KP integrability in topological recursion through the x-y swap relation

Alexander Alexandrov

IBS Center for Geometry and Physics, Pohang, Korea

September 26, 2024

Noncommutative Geometry Meets Topological Recursion

Based on a joint work with Boris Bychkov, Petr Dunin-Barkowski, Maxim Kazarian, and Sergey Shadrin

Table of Contents

Table of Contents

KP HIERARCHY

The theory of integrable hierarchies was actively developed by the Kyoto school of Date, Jimbo, Kashiwara, Miwa, and Sato in the 80s of the last century. They found a fundamental relation between integrable hierarchies, representation theory of the infinite dimensional Lie algebras, and free field formalism.

The **Kadomtsev–Petviashvili (KP) hierarchy** was introduced by **[Sato '81]**. It can be represented in terms of **tau-functions**

$$\tau(\mathbf{t}) \in \mathbb{C}[\![t_1, t_2, t_3, \dots]\!],$$

which are nothing but the vacuum expectation values of some group elements in the free field formalism, by the **Hirota bilinear identity**

$$\oint_\infty e^{\sum_{k>0}(t_k - t'_k)z^k} \tau(\mathbf{t} - [z^{-1}])\tau(\mathbf{t}' + [z^{-1}])dz = 0.$$

Here we use the short-hand notation $\mathbf{t} \pm [z^{-1}] := \{t_1 \pm z^{-1}, t_2 \pm \frac{1}{2}z^{-2}, t_3 \pm \frac{1}{3}z^{-3}, \dots\}$. This bilinear identity encodes all nonlinear PDEs of the KP hierarchy.

The first nontrivial term in the expansion of the l.h.s. of the Hirota bilinear identity gives the **KP equation**

$$\tau\tau_{1111} - 4\tau_1\tau_{111} + 3\left(\tau_{11}\right)^2 + 3\tau\tau_{22} - 3\left(\tau_2\right)^2 - 4\tau\tau_{13} + 4\tau_1\tau_3 = 0,$$

where

$$\tau_{i_1 i_2 \ldots} = \frac{\partial}{\partial t_{i_1}} \frac{\partial}{\partial t_{i_2}} \ldots \tau.$$

The second derivative of this equation with respect to t_1 gives the standard KP equation

$$3u_{22} = \left(4u_3 - 12uu_1 - u_{111}\right)_1,$$

where $u = \frac{\partial^2}{\partial {t_1}^2} \log(\tau)$.

Sato Grassmannian and KdV reduction

The KP hierarchy can be written in several different ways. In particular, it has an interpretation as Plücker relations for the semi-infinite **Sato Grassmannian** which is equivalent to the **free charged fermion description**. These relations ensure that a particular vector $v \in \bigoplus_\lambda v_\lambda$ (here $v_\lambda = z^{\lambda_1 - 1} \wedge z^{\lambda_2 - 2} \wedge z^{\lambda_3 - 3} \wedge \dots$ and the sum is taken over all partitions $\lambda \vdash d$, $d \in \mathbb{Z}_{\geq 0}$ arranged as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq$ ...) can be represented as

$$v = \Phi_1 \wedge \Phi_2 \wedge \Phi_3 \wedge \dots ,$$

where $\Phi_i = z^{-i}(1 + O(z))$. This description is equivalent to the Hirota bilinear identity, and the tau-function in the **Miwa parametrization** is given by

$$\tau \Big|_{t_k = \frac{1}{k}(z_1^k + \dots + z_n^k)} = \frac{\det_{i,j \leq n} z_j \Phi_i(z_j)}{\Delta(z^{-1})}.$$

If a tau-function of the KP hierarchy does not depend on the even time variables,

$$\frac{\partial}{\partial t_{2k}} \tau_{\mathrm{KdV}}(\mathbf{t}) = 0 \qquad \forall\, k > 0,$$

than it is a tau-function of the **Korteweg–De Vries (KdV) hierarchy**.

Table of Contents

Topological recursion

Topological recursion **[Chekhov, Eynard, Orantin, '06-'08]** associates to a small collection of input data that consists of two meromorphic functions x and y defined on a compact Riemann surface Σ (the triple (Σ, x, y) is traditionally called the **spectral curve**) a system

$$\{\omega_n^{(g)}\}_{\substack{g\geq 0, n\geq 1\\ 2g-2+n>0}}$$

of symmetric meromorphic n-differentials $\omega_n^{(g)}$ on Σ^n, $n \geq 1$, via an explicitly given recursive procedure.

To motivate its study it is enough to mention that it provides a unifying interface between matrix models, mirror symmetry, enumerative geometry, enumerative combinatorics, quantum knot invariants, free probability theory, topological string amplitudes based on topological vertex, and many further areas of mathematics and mathematical physics.

Topological recursion originates from the Virasoro constraints and spectral curve description of correlation functions in matrix models developed in '80s-'00s with the roots in Wigner's semicircular distribution **[Ambjørn, Chekhov, Kristjansen, Makeenko, Jurkiewicz, Marshakov, Mironov, Morozov]**.

Question: What is the general relation between topological recursion and KP?

