* $\gamma \in \Gamma$ (up to sign)

BPS invariants

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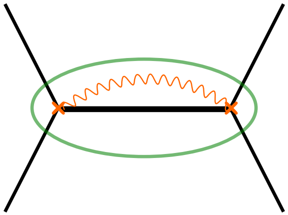
For example, if both endpoints simple zeroes:



Refined BPS structure

Definition. We define $\Omega(\gamma)$ of φ below for $\gamma \in \Gamma$ appearing as canonical lifts of saddles or ring domains in $\mathcal{F}_\vartheta(\varphi)$

$$\Omega(\gamma) = \begin{cases} +1 & \text{type I} \\ q^{\frac{1}{2}} + q^{-\frac{1}{2}} & \text{type II} \\ q + 2 + q^{-1} & \text{type III} \\ -q^{\frac{1}{2}} & \text{deg. ring domain} \\ -(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) & \text{nondeg. ring domain} \end{cases}$$

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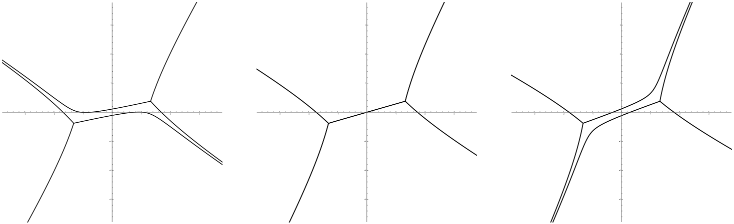
When we send $q^{\frac{1}{2}} \to 1$, we recover the unrefined invariants.
Note: interpretation is not clear, and some shifts are allowed.

BPS invariants

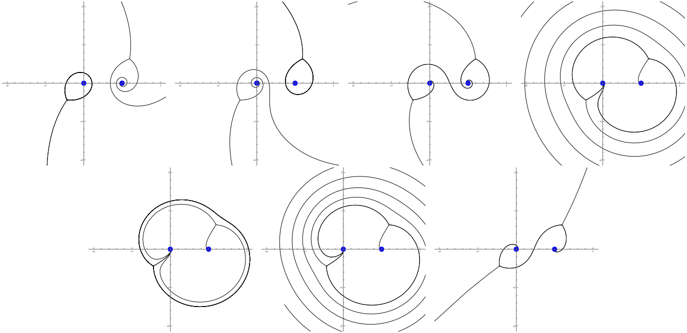
Simple example (Weber): $Q_{\text{Web}}(x) = \frac{1}{4}x^2 - m_\infty^2$



$$\Omega(\gamma_{\text{BPS}}) = 1$$

BPS structure

Main example: $Q_{\mathrm{HG}}(x) = \dfrac{m_\infty^2 x^2 - (m_\infty^2 - m_1^2 + m_0^2)x + m_0^2}{x^2(x-1)^2}$



$\Omega(\gamma_{\mathrm{BPS}}) = \pm 1$ 7 ($\times 2$) BPS states

Free energy

[Iwaki-Koike-Takei] showed (for example):

$$F_g^{\mathrm{HG}}(\boldsymbol{m}) = \frac{B_{2g}}{2g(2g-2)}\left(\frac{1}{(m_0+m_1+m_\infty)^{2g-2}} + \frac{1}{(m_0+m_1-m_\infty)^{2g-2}} + \frac{1}{(m_0-m_1+m_\infty)^{2g-2}} + \frac{1}{(m_0-m_1-m_\infty)^{2g-2}} - \frac{1}{(2m_0)^{2g-2}} - \frac{1}{(2m_1)^{2g-2}} - \frac{1}{(2m_\infty)^{2g-2}}\right).$$

Free energy

Theorem. [Iwaki-K, Adv. Math. 2022] *For the spectral curves of hypergeometric type,* $\boldsymbol{m}$ *generic, we have*

$$F_g(\boldsymbol{m}) = \frac{B_{2g}}{4g(2g-2)} \sum_{\gamma \in \Gamma} \Omega(\gamma) \left(\frac{2\pi i}{Z(\gamma)} \right)^{2g-2}, \quad g \geq 2$$

Result (reformulated)

Theorem. [K-Osuga] *For Whittaker, Weber, degenerate Bessel, and Airy refined spectral curves, we have*

$$F_g(m,\mu) = c_g \sum_{\gamma\in\Gamma}\sum_{n\in\mathbb{Z}} B_{2,2g}\left(\frac{Z^{\mathrm{reg}}_{\frac{1}{2}}(\gamma)}{2\pi i} + (n-1)\frac{\mathcal{Q}}{2}\right)\Omega_n(\gamma)\left(\frac{2\pi i}{Z(\gamma)}\right)^{2g-2}$$

where $c_g = \frac{(-1)^{2g-2}}{4g(2g-1)(2g-2)}$ and $Z^{\mathrm{reg}}_{\frac{1}{2}}(\gamma) := \int_\gamma \omega_{\frac{1}{2},1}^{\mathrm{odd}}$.

Conjecture. This holds for all refined spectral curves of hypergeometric type.