Remodeling Conjecture with Descendants

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Noncommutative Geometry Meets Topological Recursion

BIRS-IASM

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This Talk is about...

all-genus mirror symmetry

of

toric Calabi-Yau 3-folds

via

topological recursion

Mirror Symmetry of Calabi-Yau 3-Folds

A-model (X, ω)		B-model $(X_q^\vee, \Omega_q)$
X: Calabi-Yau 3-fold ω: symplectic form Kähler parameters τ	$\overset{\text{mirror}}{\underset{\text{map}}{\longleftrightarrow}}$	$X_q^\vee$: mirror Calabi-Yau 3-fold Ω_q: holomorphic 3-form complex parameters q
A-VHS, Dubrovin connection		B-VHS, Gauss-Manin connection
Gromov-Witten potential		period integrals, BCOV, etc.
$\vdots$		$\vdots$

Mirror Symmetry of Calabi-Yau 3-Folds

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A-VHS, Dubrovin connection		B-VHS, Gauss-Manin connection
Gromov-Witten potential		period integrals, BCOV, etc.
$\vdots$		$\vdots$

- Higher-genus mirror symmetry is usually quite challenging
- For X toric, ideas:
    - mirror B-model can be given by families of curves
    - topological recursion provides quantization to higher genus

Remodeling the B-Model

- For X toric, can use Hori-Vafa mirror

$$X_q^\vee = \{uv = H(X, Y, q)\} \subset \mathbb{C}^2_{(u,v)} \times (\mathbb{C}^*)^2_{(X,Y)}$$

  - $H(X, Y, q)$: Laurent polynomial in X, Y depending on q

Remodeling the B-Model

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- Idea 1: reduce genus-0 B-model on 3d Hori-Vafa mirror to 1d mirror curves

$$C_q = \{0 = H(X,Y,q)\} \subset (\mathbb{C}^*)^2_{(X,Y)}$$

- Idea 2: use [Chekhov-Eynard-Orantin] topological recursion invariants

$$\omega_{g,n}$$

  on C_q as higher-genus B-model

- Remodeling Conjecture [Bouchard-Klemm-Mariño-Pasquetti]

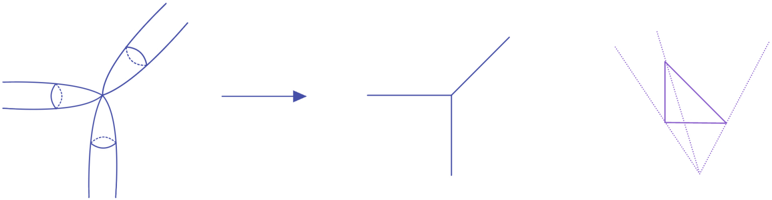
Remodeling Conjecture

A-model (X, ω)		B-model $(C_q, \omega_{0,1})$
X: toric Calabi-Yau 3-fold ω: symplectic form Kähler parameters τ	$\xleftrightarrow[\text{map}]{\text{mirror}}$	C_q: mirror curve $\omega_{0,1}$: 1-form complex parameters q
all-genus Gromov-Witten invariants		topological recursion invariants $\omega_{g,n}$
open-closed GW invariants	$\leftarrow$ I $\rightarrow$	local expansions of $\omega_{g,n}$
descendant GW invariants	$\leftarrow$ II $\rightarrow$	Laplace transforms of $\omega_{g,n}$

- I: original BKMP proposal [Eynard-Orantin, Fang-Liu-Zong]
- II: this work [Fang-Liu-Y-Zong]

Toric Calabi-Yau 3-Folds

- X: complex toric Calabi-Yau 3-fold (manifold or orbifold)
  - E.g. $\mathbb{C}^3$



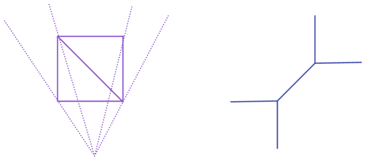
$\mathbb{C}$ coordinate axes

toric graph
(image under moment map
of real CY 2-subtorus $T'_{\mathbb{R}}$)