Topological recursion

Topological recursion (TR) have several forms and can be defined on a spectral curve of any genus. I will consider only genus zero case, $\Sigma = \mathbb{C}P^1$.

Let x and y be rational functions. Assume that all zeros $\{p_1, \dots, p_N\}$ of dx are simple and dy is regular and non-zero at these points. Let σ_i be the deck transformation with respect to x near p_i. We set $\omega_1^{(0)}(z_1) := -y(z_1)dx(z_1)$ and $\omega_2^{(0)}(z_1, z_2) := \frac{dz_1 dz_2}{(z_1 - z_2)^2}$.

The **correlation differentials** $\omega_n^{(g)}$, $g \geq 0$, $n \geq 1$, $2g - 2 + n > 0$, are computed recursively as

$$\omega_n^{(g)}(z_{[\![n]\!]}) = \frac{1}{2} \sum_{i=1}^{N} \operatorname*{res}_{z=p_i} \frac{\int_z^{\sigma_i(z)} \omega_2^{(0)}(z_1, \cdot)}{\omega_1^{(0)}(\sigma_i(z)) - \omega_1^{(0)}(z)} \Bigg(\omega_{n+1}^{(g-1)}(z, \sigma_i(z), z_{[\![n]\!] \setminus \{1\}}) + \sum_{\substack{g_1 + g_2 = g \\ I_1 \sqcup I_2 = [\![n]\!] \setminus \{1\} \\ (g_i, |I_i|) \neq (0,0)}} \omega_{|I_1|+1}^{(g_1)}(z, z_{I_1}) \omega_{|I_2|+1}^{(g_2)}(\sigma_i(z), z_{I_2}) \Bigg).$$

Here $[\![n]\!]$ denotes $\{1, \dots, n\}$.

The role of functions x and y in topological recursion is not symmetric.

Let us have completely symmetric assumptions for x and y: both are meromorphic, the zeros of dx and dy are simple and disjoint, dy is regular at the zero locus of dx and vice versa. Then we have two instances of topological recursion on Σ: for the input given by the pair (x, y) and for the input given by $(x^\vee, y^\vee) := (y, x)$. Denote the **dual correlation differentials** produced by the latter input by $\{\omega_n^{\vee,(g)}\}$.

There is a universal differential-algebraic relation that captures the exchange of x and y in the topological recursion procedure. It was conjectured in **[Borot, Charbonnier, Garcia-Failde, Leid, Shadrin '21]**, and proved in genus zero in **[Hock '22]**.

THEOREM (ABDKS '22)

$$\frac{\omega_n^{\vee,(g)}(z_{[\![n]\!]})}{\prod_{i=1}^n dx_i^\vee} = (-1)^n\, [\hbar^{2g}] \sum_\Gamma \frac{\hbar^{2g(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \prod_{i=1}^n \sum_{k_i=0}^\infty \partial_{y_i}^{k_i} [u_i^{k_i}] \frac{dx_i}{dy_i}$$

$$\frac{1}{u_i} e^{u_i \mathcal{S}(\hbar u_i \partial_{x_i}) \sum_{\tilde g=0}^\infty \hbar^{2\tilde g} \frac{\omega_1^{(\tilde g)}(z_i)}{dx_i} - u_i \frac{\omega_1^{(0)}(z_i)}{dx_i}}$$

$$\prod_{e \in E(\Gamma)} \prod_{j=1}^{|e|\geq 2} \lfloor_{(\tilde u_j, \tilde x_j) \to (u_{e(j)}, x_{e(j)})} \tilde u_j \mathcal{S}(\hbar \tilde u_j \partial_{\tilde x_j}) \sum_{\tilde g=0}^\infty \hbar^{2\tilde g} \frac{\tilde\omega_{|e|}^{(\tilde g)}(\tilde z_{[\![|e|]\!]})}{\prod_{j=1}^{|e|} d\tilde x_j} + \delta_{(g,n),(0,1)}(-x_1).$$

Description of the $x - y$ swap formula

- The sum is taken over all connected graphs Γ with n labeled vertices and multiedges of index ≥ 2, where the index of a multiedge is the number of its legs' and we denote it by $|e|$. By $g(\Gamma)$ we denote the first Betti number of Γ.
- For a multiedge e with index $|e|$ we control its attachment to the vertices by the associated map $e\colon [\![|e|]\!] \to [\![n]\!]$ that we denote also by e, abusing notation (so $e(j)$ is the label of the vertex to which the j-th "leg" of the multiedge e is attached). Do note that this map can be an arbitrary map from $[\![|e|]\!]$ to $[\![n]\!]$; in particular, it might not be injective, i.e. we allow a given multiedge to connect to a given vertex with several of its "legs".
- $\tilde\omega_2^{(0)}(\tilde x_1, \tilde x_2) = \omega_2^{(0)}(\tilde x_1, \tilde x_2) - \frac{d\tilde x_1 d\tilde x_2}{(\tilde x_1 - \tilde x_2)^2}$ if $e(1) = e(2)$, and $\tilde\omega_2^{(0)}(\tilde x_1, \tilde x_2) = \omega_2^{(0)}(\tilde x_1, \tilde x_2)$ otherwise. For all $(g,n) \neq (0,2)$ we simply have $\tilde\omega_n^{(g)} = \omega_n^{(g)}$.
- $[x^m] \sum_{i=-\infty}^{\infty} a_i x^i := a_m$.
- By $\lfloor_{a \to b}$ we denote the operator of substitution $a \to b$, that is, $\lfloor_{a\to b} f(a) = f(b)$ for any function f.
- The function $\mathcal{S}(z)$ is defined as $\mathcal{S}(z) := \frac{e^{z/2} - e^{-z/2}}{z}$.

For each $g \geq 0$, $n \geq 1$, equation is manifestly a finite sum of finite products of differential operators applied to $\omega_m^{(\tilde g)}$ for $2\tilde g - 2 + m \geq 0$

Table of Contents

Intersection theory on the Moduli spaces of Riemann surfaces

Denote by $\overline{\mathcal{M}}_{g,n}$ the **Deligne–Mumford compactification of the moduli space** $\mathcal{M}_{g,n}$ of all compact Riemann surfaces S of genus g with n distinct marked points. It is a non-singular complex orbifold of dimension $3g-3+n$, which is empty unless the **stability condition**

$$2g-2+n>0$$

is satisfied. Intersection theory on $\overline{\mathcal{M}}_{g,n}$ " $=$ " Two-dimensional topological gravity **[Witten '91]**.

For each marking index i consider the cotangent line bundle $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$, whose fiber over a point $[S, z_1, \ldots, z_n] \in \overline{\mathcal{M}}_{g,n}$ is the complex cotangent space $T^*_{z_i}S$ of S at z_i. Let $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ denote the first Chern class of $\mathbb{L}_i$. We consider the intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n} \in \mathbb{Q}.$$

The integral on the right-hand side vanishes unless the stability condition is satisfied, all k_i are non-negative integers, and the **dimension constraint** holds true,

$$3g-3+n=\sum_{i=1}^{n} k_i.$$

Let T_i, $i \geq 0$, be formal variables. We introduce the **Kontsevich–Witten tau-function**

$$\tau_{\mathrm{KW}} := \exp\left(\sum_{g,n} \hbar^{2g-2+n} \sum_{k_1,\dots,k_n \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n} \frac{\prod T_{k_i}}{n!}\right) \in \mathbb{Q}[\mathbf{T}][\![\hbar]\!],$$

that can be considered as a generating function of the Gromov–Witten invariants of a point. A parameter $\hbar$ here is introduced to trace the Euler characteristic of the punctured curve Σ (**topological expansion**).

Witten's conjecture, proved by Kontsevich, states that the partition function τ_{KW} becomes a tau-function of the KdV hierarchy after the change of variables $T_k = (2k+1)!!t_{2k+1}$.

THEOREM (KONTSEVICH '92)

The generating function τ_{KW} is a tau-function of the KdV hierarchy in the variables t_k.

BRÉZIN–GROSS–WITTEN TAU-FUNCTION

The **Brézin–Gross–Witten (BGW)** model was introduced in lattice gauge theory 40 years ago. It can be defined in terms of the asymptotic expansion of a matrix integral.

This tau-function has a natural enumerative geometry interpretation given by the intersection theory of Norbury's Θ-classes, also related to super Riemann surfaces. Norbury's Θ-classes are the cohomology classes, $\Theta_{g,n} \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$. Consider the generating function of the intersection numbers

$$\tau_\Theta := \exp\left(\sum_{g,n} \hbar^{2g-2+n} \sum_{k_1,\dots,k_n \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{k_1} \psi_2^{k_2} \cdots \psi_n^{k_n} \frac{\prod T_{k_i}}{n!}\right).$$

Genus zero part is trivial! For this generating function **[Norbury '17]** has suggested a direct analog of Witten's conjecture.

THEOREM (CHIDAMBARAM, GARCIA-FAILDE, GIACCHETTO '22)

The generating function τ_Θ is the BGW tau-function of the KdV hierarchy in the variables $t_{2k+1} = \frac{T_k}{(2k+1)!!}$,

$$\tau_\Theta = \tau_{\mathrm{BGW}}.$$

$$\log \tau_{\mathrm{KW}} = \left(\frac{{t_1}^3}{6} + \frac{t_3}{8}\right)\hbar + \left(\frac{t_3{t_1}^3}{2} + \frac{5\,t_5t_1}{8} + \frac{3\,{t_3}^2}{16}\right)\hbar^2 + \left(\frac{15\,t_1t_3t_5}{4} + \frac{3\,{t_3}^2{t_1}^3}{2} + \frac{5\,t_5{t_1}^4}{8} + \frac{35\,t_7{t_1}^2}{16} + \frac{3\,{t_3}^3}{8} + \frac{105\,t_9}{128}\right)\hbar^3 + O(\hbar^4),$$

$$\log \tau_{\mathrm{BGW}} = \frac{t_1}{8}\hbar + \frac{{t_1}^2}{16}\hbar^2 + \left(\frac{{t_1}^3}{24} + \frac{9\,t_3}{128}\right)\hbar^3 + \left(\frac{{t_1}^4}{32} + \frac{27\,t_3t_1}{128}\right)\hbar^4 + \left(\frac{{t_1}^5}{40} + \frac{27\,t_3{t_1}^2}{64} + \frac{225\,t_5}{1024}\right)\hbar^5 + \left(\frac{{t_1}^6}{48} + \frac{45\,t_3{t_1}^3}{64} + \frac{1125\,t_5t_1}{1024} + \frac{567\,{t_3}^2}{1024}\right)\hbar^6 + O(\hbar^7).$$