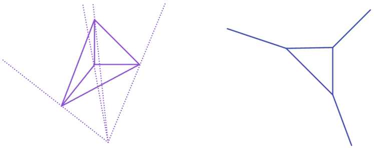
toric fan

Toric Calabi-Yau 3-Folds

- Additional examples



resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{P}^1$



$\mathcal{O}_{\mathbb{P}^2}(-3)$

  - more general examples come from canonical bundles of toric surfaces
  - toric fan = cone over polytope with regular triangulation
  - toric graph = dual graph of triangulated polytope

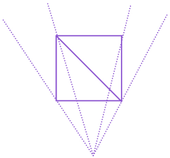
- We assume that X is semi-projective, i.e. a GIT quotient
  - equivalently: X is a symplectic quotient, support of fan is convex

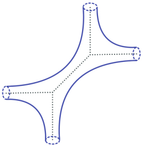
Mirror Curves

- Affine algebraic curve

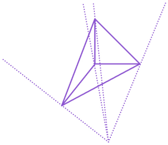
$$C_q = \{0 = H(X, Y, q)\} \subset (\mathbb{C}^*)^2_{(X,Y)}$$

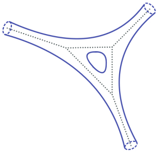
- Examples





$$H = X + Y + 1 + qXY$$



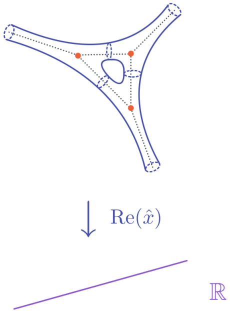


$$H = X + Y + 1 + qX^{-1}Y^{-1}$$

  - tropicalization of C_q $\leftrightarrow$ toric graph of X
  - $\mathfrak{g}$ = genus of C_q = # compact divisors in X
  - $\mathfrak{n}$ = # punctures of C_q = # non-compact axes in X
  - N = # patches of C_q = # torus fixed points in X = $\dim H^*(X; \mathbb{C})$

$$\boxed{N = 2\mathfrak{g} - 2 + \mathfrak{n}}$$

Topological Recursion - $(0,1)$



- Take $(\mathrm{u}_1, \mathrm{u}_2) \in \mathbb{R}^2 \setminus \{0\}$ and let

$$\hat{x} = -\mathrm{u}_1 \log X - \mathrm{u}_2 \log Y$$

  - meromorphic (multivalued) function on C_q
  - N simple ramification points, one on each patch
  - later, $\mathrm{u}_1, \mathrm{u}_2$ will serve as equivariant parameters of Calabi-Yau torus T'
  - $\mathrm{Re}(\hat{x})$ will serve as superpotential of T'-equivariant Landau-Ginzburg model

- Let

$$\hat{y} = -\frac{\log Y}{\mathrm{u}_1}, \qquad \omega_{0,1} = \hat{y}\, d\hat{x}$$

Topological Recursion - $(0, 2)$

- $\omega_{0,2}$ = fundamental bidifferential of second kind of compactified mirror curve $\overline{C}_q$
  - symmetric, meromorphic form on $\overline{C}_q^2$ with double pole along diagonal, locally

$$\omega_{0,2}(p_1, p_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{hol'c} \qquad \text{as } p_1 \to p_2$$

  - Torelli marking: pick symplectic basis

$$A_1, \dots, A_{\mathfrak{g}}, \quad B_1, \dots, B_{\mathfrak{g}}$$

of $H_1(\overline{C}_q; \mathbb{Z})$ such that the "A-periods" give mirror maps

$$\int_{A_i} \omega_{0,1} = \tau_i = \log q_i + O(q)$$

and use the A-cycles to normalize $\omega_{0,2}$

$$\int_{p_1 \in A_i} \omega_{0,2}(p_1, p_2) = 0$$

Topological Recursion - (g, n)