Kontsevich-type matrix integral

The asymptotic expansion of the **Kontsevich matrix model** describes the Kontsevich–Witten tau-function in the **Miwa parametrization**

$$\tau_{\mathrm{KW}}\Big|_{t_k=\frac{1}{k}\operatorname{Tr}\Lambda^{-k}} = \mathcal{C}^{-1}\int_{\mathcal{H}_N}[d\Phi]\exp\left(\frac{1}{\hbar}\operatorname{Tr}\left(\frac{\Phi^3}{3!}-\frac{\Lambda\Phi^2}{2}\right)\right),$$

where $\mathcal{H}_N$ is the space $N\times N$ Hermitian matrices and $[d\Phi]$ is the standard measure on it.
KdV integrability of the BGW model follows from the relation to the generalized Kontsevich-type model **[Mironov, Morozov, and Semenoff '96]**:

$$\tau_{\mathrm{BGW}}\Big|_{t_k=\frac{1}{k}\operatorname{Tr}\Lambda^{-k}} = \tilde{\mathcal{C}}^{-1}\int_{\mathcal{H}_N}[d\Phi]\exp\left(\frac{1}{2\hbar}\operatorname{Tr}\left(\Lambda^2\Phi+\Phi^{-1}-2\hbar N\log\Phi\right)\right).$$

Asymptotic expansion near the critical point $\Phi=\Lambda$.

TR for the KW and BGW tau-functions

The Kontsevich–Witten tau-function can be described by TR on the **Airy spectral curve** **[Eynard, Orantin ’07; Zhou ’13]**

$$x = -\frac{z^2}{2}, \quad y = z.$$

The correlation differentials generate the intersection numbers

$$\omega_n^{(g)}(z_{[\![n]\!]}) = \sum_{k_1+\cdots+k_n=3g-3+n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n} \frac{(2k_1+1)!!dz_1}{z_1^{2k_1+2}} \cdots \frac{(2k_n+1)!!dz_n}{z_n^{2k_n+2}}.$$

The Brézin–Gross–Witten tau-function corresponds to the **Bessel spectral curve** **[A. ’18; Do, Norbury ’18]**

$$x = -\frac{z^2}{2}, \quad y = z^{-1}.$$

One critical point of $dx = -zdz$, $z = 0$. The deck transformation is $\sigma(z) = -z$.

For both curve the dual TRs are trivial, because $dy = dz$ and $dy = dz^{-1}$ have no zeroes on Σ, therefore for $2g-2+n>0$ we have $\omega_n^{\vee,(g)} = 0$.

Table of Contents

n-POINT DIFFERENTIALS

Let Σ be a smooth complex curve, we call it the **spectral curve**. Consider a collection of n-differentials $\omega_n^{(g)}$ on Σ^n defined for all $g \geq 0$, $n \geq 1$.

We do not assume that these differentials satisfy TR relations!

We assume that all $\omega_n^{(g)}$'s are symmetric and meromorphic with no poles on the diagonals for $(g, n) \neq (0, 2)$, and $\omega_2^{(0)}$ is also symmetric and meromorphic but it has a second order pole on the diagonal with biresidue 1. It will be convenient to arrange the n-differentials $\omega_n^{(g)}$, $g \geq 0$, for each fixed n into generating series:

$$\omega_n := \sum_{g=0}^{\infty} \hbar^{2g-2+n} \omega_n^{(g)}.$$

Systems of n-point differentials provide a natural and convenient way to assemble the answers to various problems in many interrelated areas of mathematics and theoretical physics.

To pass from a system of differentials to the KP tau-functions, we need the following definition:

Definition

A point $o \in \Sigma$ is called **regular** for the system of differentials $\{\omega_n^{(g)}\}$ if $\omega_n^{(g)} - \delta_{(g,n),(0,2)} \frac{dx_1 dx_2}{(x_1-x_2)^2}$ is regular at $(o,\dots,o) \in \Sigma^n$ for all $(g,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$, $(g,n) \neq (0,1)$, where x is a local coordinate on Σ at o.

Note that the regularity condition for $\{\omega_n^{(g)}\}$ is independent of a choice of local coordinate and that we have no condition for $\omega_1^{(0)}$.

For a regular point o, and an arbitrary local coordinate x, the forms $\omega_n^{(g)}$ can be expanded

$$\omega_n = \sum_{g=0}^{\infty} \hbar^{2g-2+n} \omega_n^{(g)} = \delta_{n,2} \frac{dx_1 dx_2}{(x_1-x_2)^2} + \sum_{k_1,\dots,k_n=1}^{\infty} f_{k_1,\dots,k_n} \prod_{i=1}^{n} x_i^{k_i-1} dx_i,$$

where the coefficients $f_{k_1,\dots,k_n}$ expand as $\sum_{g=0}^{\infty} \hbar^{2g-2+n} f^{(g)}_{k_1,\dots,k_n}$. Introduce the associated partition function $F = F_{o,x}$ as

$$F(t_1,t_2,\dots) := \sum_{g,n} \hbar^{2g-2+n} F_n^{(g)} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k_1,\dots,k_n=1}^{\infty} f_{k_1,\dots,k_n} t_{k_1} \dots t_{k_n},$$

KP integrability as a property of a system of differentials

This way we associate to a given collection of differentials $\omega_n^{(g)}$ a large family of potentials $F_{o,x}$: the freedom in its definition consists of the choice of a regular point o and the choice of a local coordinate x at o. We would like to study conditions assuring that the partition function

$$Z_{o,x} := \exp(F_{o,x})$$

is a tau-function of the KP hierarchy.

Theorem (ABDKS '23)

The KP integrability is an internal property of the collection of differentials: if $Z_{o,x}$ is a tau-function of the KP hierarchy for some choice of a regular point o and a coordinate x at this point, then it is a tau-function of the KP hierarchy for any other choice of a regular point and a local coordinate at that point.

The choice of $\omega_1^{(0)}$ does not affect the KP integrability property, it can be chosen in an arbitrary way. For definiteness, we do not include the terms with $(g,n)=(0,1)$ into the generating functions.

A change of the local spectral parameter and the regular point provide a symmetry of the KP hierarchy, given by an element of the Virasoro subgroup of $\widehat{GL_\infty}$.

For a given formal power series $\tau(\mathbf{t})$ the kernel $K(x_1,x_2)$ is defined as

$$K(x_1,x_2) := \frac{1}{x_1-x_2}\tau\Big|_{t_k=\frac{1}{k}(x_1^k-x_2^k)}.$$

If τ is a tau-function of the KP hierarchy, then K is called the **Baker–Akhiezer kernel**. It is a sum of one specific term $(x_1-x_2)^{-1}$ and some formal power series in x_1,x_2. Under the assumption of KP integrability it characterizes the solution uniquely and can be used to express the n-point functions W_n associated to τ by the following formulas. Let

$$W_n(x_1,\dots,x_n) := \Big(\prod_{i=1}^{n}\sum_{k=1}^{\infty} x_i^{k-1}\partial_{t_k}\Big)\log\tau\Big|_{\mathbf{t}=\mathbf{0}},$$

then we have the **determinantal formulas**:

$$W_1(x_1) = \lim_{x_1'\to x_1}\Big(K(x_1,x_1') - \frac{1}{x_1-x_1'}\Big);$$

$$W_2(x_1,x_2) = -K(x_1,x_2)K(x_2,x_1) - \frac{1}{(x_1-x_2)^2};$$

$$W_n(x_{[\![n]\!]}) = (-1)^{n-1}\sum_{\sigma\in C_n}\prod_{i=1}^{n}K(x_i,x_{\sigma(i)}), \qquad n\geq 3,$$

From determinantal formulas to KP integrability

These formulas are direct consequences of the Wick formula.

Lemma (Zhou '15)

A formal power series τ is a KP tau-function if and only if the determinantal formulas hold.

In the context of KP integrability as a property of a system of differentials, we consider a given system of differentials $\{\omega_n^{(g)}\}$, and we assume that the tau-function and the corresponding kernel K are associated with a particular choice of the regular point o of the spectral curve and the local coordinate x at this point. We introduce an invariant version of the Baker–Akhiezer kernel as the bi-half-differential

$$\mathbb{K}(x_1,x_2) := K(x_1,x_2)\sqrt{dx_1 dx_2}$$

$$= \frac{\sqrt{dx_1\, dx_2}}{x_1 - x_2} \exp\left(\sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} \int_{z_2}^{z_1} \cdots \int_{z_2}^{z_1} \omega_n^{(g)} + \frac{1}{2} \int_{z_2}^{z_1}\int_{z_2}^{z_1} (\omega_2^{(0)} - \omega_2^{(0),\text{sing}}) \right).$$

then the determinantal formulas take the form

$$\omega_n(x_{[\![n]\!]}) = (-1)^{n-1} \sum_{\sigma \in C_n} \prod_{i=1}^{n} \mathbb{K}(x_i, x_{\sigma(i)}), \qquad n \geq 2.$$

$x - y$ swap relation, KP integrability, and spectral curve

One of the main results about the $x - y$ swap relation can be formulated as follows:

Theorem (ABDKS '23)

The $x - y$ swap relation preserves KP integrability: the original system of differentials $\omega_n^{(g)}$ is KP integrable if and only if the dual system of differentials $\omega_n^{\vee(g)}$ is KP integrable.