- On spectral curve

$$(C_q, \hat{x}, \hat{y}, \omega_{0,2})$$

run [Chekhov-Eynard-Orantin] topological recursion to obtain

  - symmetric, meromorphic form on C_q^n

$$\omega_{g,n}, \qquad 2g - 2 + n > 0$$

  - free energy

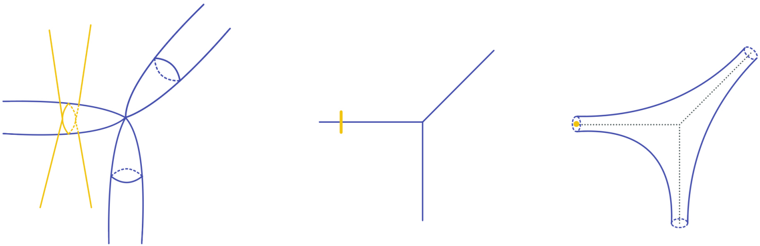
$$\omega_g, \qquad g \geq 0$$

Topological Recursion - (g, n)

- On spectral curve
$$(C_q, \hat{x}, \hat{y}, \omega_{0,2})$$
run [Chekhov-Eynard-Orantin] topological recursion to obtain
  - symmetric, meromorphic form on C_q^n
$$\omega_{g,n}, \qquad 2g - 2 + n > 0$$
  - free energy
$$\omega_g, \qquad g \geq 0$$
- B-model graph sum [Dunin-Barkowski–Orantin–Shadrin–Spitz]
$$\omega_{g,n} = \sum_{\vec{\Gamma} \in G_{g,n}} \frac{w_B(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}$$
  - $G_{g,n}$: collection of decorated stable graphs of genus g and with n leaves
  - w_B: B-model weight of graphs
  - quantization of genus-0 B-model

Open-Closed Gromov-Witten Invariants

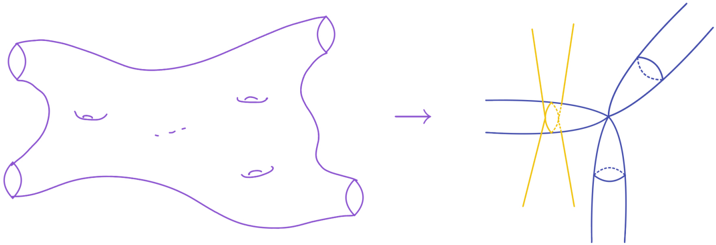
- Remodeling Conjecture relates $\omega_{g,n}$, ω_g to Gromov-Witten theory of X
- Boundary condition for open theory: Aganagic-Vafa Lagrangian L



  - preserved under action of real CY 2-subtorus $T'_{\mathbb{R}}$
  - non-compact, topologically $S^1 \times \mathbb{R}^2$
  - intersects a unique axis in X
  - we assume this axis is non-compact, corresponding to a puncture on C_q

Open-Closed Gromov-Witten Invariants

- Open Gromov-Witten invariants are virtual counts of stable maps from bordered Riemann surfaces to (X, L)



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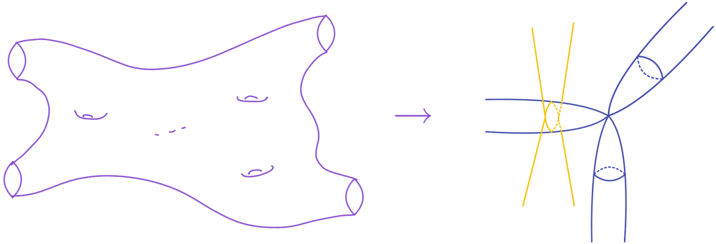


- Consider generating function

$$F_{g,n}^{X,L}$$

of invariants from genus-g domains with n boundary components

- Consider also generating function

$$F_g^X$$

of closed invariants counting stable maps from genus-g borderless domains

Open-Closed Gromov-Witten Invariants

- Consider generating function

$$F^{X,L}_{g,n}$$

of invariants from genus-g domains with n boundary components

- A-model graph sum [Givental, Teleman, Zong]

$$F^{X,L}_{g,n} = \sum_{\vec{\Gamma}\in G_{g,n}} \frac{w_A(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}$$