KP integrability restricts the topology of the spectral curve:

Theorem (ABDKS '23)

If $\omega_2^{(0)}$ has no other poles than on the diagonal and $\{\omega_n^{(g)}\}$ satisfy the KP integrability property, then the spectral curve is rational.

This condition on $\omega_2^{(0)}$ is always satisfied, by definition, in the set-up of topological recursion:

Theorem (ABDKS '23)

If the differentials $\{\omega_n^{(g)}\}$ produced by topological recursion are KP integrable, then the spectral curve Σ is rational.

Table of Contents

Integral transform

Let us consider the integral transform of a function $f^\vee(z)$,

$$f(z) = \frac{\mathrm{i}}{\sqrt{2\pi\,\hbar}} \int f^\vee(\chi)\, y'(\chi)\, e^{\frac{1}{\hbar}\left(x(z)(y(z)-y(\chi))+\int_z^\chi x\,dy\right)}\, d\chi.$$

Applying the Taylor expansion of the exponent at the point $\chi = z$ we compute

$$\frac{1}{\hbar}\left(x(z)(y(z)-y(\chi)) + \int_z^\chi x\, y'\, dz\right) = \frac{1}{2} x'(z) y'(z) \frac{(\chi-z)^2}{\hbar} + O((\chi-z)^3).$$

This expression is a quadratic form in $\frac{\chi-z}{\sqrt{\hbar}}$ plus higher order corrections involving positive powers of $\sqrt{\hbar}$. Therefore, applying the shift $\chi = z + \sqrt{\hbar}\xi/\sqrt{x'(z)y'(z)}$ we can rewrite (and, actually, define) this integral as

$$f(z) = \frac{\mathrm{i}}{\sqrt{2\pi}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{f^\vee\left(z+\frac{\sqrt{\hbar}\,\xi}{\sqrt{x'(z)y'(z)}}\right)\; y'\left(z+\frac{\sqrt{\hbar}\,\xi}{\sqrt{x'(z)y'(z)}}\right)}{\sqrt{x'(z)y'(z)}}\; e^{\frac{\xi^2}{2}+\sum_{k\geq 3} \hbar^{\frac{k}{2}-1} a_k(z) \frac{\xi^k}{k!}}\, d\xi,$$

where $a_k(z) := (\partial_z^{k-1}(xy') - xy^{(k)})/(x'(z)y'(z))^{k/2}$. The coefficients of the half-integer powers of $\hbar$ involve odd powers of ξ only and do not contribute to the integral. As a result, $f(z)$ is a series in nonnegative integer powers of $\hbar$ whose coefficients are certain rational combinations of derivatives of different order of the functions $f^\vee$, x, and y.

$x - y$ swap for Baker–Akhiezer kernel

If the systems of differentials $\{\omega_n^{(g)}\}$ and hence $\{\omega_n^{\vee,(g)}\}$ are KP integrable, in the complicated combinatorial expression of $x - y$ duality they both can be replaced by a simpler determinantal formulas. This observation raises a natural question: what happens with the Baker–Akhiezer kernel under the $x - y$ swap? It proves out that there are explicit formulas relating $x - y$ dual kernels.

Let us consider two integral transforms which are inverse to each other. Let for some function $f^\vee(z)$

$$f(z) = \frac{\mathrm{i}}{\sqrt{2\pi\,\hbar}} \int f^\vee(\chi)\, y'(\chi)\, e^{\frac{1}{\hbar}\left(x(z)(y(z)-y(\chi))+\int_z^\chi x\,dy\right)}\, d\chi.$$

Then

$$f^\vee(z) = \frac{1}{\sqrt{2\pi\,\hbar}} \int f(\chi)\, x'(\chi)\, e^{-\frac{1}{\hbar}\left(y(z)(x(z)-x(\chi))+\int_z^\chi y\,dx\right)}\, d\chi.$$

All integrals of this type are understood purely formally in the sense of asymptotic expansions for small absolute value of $\hbar$ near the critical points $\chi = z$. With this convention the expressions become formal Gaussian integrals and the coefficient of each power of $\hbar$ of this integral is a finite order differential operator applied to $f^\vee$ or f, respectively.

This type of transformation connects the Baker–Akhiezer kernels on the two sides of the $x-y$ duality.

THEOREM (ABDKS '23)

The Baker–Akhiezer kernels $\mathbb{K}$ and $\mathbb{K}^\vee$ related by the $x-y$ swap are expressed in terms of one another by the following double integrals:

$$K(z_1,z_2) = \frac{-\mathrm{i}}{2\pi\hbar}\iint K^\vee(\chi_1,\chi_2)\, y'(\chi_1)y'(\chi_2)\,d\chi_1 d\chi_2 \times \\ e^{-\frac{1}{\hbar}\left(x(z_2)(y(z_2)-y(\chi_1))+\int_{z_2}^{\chi_1} x\,dy\right)} e^{\frac{1}{\hbar}\left(x(z_1)(y(z_1)-y(\chi_2))+\int_{z_1}^{\chi_2} x\,dy\right)};$$

$$K^\vee(z_1,z_2) = \frac{-\mathrm{i}}{2\pi\hbar}\iint K(\chi_1,\chi_2)\, x'(\chi_1)x'(\chi_2)d\chi_1 d\chi_2 \times \\ e^{-\frac{1}{\hbar}\left(y(z_2)(x(z_2)-x(\chi_1))+\int_{z_2}^{\chi_1} y\,dx\right)} e^{\frac{1}{\hbar}\left(y(z_1)(x(z_1)-x(\chi_2))+\int_{z_1}^{\chi_2} y\,dx\right)}.$$