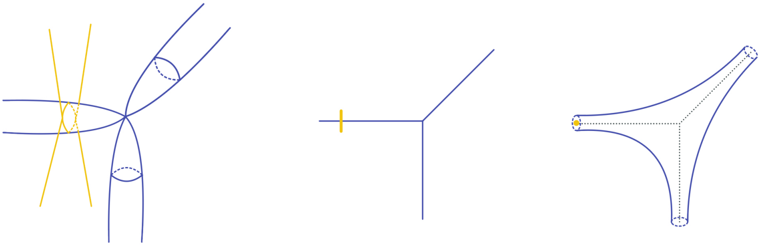
  - $G_{g,n}$: collection of decorated stable graphs of genus g and n leaves
  - w_A: A-model weight of graphs
  - quantization of genus-0 A-model

Remodeling Conjecture

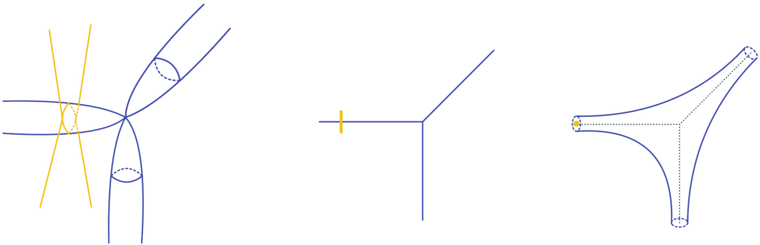
Theorem [Bouchard-Klemm-Mariño-Pasquetti, Eynard-Orantin, Fang-Liu-Zong]

Under the mirror map:

- $F_{g,n}^{X,L}$ can be obtained from local expansion of $\omega_{g,n}$ near the puncture on C_q corresponding to L
- F_g^X can be obtained from free energy ω_g



Remodeling Conjecture



- Proof idea: relate $F^{X,L}_{g,n}$ and $\omega_{g,n}$ by comparing A-/B-model graph sums
- Leaf term is given by $(0,1)$-case, i.e. disk mirror theorem

$$dF^{X,L}_{0,1} \quad = \quad \omega_{0,1}$$

  - [Graber-Zaslow, Fang-Liu, Fang-Liu-Tseng]
  - based on genus-0 toric mirror theorem [Givental, Coates-Corti-Iritani-Tseng, Cheong–Ciocan-Fontanine–Kim]

Remodeling Conjecture with Descendants

A-model (X, ω)		B-model $(C_q, \omega_{0,1})$
all-genus Gromov-Witten invariants		topological recursion invariants $\omega_{g,n}$
open-closed GW invariants	← BKMP →	local expansions of $\omega_{g,n}$
descendant GW invariants	← next →	Laplace transforms of $\omega_{g,n}$

Descendant Gromov-Witten Invariants

- For T'-equivariant insertions $\gamma_1, \dots, \gamma_n \in H^*_{T'}(X; \mathbb{C})$ consider

$$\left\langle\!\!\left\langle \frac{\gamma_1}{z_1 + \psi_1}, \dots, \frac{\gamma_n}{z_n + \psi_n} \right\rangle\!\!\right\rangle_{g,n}^{X,T'}$$

generating function of genus-g, n-pointed descendant Gromov-Witten invariants

  - ψ_i: ψ-classes
  - z_i: formal variables

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  generating function of genus-g, n-pointed descendant Gromov-Witten invariants
    - ψ_i: ψ-classes
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- Can also be expressed by A-model graph sum formula

- Leaf term is again given by $(0,1)$-case

$$\left\langle\!\!\left\langle \frac{\gamma}{z + \psi} \right\rangle\!\!\right\rangle^{X,T'}_{0,1}$$

- Goal: obtain $\langle\!\langle \cdots \rangle\!\rangle^{X,T'}_{g,n}$ from Laplace transform of $\omega_{g,n}$ along suitable cycles

Relative Cycles

- Consider exponential map $\mathbb{C}^2_{(-\log X, -\log Y)} \to (\mathbb{C}^*)^2_{(X,Y)}$ and induced $\mathbb{Z}^2$-covering

$$\widetilde{C}_q = C_q \times_{(\mathbb{C}^*)^2} \mathbb{C}^2$$

  - $\mathbb{Z}^2$ = character lattice $\mathrm{Hom}(T', \mathbb{C}^*)$

- $\hat{x} = -\mathrm{u}_1 \log X - \mathrm{u}_2 \log Y$ is well-defined on $\widetilde{C}_q \Rightarrow T'$-equivariant superpotential