Here $\mathbb{K}(z_1,z_2) = K(z_1,z_2)\sqrt{dx_1 dx_2}$, $\mathbb{K}^\vee(z_1,z_2) = K^\vee(z_1,z_2)\sqrt{dy_1 dy_2}$, $x' = \partial_z x$, $y' = \partial_z y$. We consider the integrals as asymptotic expansions in $\sqrt{\hbar}$ near the critical point, which is $\chi_1 = z_2$, $\chi_2 = z_1$.

These integral transforms are inspired by and closely related to the Kontsevich matrix model and its generalizations.

$x - y$ DUAL OF A TRIVIAL CASE

An important special case concerns the situation when the $x-y$ dual side is known to be trivial.

We assume that $\omega_0^{\vee,(2)} = \frac{dz_1 dz_2}{(z_1-z_2)^2}$ and for all $(g,n) \neq (0,2)$

$$\omega_n^{\vee,(g)} = 0.$$

Here z is a meromorphic function on Σ that can serve as a coordinate at the point $o \in \Sigma$. In this case $Z^{\vee}_{o,z} = 1$ is a tau-function of the KP hierarchy, and under the $x-y$ duality relation the Baker–Akhiezer kernel is given by the double integral:

$$\frac{\mathbb{K}(z_1,z_2)}{\sqrt{dz_1 dz_2}} = \frac{-\mathrm{i}\sqrt{x'(z_1)x'(z_2)}}{2\pi\hbar} \iint \frac{\sqrt{y'(\chi_1)y'(\chi_2)}}{\chi_1-\chi_2} d\chi_1 d\chi_2 \times$$
$$e^{-\frac{1}{\hbar}\left(x(z_2)(y(z_2)-y(\chi_1))+\int_{z_2}^{\chi_1} x\,dy\right)} e^{\frac{1}{\hbar}\left(x(z_1)(y(z_1)-y(\chi_2))+\int_{z_1}^{\chi_2} x\,dy\right)}.$$

Note that in the KP integrable case the kernel $\mathbb{K}$ carries the complete information on the corresponding KP tau-functions. Let (o, z) be a regular point for all differentials and a local coordinate at this point, respectively. Then, the semi-infinite plane corresponding to the tau-function $Z = Z_{o,z}$ is spanned by the Taylor coefficients of the expansion of $\mathbb{K}(z_1, z_2)/\sqrt{dz_1 dz_2}$ in z_2.

THEOREM (ABDKS '23)

Assume $x'y'$ has a pole of degree ≥ 3 at $z = 0$. Let $\Phi_i^{*,\vee}(z)$, $i \geq 1$ have the form $z^{-i}(1 + O(z))$ and generate the semi-infinite plane corresponding to the tau-function $Z^{*,\vee}(\mathbf{t}) = Z^{\vee}(-\mathbf{t})$, where $Z^{\vee}$ is $x - y$ dual to the tau-function $Z = Z_{o,z}$. Then the following vectors Φ_i, $i \geq 1$, have the form $\Phi_i(z) = z^{-1}(1 + O(z))$ and generate the semi-infinite plane corresponding to the tau-function Z:

$$\Phi_i(z) = \sqrt{\frac{-x'(z)}{2\pi\hbar}} \int \Phi_i^{*,\vee}(\chi) \sqrt{y'(\chi)}\, e^{\frac{1}{\hbar}\left(x(z)(y(z)-y(\chi)) + \int_z^{\chi} x\,dy\right)} d\chi, \qquad i \geq 1.$$

The so-called $p - q$ duality describes a relation between two minimal models coupled to two-dimensional topological gravity. In our language it should correspond to the swap of x and y defined to be polynomials of fixed finite degrees p and q, respectively. We claim that in these special cases the our statements provide the $\mathbf{p} - \mathbf{q}$ **duality** proposed in **[Kharchev, Marshakov '95]**

Table of Contents

A typical source of situations with trivial dual differentials comes from the theory of topological recursion. Assume z is a global coordinate on $\mathbb{C}P^1$, and $y(z) = (az+b)/(cz+d)$ with $ad - bc \neq 0$, and $x(z)$ is rational. Then the theory of topological recursion assigns to this input a system of differentials $\{\omega_n^{(g)}\}$. These differentials are regular at the points $o \in \mathbb{C}P^1$ where $dx \neq 0$.

THEOREM (ABDKS '23)

Let $\{\omega_n^{(g)}\}$ be a system of differentials constructed by topological recursion for the input consisting of $\Sigma = \mathbb{C}P^1$ with a global coordinate z and $y(z) = (az+b)/(cz+d)$. Let $z = 0$ be a regular point for the system of differentials. Then $Z_{0,z}$ is a KP tau-function. Moreover, if $x'y'$ has a pole of degree at least 3 at $z = 0$, then the tau-function $Z_{0,z}$ is described by the point of the semi-infinite Grassmannian spanned by

$$\Phi_i(z) = \sqrt{\frac{-x'(z)}{2\pi\hbar}} \int \chi^{-i} \sqrt{y'(\chi)}\, e^{\frac{1}{\hbar}\left(x(z)(y(z)-y(\chi)) + \int_z^{\chi} x\,dy\right)} d\chi, \qquad i \geq 1.$$

As a corollary of this theorem, we obtain the following explicit formulas in two important sets of examples of topological recursion.

(1) Let $x = z^{-r}/r$, $y = -z^{-1}$, $r \geq 2$. In this case the associated partition function $Z_{0,z}$ is known to be the string solution of the r-th Gelfand–Dickey hierarchy, governed the intersection theory of **Witten's class**. The corresponding point on the Sato Grassmannian is given by

$$\Phi_i(z) = \frac{1}{\sqrt{2\pi\hbar z^{r+1}}} \int \chi^{-i-1} e^{\frac{1}{r\,(r+1)\,\hbar}\left(z^{-r} - rz^{-r-1}(\chi - z) - \chi^{-r}\right)} d\chi, \qquad i \geq 1.$$

(2) Let $x = z^{-r}/r$, $y = -z$, $r \geq 2$. The corresponding point on the Sato Grassmannian is given by

$$\Phi_i(z) = \frac{\mathrm{i}}{\sqrt{2\pi\hbar z^{r+1}}} \int \chi^{-i} e^{\frac{1}{r\,(r-1)\,\hbar}\left(-z^{1-r} + (r-1)z^{-r}(\chi - z) + \chi^{1-r}\right)} d\chi, \qquad i \geq 1.$$

In these examples the limit procedure on the side of topological recursion is described in detail and proved by **[Charbonnier, Chidambaram, Garcia-Failde, Giacchetto '22]**.

For $x = z^r/r$ and $y = z^{-1}$ the point $z = \infty$ is regular, so we consider the partition function $Z_{\infty,z^{-1}}$.

THEOREM (ABDKS '23)

The partition function $Z_{\infty,z^{-1}}$ is given by the following matrix integral:

$$Z_{\infty,z^{-1}} = \frac{\int_{\mathcal{H}_N} [dM] \exp\left(\frac{1}{\hbar}\operatorname{Tr}\left(\frac{\Lambda^r M}{r} + \frac{M^{1-r}}{r(r-1)} - \hbar N \ln M\right)\right)}{\int_{\mathcal{H}_N} [dM] \exp\left(\frac{1}{\hbar}\operatorname{Tr}\left(\frac{M^{1-r}}{r(r-1)} - \hbar N \ln M\right)\right)},$$

where $\mathcal{H}_N$ is the space $N \times N$ Hermitian matrices and $[dM]$ is the standard measure on it, and the variables t_k are the Miwa variables, $t_k = \frac{1}{k}\operatorname{Tr}\Lambda^{-k}$, $i = 1, 2, \dots$.

The latter matrix integral was introduced in **[Mironov, Morozov, Semenoff '96]** and is knows as a higher Brézin–Gross–Witten tau-function. The statement of this theorem was conjectured in **[Alexandrov, Dhara '22; Chidambaram, Garcia-Failde, Giacchetto '22]**.

Partition function $Z_{\infty,z^{-1}}$ has a geometric meaning. The formal power series expansion of its logarithm gives the intersection numbers of the ψ-classes with the so-called r-theta classes $\Theta^r_{g,n}$ on the moduli spaces of curves $\overline{\mathcal{M}}_{g,n}$. In particular, for $r = 2$ it is reduced to a generating series of ψ- and a combination of κ-classes **[Kazarian, Norbury '21]**.

Table of Contents

Any TR on a rational spectral curve is KP integrable

- Σ is of genus zero, that is, $\Sigma = \mathbb{C}P^1$;
- B is the canonical Bergman kernel, that is, $B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$, where z is a global affine coordinate on $\Sigma = \mathbb{C}P^1$;
- dx and dy are any meromorphic differentials (It is sufficient to assume that dx and dy are defined in a vicinity of the points $q_1, \dots, q_n$).

Theorem (ABDKS '24)

The system of generalized TR differentials for the input data as above possesses KP integrability property.

The upshot of this Theorem and the restriction of the KP integrability to genus zero spectral curve can be then formulated as follows:

Corollary

A system of differentials $\{\omega_n^{(g)}\}$ produced by generalized topological recursion is KP integrable if and only if the spectral curve is rational.

Open questions

- TR and integrability in the non-regular points. Multi-component KP.
- Family of tau-functions generated by topological recursion.
- BKP hierarchy in special points. Other integrable systems.
- Matrix models from TR
- Integrability for the higher genera spectral curves: Θ-function deformation.

Work in progress with Boris Bychkov, Petr Dunin-Barkowski, Maxim Kazarian, and Sergey Shadrin

Thank you!