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- $\hat{x} = -\mathrm{u}_1 \log X - \mathrm{u}_2 \log Y$ is well-defined on $\widetilde{C}_q \Rightarrow T'$-equivariant superpotential

- Consider group of integral relative 1-cycles

$$H_1(\widetilde{C}_q, \mathrm{Re}(\hat{x}) \gg 0; \mathbb{Z})$$

- Laplace transform/oscillatory integral

$$\int_\Gamma e^{-\frac{\hat{x}}{z}} \omega_{0,1}$$

  provides B-model integral structure

Equivariant K-groups and Central Charges

- For A-model, we will consider certain T'-equivariant K-group

$$K^{+}_{T'}(X)$$

with framing (c.f. [Iritani])

$$\kappa(\mathcal{E}) \approx \widetilde{\mathrm{ch}}_z(\mathcal{E})\hat{\Gamma}^z_X \qquad \in H^*_{T'}(X;\mathbb{C})[\![\mathsf{u}_1, \mathsf{u}_2, z^{-1}]\!]$$

  - $\widetilde{\mathrm{ch}}_z$: ($z$-modified) Chern character
  - $\hat{\Gamma}^z_X$: (z-modified) Gamma class of X

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  - $\widetilde{\mathrm{ch}}_z$: ($z$-modified) Chern character
  - $\hat{\Gamma}^z_X$: (z-modified) Gamma class of X

- Descendant potential/central charge

$$\left\langle\!\!\left\langle \frac{\kappa(\mathcal{E})}{z+\psi} \right\rangle\!\!\right\rangle^{X,T'}_{0,1}$$

provides A-model integral structure

Remodeling Conjecture with Descendants

Theorem [Fang-Liu-Y-Zong]

There is an isomorphism

$$\mathrm{mir}^{+} : K^{+}_{T'}(X) \to H_1(\widetilde{C}_q, \mathrm{Re}(\hat{x}) \gg 0; \mathbb{Z})$$

such that for $\mathcal{E}_1, \dots, \mathcal{E}_n \in K^{+}_{T'}(X)$, under the mirror map

$$\left\langle\!\!\left\langle \frac{\kappa(\mathcal{E}_1)}{z_1 + \psi_1}, \dots, \frac{\kappa(\mathcal{E}_n)}{z_n + \psi_n} \right\rangle\!\!\right\rangle_{g,n}^{X,T'} = \int_{p_1 \in \mathrm{mir}^+(\mathcal{E}_1)} \cdots \int_{p_n \in \mathrm{mir}^+(\mathcal{E}_n)} e^{-\frac{\hat{x}(p_1)}{z_1} - \dots - \frac{\hat{x}(p_n)}{z_n}} \omega_{g,n}(p_1, \dots, p_n)$$

Remodeling Conjecture with Descendants

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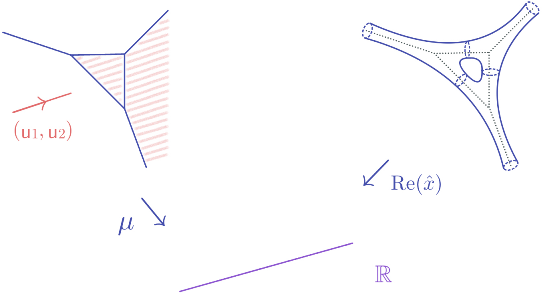
- mir^+ identifies the A-/B-model integral structures
- Proof is again based on comparing A-/B-model graph sums
- Reduces to comparing leaf terms

$$\left\langle\!\!\left\langle \frac{\kappa(\mathcal{E})}{z + \psi} \right\rangle\!\!\right\rangle^{X,T'}_{0,1} = \int_{\mathrm{mir}^+(\mathcal{E})} e^{-\frac{\hat{x}}{z}} \omega_{0,1}$$

  - previously obtained for compact toric case [Iritani, Fang]

More on Equivariant K-groups

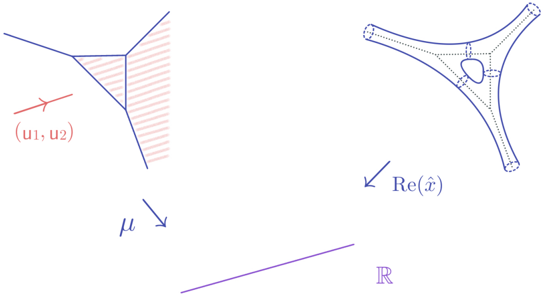
- $K^+_{T'}(X)$ accounts for T'-equivariant coherent sheaves with "bounded below support"



  - choice of (u_1, u_2) determines 1-subtorus of T' $\Rightarrow$ moment map $\mu : X \to \mathbb{R}$
  - take $K^+_{T'}(X)$ = Grothendieck group of certain subcategory of $D^b(\mathrm{Coh}_{T'}(X))$ of sheaves whose support is bounded below under μ
  - generalizes [Borisov-Horja] on K-groups with compact support

More on Equivariant K-groups

- Lemma: $K^+_{T'}(X)$ is generated by T'-equivariant line bundles on toric divisors and curves in X that are bounded below under μ



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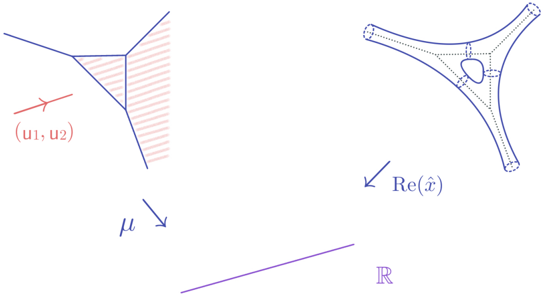
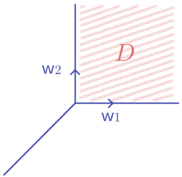
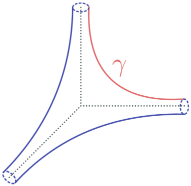


- To construct mir^{+}, we construct mirror cycles of these generators
  - here we focus on line bundles on toric divisors
- Intuition from toric geometry
  - toric graph of X $\leftrightarrow$ tropicalization of C_q
  - $\mathbb{C}^3$ charts of X $\leftrightarrow$ pair-of-pants decomposition of C_q

Construction of Mirror Cycles

- First consider local picture at a vertex
  - $\mathbb{C}^3$ chart $\leftrightarrow$ pair of pants in C_q
  - toric divisor D $\leftrightarrow$ cycle γ





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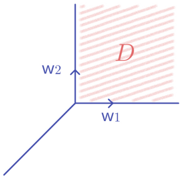
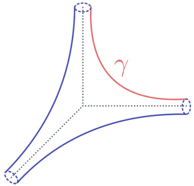




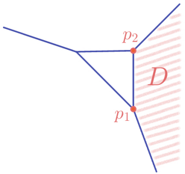
- Take lift Γ of γ to cover $\widetilde{C}_q$

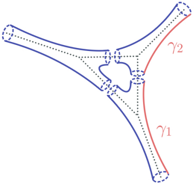
$$\int_\Gamma e^{-\frac{\hat{x}}{z}}\omega_{0,1} \quad = \quad e^{-\frac{\sqrt{-1}}{z}(a\mathrm{w}_1+b\mathrm{w}_2)}\frac{\Gamma(\frac{\mathrm{w}_1}{z})\Gamma(\frac{\mathrm{w}_2}{z})}{\Gamma(1+\frac{\mathrm{w}_1+\mathrm{w}_2}{z})}$$

  - matches restriction of central charge of twist of $\mathcal{O}_D$ to vertex
  - $\mathbb{Z}^2$ choices of lift Γ, coordinate has constant imaginary part (a, b)
  - correspond to $\mathbb{Z}^2$ choices of T'-equivariant line bundles twisting $\mathcal{O}_D$

Construction of Mirror Cycles

- Back to global picture, to construct $\text{mir}^+(\mathcal{O}_D \otimes \mathcal{L})$, patch local cycles





  - T'-equivariant line bundle $\mathcal{L}$ is determined by T'-characters after restriction to vertices and compatibility condition at adjacent vertices
  - locally, take cycles mirror to characters
  - compatibility condition $\Rightarrow$ cycles can be patched together

- Construction for line bundles on toric curves is similar

Thank you!